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The Biot coefficient for an elasto-plastic material

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ABSTRACT

This paper investigates the influence of the development of elasto-plastic failure on the evolution of the Biot coefficient for a fluid-saturated geomaterial. Attention is restricted to the study of an elasto-plastic porous skeleton that has a solid phase with failure characteristics corresponding to an isotropic medium with a von-Mises-type failure criterion and an associated flow rule. The evolution of the Biot coefficient with the development of failure is illustrated through specific examples. The assumptions implicit in standard computational approaches for examining poroelastic behavior in the light of these developments are also discussed.

1. Introduction

The classical theory of poroelasticity developed by Biot (1941) is one of the key developments in geomechanics that describes the mechanical behavior of a porous elastic solid saturated by an ideal fluid (Cheng, 2015; Rice & Cleary, 1976; Selvadurai, 1996, 2007; Selvadurai & Suvorov, 2016; Verruijt, 2015). The theory was initially developed for the specific purpose of examining the mechanical behavior of fluid-saturated geomaterials. The use of the theoretical concepts has reached beyond the initial objectives and the theory has been applied to the study of the mechanics of a wide range of materials including bone (Cowin, 2001) and soft tissues such as biological materials including brain tissue and arterial materials. Extensive discussions, reviews and applications of poro-hyperelasticity are given in recent articles by Selvadurai and Suvorov (2016, 2017, 2018) and Suvorov and Selvadurai (2016).

Unlike the classical theory of soil consolidation proposed by Terzaghi (1925) and further discussed by Fillunger (1936) and others (see e.g. de Boer, 1999, 2000), Biot's theory of poroelasticity addresses the partitioning of the total stresses between the stress carried by the porous skeleton and the pore fluid pressure in the correct fashion, taking into consideration the constitutive behaviors of both the porous skeleton and the pore fluid and ensuring the compatibility of deformations. This aspect has recently been used quite successfully for the estimation of the Biot coefficient for limestone with extremely low permeability (Selvadurai, 2018), where saturation of the pore space necessary for experimental evaluation of the Biot coefficient requires an inordinate amount of time. The general formulation developed by Biot can address the stress partitioning and the stresses transmitted to the porous skeleton will ultimately govern its failure. The development of failure of the porous skeleton can have a significant influence in terms of alteration of the permeability of the porous medium; this can influence the rate of consolidation and, more significantly, the development of failure in the porous skeleton can alter its stiffness characteristics, which in turn will influence the partitioning of stresses (Selvadurai, 2004; Selvadurai & Shirazi, 2004, 2005). Therefore, an assessment of the manner in which the Biot coefficient can change

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due to the development of failure in the porous skeleton is relevant to the examination of poro-elasto-plastic materials, where it is implicitly assumed that the stress partitioning is based on the initial poroelastic estimate of the Biot coefficient (Pariseau, 1999; Selvadurai & Suvorov, 2012, 2014).

The present paper examines the problem of elasto-plastic failure of the porous skeleton in which the material of the solid phase has a failure criterion of the von-Mises type and an associated flow rule. The basic model used in the studies deals with the problem of an elasto-plastic spherical shell with an internal cavity filled with a fluid. This model is well known in the mechanics of composite materials and is usually referred to as the composite sphere assemblage model. It is useful to note that the composite sphere assemblage model (with empty pores) provides the estimate of the overall shear modulus that is noticeably smaller than the overall bulk modulus (Christensen, 1979). This is due to the fact that the overall Poisson's ratio of the composite sphere assemblage is close to 0.5. But for most rocks, the values of the bulk and shear moduli of the drained skeleton are close to each other. For example, Zimmerman (2000) reports that the values of the overall bulk modulus and shear modulus for the drained skeleton of a sandstone are 8.3 GPa and 6.3 GPa, respectively. Thus, the composite sphere assemblage is not the most appropriate micromechanical model for modeling *elastic* behavior of rocks. However, here we still use this model for studying the *elasto-plastic* response because of its simplicity and the possibility of deriving an analytical solution and making observations concerning, in particular, changes in the Biot coefficient during plastic flow.

It should be mentioned that the choice of a plasticity model for the material of the solid phase is not restricted to von-Mises type plasticity, if an analytical solution is desired. Several analytical solutions were developed for different composite spheres in which the Drucker–Prager plasticity model was used for the material of the outer shell (Chadwick, 1959; Narasimhan, 2004; Quang & He, 2008). However, these solutions are much more involved and were not used here because we wanted to facilitate an understanding of the essential features of a poro-elasto-plastic material response in the simplest way possible.

2. Two definitions of effective stress for elasto-plastic rocks

Let σ'_{ij} and σ''_{ij} denote two possible definitions of effective stress in the rock. It was noted by Rice (1977) that to find the plastic strain increment in rocks it is more appropriate to use Terzaghi's definition of effective stress (see also Makhnenko & Labuz, 2016), i.e.

$$\sigma'_{ij} = \sigma_{ij} + p\delta_{ij} \tag{2.1}$$

rather than Biot's effective stress

$$\sigma''_{ij} = \sigma_{ij} + B_{el} p\delta_{ij}. \tag{2.2}$$

Here B_{el} is elastic Biot coefficient, p is the fluid pressure and σ_{ij} are components of the (total) stress tensor. The standard sign convention for the stresses and fluid pressure is used here: the stresses are positive when they are tensile while the fluid pressure is positive when the fluid is in compression. Note the connection between the two effective stresses σ'_{ij} and σ''_{ij} that follows from (2.1) and (2.2):

$$\sigma'_{ij} = \sigma''_{ij} + (1 - B_{el})p \tag{2.3}$$

We will show that these two definitions of effective stress lead to different elasto-plastic constitutive equations, and in particular, to different tangent Biot coefficients.

We will prove this for the case of isotropic compression only, i.e., when $\sigma_{11} = \sigma_{22} = \sigma_{33}$ and shear stresses are zero.

2.1. First, assume that the yield surface is written in terms of Terzaghi's effective stress σ'_{ij} (Davis & Selvadurai, 2002), i.e., the yield surface (failure surface) is written as

$$f = \frac{\sigma'_{11} + \sigma'_{22} + \sigma'_{33}}{3} - F(\gamma). \tag{2.4}$$

Here $F(\gamma)$ is the function responsible for hardening of the material and γ is the hardening parameter. Note that the hardening parameter γ also depends on σ'_{ij}.

The plastic potential can be written as

$$g = \frac{\sigma'_{11} + \sigma'_{22} + \sigma'_{33}}{3}. \tag{2.5}$$

The plastic strain ε^{pl}_{11} is evaluated according to the flow rule

$$d\varepsilon^{pl}_{11} = d\lambda \frac{\partial g}{\partial \sigma'_{11}} = \frac{1}{3} d\lambda. \tag{2.6}$$

Next, we use the consistency condition:

$$\frac{\partial f}{\partial \sigma'_{11}} d\sigma'_{11} + \frac{\partial f}{\partial \sigma'_{22}} d\sigma'_{22} + \frac{\partial f}{\partial \sigma'_{33}} d\sigma'_{33} + \frac{\partial f}{\partial \gamma} d\gamma = 0. \tag{2.7}$$

Using (2.4) and (2.6), we can rewrite (2.7) as

$$d\sigma'_{11} + \frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\lambda}d\lambda = d\sigma'_{11} + \frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}d\varepsilon_{11}^{pl} = 0. \tag{2.8}$$

Using the connection between the two effective stresses (2.3), the consistency condition (2.8) becomes

$$d\sigma''_{11} + (1 - B_{el})dp + \frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}d\varepsilon_{11}^{pl} = 0. \tag{2.9}$$

It is known that the constitutive relationship for an elasto-plastic porous medium can be written as

$$\sigma_{11} = 3K_D\left(\varepsilon_{11} - \varepsilon_{11}^{pl}\right) - B_{el}p, \tag{2.10}$$

where K_D is the elastic bulk modulus of the drained porous medium. Therefore, using the definition of the Biot's effective stress (2.2), we obtain

$$\sigma''_{11} = 3K_D\left(\varepsilon_{11} - \varepsilon_{11}^{pl}\right). \tag{2.11}$$

Substitution of the incremental form of (2.11) into (2.9) gives

$$3K_D\left(d\varepsilon_{11} - d\varepsilon_{11}^{pl}\right) + (1 - B_{el})dp + \frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}d\varepsilon_{11}^{pl} = 0. \tag{2.12}$$

From Eq. (2.12) we can find the plastic strain increment as

$$d\varepsilon_{11}^{pl} = \frac{3K_D d\varepsilon_{11} + (1 - B_{el})dp}{3K_D - \frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}}. \tag{2.13}$$

Consequently, by substituting the plastic strain increment (2.13) into the incremental form of (2.10), the increment of the total stress can be obtained as

$$d\sigma_{11} = 3K_D\left(1 - \frac{K_D}{K_D - \frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}}\right)d\varepsilon_{11} - \left(B_{el} + K_D\frac{1 - B_{el}}{K_D - \frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}}\right)dp. \tag{2.14}$$

From this relationship, we can identify the tangent bulk modulus

$$K_t = K_D\left(1 - \frac{K_D}{K_D - \frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}}\right) = K_D\frac{-\frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}}{K_D - \frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}} \tag{2.15}$$

and tangent Biot coefficient

$$B_t = B_{el} + K_D\frac{1 - B_{el}}{K_D - \frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}} = \frac{K_D - B_{el}\frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}}{K_D - \frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon_{11}^{pl}}}. \tag{2.16}$$

It can be easily verified that

$$B_t = 1 - \frac{K_t}{K_s} = 1 - \frac{K_t(1 - B_{el})}{K_D}, \tag{2.17}$$

where K_s is the elastic bulk modulus of solid phase. Therefore, we have the usual connection between tangent Biot coefficient and tangent bulk modulus:

The tangent Biot coefficient B_t is defined in terms of the tangent bulk modulus K_t of the drained porous medium in the same way as the elastic Biot coefficient B_{el} is defined in terms of effective bulk modulus of the drained medium K_D.

2.2. Now we assume that the yield surface and plastic potential are expressed in terms of the Biot's effective stress

$$f = \frac{\sigma''_{11} + \sigma''_{22} + \sigma''_{33}}{3} - F(\gamma), \qquad g = \frac{\sigma''_{11} + \sigma''_{22} + \sigma''_{33}}{3}. \tag{2.18}$$

Note that the hardening parameter γ here depends on σ''_{ij} and thus it is not identical to γ defined for the previous case. The consistency condition is now given by

$$\frac{\partial f}{\partial \sigma''_{11}}d\sigma''_{11} + \frac{\partial f}{\partial \sigma''_{22}}d\sigma''_{22} + \frac{\partial f}{\partial \sigma''_{33}}d\sigma''_{33} + \frac{\partial f}{\partial \gamma}d\gamma = 0. \tag{2.19}$$

Table 1
Two definitions of effective stress used in the yield surface, plastic potential.

Terzaghi's definition	Biot's definition
Tangent Biot coefficient is non-constant and related to the tangent bulk modulus This definition is used in ABAQUS™ This definition is consistent with the exact analytical solution for the elasto-plastic composite sphere assemblage	Tangent Biot coefficient is constant and equal to the elastic Biot coefficient

By using (2.18) in (2.19), we can obtain

$$d\sigma''_{11} + \frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\lambda}d\lambda = d\sigma''_{11} + \frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon^{pl}_{11}}d\varepsilon^{pl}_{11} = 0. \tag{2.20}$$

After substituting the incremental form of (2.11)

$$d\sigma''_{11} = 3K_D\left(d\varepsilon_{11} - d\varepsilon^{pl}_{11}\right) \tag{2.21}$$

into (2.20), we obtain the plastic strain increment as

$$d\varepsilon^{pl}_{11} = \frac{3K_D d\varepsilon_{11}}{3K_D - \frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon^{pl}_{11}}}. \tag{2.22}$$

By inserting this plastic strain increment into the incremental form of the stress-strain constitutive Eq. (2.10), we will find the total stress increment

$$d\sigma_{11} = 3K_D\left(1 - \frac{K_D}{K_D - \frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon^{pl}_{11}}}\right)d\varepsilon_{11} - B_{el}dp. \tag{2.23}$$

Thus, the tangent bulk modulus K_t is formally given by the same type of expression as before

$$K_t = K_D\frac{-\frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon^{pl}_{11}}}{K_D - \frac{1}{3}\frac{\partial f}{\partial \gamma}\frac{d\gamma}{d\varepsilon^{pl}_{11}}} \tag{2.24}$$

but the Biot coefficient remains unchanged during plastic flow, i.e., it is equal to the elastic Biot coefficient. Note that the tangent bulk modulus depends here on σ''_{ij} and thus it is not the same as in the previous case.

We can summarize the concepts described above in the form of a table (Table 1), in which we also include some additional observations concerning the definition of effective stress given in the ABAQUS™ finite element code and the analytical solution of the problem of isotropic compression of a composite sphere assemblage.

3. Isotropic compression of an elasto-plastic composite sphere assemblage

Studies related to the composite sphere problem in elasto-plastic media have been central to the development of the classical theory of plasticity and have been extensively documented in the noteworthy volumes and articles by Nadai (1930), Sokolovsky (1946), Hopkins (1960), and Kachanov (1974), and in volumes by Drucker (1967), Chakrabarty (1998), and Davis and Selvadurai (2002). The composite sphere assemblage (CSA) model is used here to represent the geometry of the fluid-saturated porous medium; the inner core of the composite sphere is filled with fluid while the outer shell of the composite sphere constitutes the elasto-plastic solid phase. The theory presented in this and subsequent chapters mainly relies on the book by Sokolovsky (1946).

Consider a typical composite sphere where the radius of the inner core is a, and the radius of the outer surface of the composite sphere is b (Fig. 1).

Porosity can be defined as

$$\varphi = \frac{a^3}{b^3}. \tag{3.1}$$

The material of the solid phase is elasto-plastic of von-Mises type. The yield condition can be written as

$$\frac{1}{\sqrt{6}}\sqrt{(\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + (\sigma_1 - \sigma_2)^2} = k = \frac{\sigma_s}{\sqrt{3}}. \tag{3.2}$$

Here k is the yield stress in shear, σ_s is the yield stress in simple compression. In this study, the material is considered to be ideally plastic without hardening effects.

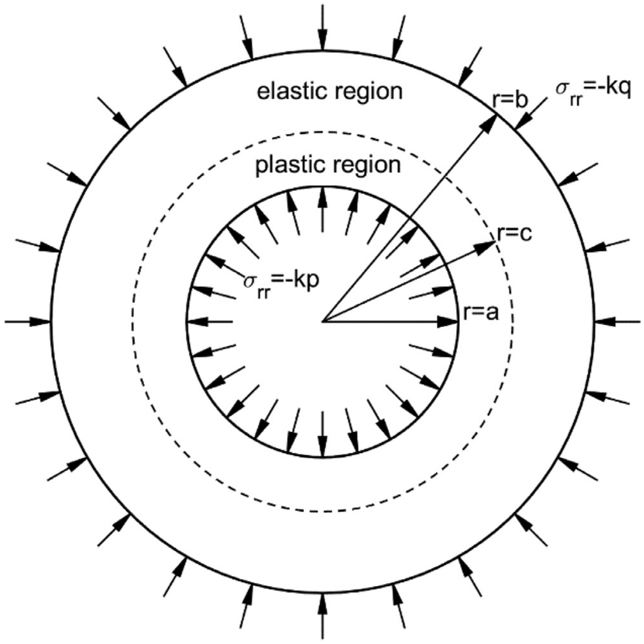

Fig. 1. Typical composite sphere in a composite sphere assemblage.

The composite sphere is subjected to an applied compressive stress on the external boundary $r = b$:

$$\sigma_{rr} = -kq, \qquad r = b. \tag{3.3}$$

Thus, q can be defined as the normalized applied pressure. The resulting radial stress on the fluid-solid phase boundary, $r = a$, is denoted by

$$\sigma_{rr} = -kp, \qquad r = a. \tag{3.4}$$

Therefore, p can be regarded as the normalized fluid pressure.

Components of the elastic strain can be evaluated according to the Hooke's law

$$\begin{cases} \varepsilon_{rr}^{el} = \frac{N}{G_s}\sigma + \frac{1}{2G_s}(\sigma_{rr} - \sigma) \\ \varepsilon_{\theta\theta}^{el} = \varepsilon_{\varphi\varphi}^{el} = \frac{N}{G_s}\sigma + \frac{1}{2G_s}(\sigma_{\theta\theta} - \sigma) \end{cases} \tag{3.5}$$

Here σ is the mean stress in the solid phase, i.e. $\sigma = \frac{1}{3}(\sigma_{rr} + 2\sigma_{\theta\theta})$, G_s is the shear modulus of the solid phase, and

$$N = \frac{1 - 2\nu_s}{2(1 + \nu_s)} = \frac{G_s}{3K_s}, \tag{3.6}$$

where K_s is the bulk modulus of the solid phase and ν_s is the Poisson's ratio of the solid phase.

The components of plastic strain are evaluated according to the following flow rule:

$$\begin{cases} \varepsilon_{rr}^{pl} = \frac{\psi - 1}{2G_s}(\sigma_{rr} - \sigma) \\ \varepsilon_{\theta\theta}^{pl} = \varepsilon_{\varphi\varphi}^{pl} = \frac{\psi - 1}{2G_s}(\sigma_{\theta\theta} - \sigma) \end{cases}. \tag{3.7}$$

Here ψ is the plasticity modulus that needs to be determined. The plastic strain of the solid phase is assumed to be incompressible.

The outer shell of the composite sphere has a plastic region that constitutes a spherical annulus $a \le r \le c$, and the elastic region $c \le r \le b$ (Fig. 1). Thus, $r = c$ is the interface between the plastic and elastic regions in the solid phase. Onset of yielding occurs when $c = a$ and the composite sphere becomes entirely plastic when $c = b$. We introduce the following dimensionless quantities

$$\rho = \frac{r}{b}, \qquad \alpha = \frac{a}{b}, \qquad \gamma = \frac{c}{b}. \tag{3.8}$$

We note that onset of yielding corresponds to $\gamma = \alpha = \varphi^{1/3}$, and the fully plastic condition corresponds to $\gamma = 1$. Therefore, the parameter γ characterizes the size of plastic region and the extent of plastic deformation, $\alpha \le \gamma \le 1$.

In addition, we normalize the radial displacement u_r and introduce the dimensionless radial displacement U_r:

$$U_r = \frac{u_r G_s}{kb}. \tag{3.9}$$

For the present radially-symmetric problem, the yield condition (3.2) can be written as

$$\sigma_{rr} - \sigma_{\theta\theta} = -\chi\sqrt{3}k, \tag{3.10}$$

where $\chi = \pm 1$.

Radial displacement in the elastic region can be found as (Sokolovsky, 1946)

$$U_r = N\rho\left\{-q + \chi\frac{2}{\sqrt{3}}\gamma^3\left(\frac{1}{4N}\frac{1}{\rho^3} + 1\right)\right\}, \qquad \gamma \le \rho \le 1. \tag{3.11}$$

Radial displacement in the plastic region is given by

$$U_r = N\rho\left\{-p + \chi\frac{2}{\sqrt{3}}\left(\frac{1+4N}{4N}\frac{\gamma^3}{\rho^3} + 3\ln\frac{\rho}{\alpha}\right)\right\}, \qquad \alpha \le \rho \le \gamma. \tag{3.12}$$

On the interface between the plastic and elastic regions, $\rho = \gamma$, these radial displacements must be equal, i.e.,

$$-q + \chi\frac{2}{\sqrt{3}}\gamma^3\left(\frac{1}{4N}\frac{1}{\gamma^3} + 1\right) = -p + \chi\frac{2}{\sqrt{3}}\left(\frac{1+4N}{4N} + 3\ln\frac{\gamma}{\alpha}\right). \tag{3.13}$$

Using the equality (3.13), we obtain the following relationship

$$p = q + \chi\frac{2}{\sqrt{3}}\left(1 - \gamma^3 + \ln\frac{\gamma^3}{\alpha^3}\right). \tag{3.14}$$

The equality (3.14) can be treated as a yield condition for the composite sphere assemblage. Now it becomes clear that the yield condition for the composite sphere assemblage is expressed in terms of Terzaghi's effective stress $k(p-q)$. Conditions for the onset of yielding can be obtained by setting $\gamma = \alpha$.

On the external boundary at $\rho = 1$, the radial displacement is obtained from the expression for U_r valid in the elastic region (3.11), i.e.,

$$U_{rb} = N\left\{-q + \chi\frac{2}{\sqrt{3}}\gamma^3\left(\frac{1}{4N} + 1\right)\right\}. \tag{3.15}$$

which gives

$$U_{rb} + Nq = N\chi\frac{2}{\sqrt{3}}\gamma^3\left(\frac{1}{4N} + 1\right). \tag{3.16}$$

Now we can represent the relationships (3.14) and (3.16) in the incremental form

$$\begin{aligned} dU_{rb} + Ndq &= N\chi\frac{2}{\sqrt{3}}\left(\frac{1}{4N} + 1\right)3\gamma^2 d\gamma, \\ dp &= dq - \chi\frac{2}{\sqrt{3}}\left(1 - \frac{1}{\gamma^3}\right)3\gamma^2 d\gamma \end{aligned} \tag{3.17}$$

By substituting $3\gamma^2 d\gamma$ from the first relationship of (3.17) into the second one and expressing dq, we can obtain the desired form of the elasto-plastic constitutive equation in incremental form

$$\frac{1 + \frac{1}{4N}}{\frac{1}{\gamma^3} + \frac{1}{4N}}dp = dq + \frac{\frac{1}{\gamma^3} - 1}{\frac{1}{\gamma^3} + \frac{1}{4N}}\frac{1}{N}dU_{rb}. \tag{3.18}$$

Note that this constitutive relationship is written in incremental form and it links three quantities: the increment of the applied pressure dq, the increment of the fluid pressure dp and the increment in the radial displacement dU_{rb} on the boundary $r = b$. The radial displacement u_r on $r = b$ is related to the overall volumetric strain ε_V^0 in the composite sphere according to

$$\varepsilon_V^0 = \frac{3u_r(r=b)}{b}. \tag{3.19}$$

Therefore, the constitutive Eq. (3.18) can also be written in terms of $d\varepsilon_V^0$, which is more common

$$\frac{1 + \frac{1}{4N}}{\frac{1}{\gamma^3} + \frac{1}{4N}}dp = dq + \frac{\frac{1}{\gamma^3} - 1}{\frac{1}{\gamma^3} + \frac{1}{4N}}\frac{G_s}{3Nk}d\varepsilon_V^0. \tag{3.20}$$

Using the constitutive Eq. (3.20), we can identify the tangent Biot coefficient B_t:

$$B_t = \frac{1 + \frac{1}{4N}}{\frac{1}{\gamma^3} + \frac{1}{4N}}. \tag{3.21}$$

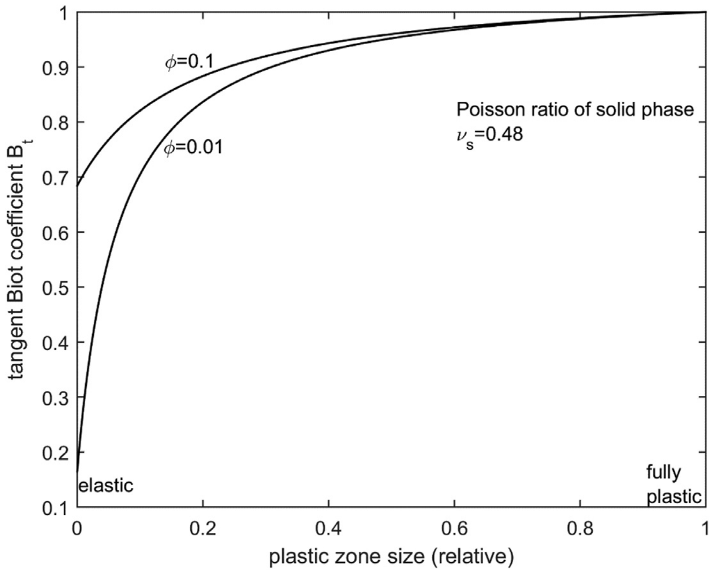

Fig. 2. Tangent Biot coefficient versus the size of plastic zone in the solid phase of the composite sphere. Porosities are 0.1 and 0.01 and Poisson's ratio of the solid phase is 0.48.

Note that the tangent Biot coefficient (3.21) depends on the plastic region size parameter γ. For the onset of yielding, B_t is equal to its elastic counterpart:

$$B_{el} = \frac{1 + \frac{1}{4N}}{\frac{1}{\alpha^3} + \frac{1}{4N}} = \frac{1 + \frac{1}{4N}}{\frac{1}{\varphi} + \frac{1}{4N}}. \tag{3.22}$$

For a fully plastic solid phase, the tangent Biot coefficient is equal to 1.

During plastic flow the tangent Biot coefficient gradually changes its value from the elastic Biot coefficient to a maximum value of 1, which occurs for an entirely plastic solid phase.

In Fig. 2 we plot the tangent Biot coefficient B_t versus the relative size of the plastic zone, defined as

$$0 \le \frac{c^3 - a^3}{b^3 - a^3} \le 1.$$

The porosities are 0.1 and 0.01 and Poisson's ratio of the solid phase is $\nu_s = 0.48$. We see that the composite sphere assemblage model predicts quite low values of the elastic Biot coefficient even for high values of Poisson's ratio ν_s. This is especially true for low porosity values. When the solid phase is fully plastic, the tangent Biot coefficient is equal to 1 in all cases, as expected.

It can also be deduced from the constitutive relationship (3.20) that the effective tangent bulk modulus K_t of the porous medium is given by

$$K_t = \frac{\frac{1}{\gamma^3} - 1}{\frac{1}{\gamma^3} + \frac{1}{4N}} \frac{G_s}{3N} = K_s \frac{\frac{1}{\gamma^3} - 1}{\frac{1}{\gamma^3} + \frac{1}{4N}}, \tag{3.23}$$

since $G_s/(3N) = K_s$. The effective elastic bulk modulus of the drained medium K_D can be obtained from the formula (3.23) by setting $\gamma = \alpha$, which results in

$$K_D = K_s \frac{\frac{1}{\alpha^3} - 1}{\frac{1}{\alpha^3} + \frac{1}{4N}} = K_s \frac{\frac{1}{\varphi} - 1}{\frac{1}{\varphi} + \frac{1}{4N}} = K_s(1 - B_{el}). \tag{3.24}$$

We now observe from (3.21) and (3.23) that

$$\frac{\frac{1}{\gamma^3} - 1}{\frac{1}{\gamma^3} + \frac{1}{4N}} = 1 - B_t \tag{3.25}$$

and therefore, the following connection holds between the tangent bulk modulus and tangent Biot coefficient of the composite sphere:

$$B_t = 1 - \frac{K_t}{K_s}. \tag{3.26}$$

Eq. (3.26) is consistent with the observation made after Eq. (2.17).

Using the results of Section 2 it can be verified that the same expressions for the tangent bulk modulus K_t and tangent Biot coefficient B_t are obtained if the yield condition (2.4) is taken in the form:

$$f = \frac{\sigma'_{11} + \sigma'_{22} + \sigma'_{33}}{3} - F(\gamma) = \frac{\sigma'_{11} + \sigma'_{22} + \sigma'_{33}}{3} - k\chi \frac{2}{\sqrt{3}}\left(1 - \gamma^3 + \ln \frac{\gamma^3}{\alpha^3}\right). \tag{3.27}$$

In this yield function the hardening function $F(\gamma)$ was taken from the yield condition (3.14) for the composite sphere assemblage and we also used the fact that $k(p-q)$ is the effective mean stress.

It is useful to derive expressions for the plastic volumetric strain and the plastic radial displacement. The plastic radial displacement U_{rb}^{pl} at the external surface $r = b$ can be defined as

$$U_{rb}^{pl} = U_{rb} - U_{rb}^{el}, \tag{3.28}$$

where U_{rb}^{el} is the elastic radial displacement of the external surface. The elastic radial displacement can be found, for example, by setting $\gamma = \alpha$ in (3.18), which gives

$$U_{rb}^{el} = \frac{\frac{1}{\alpha^3} + \frac{1}{4N}}{1 - \frac{1}{\alpha^3}} Nq - \frac{1 + \frac{1}{4N}}{1 - \frac{1}{\alpha^3}} Np. \tag{3.29}$$

The same expression is obtained if we use definitions of the elastic bulk modulus of a drained composite sphere (3.24) and the elastic Biot coefficient (3.22).

Recall that the total radial displacement for the elasto-plastic sphere is given by (3.16):

$$U_{rb} + Nq = N\chi \frac{2}{\sqrt{3}} \gamma^3 \left(\frac{1}{4N} + 1\right).$$

Therefore, after some algebraic manipulations, using definition (3.28) and the yield condition (3.14), we can derive the radial plastic displacement as

$$U_{rb}^{pl} = N\chi \frac{2}{\sqrt{3}} \left(\frac{1}{4N} + 1\right)\left[\gamma^3 - \left(1 - \gamma^3 + \ln \frac{\gamma^3}{\alpha^3}\right) \frac{\alpha^3}{1 - \alpha^3}\right]. \tag{3.30}$$

The increment of the radial plastic displacement can be obtained from (3.30) as

$$dU_{rb}^{pl} = N\chi \frac{2}{\sqrt{3}} \left(\frac{1}{4N} + 1\right) \frac{1 - \frac{\alpha^3}{\gamma^3}}{1 - \alpha^3} 3\gamma^2 d\gamma. \tag{3.31}$$

The expression for the radial plastic displacement (3.30) is valid for any value of the fluid pressure p and depends only on the plastic zone size parameter γ. But, according to the yield condition (3.14) for the composite sphere assemblage, the plastic parameter γ can be uniquely defined in terms of Terzaghi's effective pressure $q - p$. Thus, the radial plastic displacement U_{rb}^{pl} also depends only on $q - p$ and thus the Rice's observation is valid for the current case of composite sphere assemblage.

It is important to note that in many research works (e.g., Khalili & Loret, 2001) it is more common to use Biot's effective pressure $q - B_{el} p$ in the yield condition and plastic potential. As we saw, this results in the presence of an elastic Biot coefficient in the incremental form of the stress-strain relationship.

4. Isotropic compression of a composite sphere assemblage – the undrained state

As an example, we consider the application of an external pressure q to the elasto-plastic composite sphere in the undrained state. To solve a poroelastic (elasto-plastic) problem, additional relationships are required for finding the fluid pressure p. For the undrained state, this relationship will indicate zero change in the fluid content. The bulk modulus of the fluid phase is denoted by K_f; then, if the radius of the pore is a, in the undrained state, we have

$$K_f \frac{3u_{ra}}{a} = -kp \tag{4.1}$$

Here, u_{ra} is the actual radial displacement of the cavity of the composite sphere (not normalized), and $-kp$ is the compressive stress in the fluid, according to the boundary condition (3.4). Thus, by using (3.9), the normalized radial displacement at the cavity wall, U_{ra}, can be obtained from (4.1) as

$$U_{ra} = -N\frac{K_s}{K_f}\alpha p \tag{4.2}$$

Recall that K_s is the bulk modulus of the solid phase.

Now we can use the expression for the radial displacement in the plastic zone of the composite sphere, given by (3.12)

$$U_r = N\rho \left\{-p + \chi \frac{2}{\sqrt{3}} \left(\frac{1 + 4N}{4N} \frac{\gamma^3}{\rho^3} + 3 \ln \frac{\rho}{\alpha}\right)\right\}.$$

Obviously, at the inner radius $\rho = \alpha$, and thus we obtain

$$U_{ra} = N\alpha\left\{-p + \chi\frac{2}{\sqrt{3}}\frac{1+4N}{4N}\frac{\gamma^3}{\alpha^3}\right\}. \tag{4.3}$$

By equating (4.2) and (4.3), we can find the desired relationship, valid only for the undrained condition,

$$p\left(1 - \frac{K_s}{K_f}\right) = \chi\frac{2}{\sqrt{3}}\frac{1+4N}{4N}\frac{\gamma^3}{\alpha^3}. \tag{4.4}$$

In the incremental form,

$$dp\left(1 - \frac{K_s}{K_f}\right) = \chi\frac{2}{\sqrt{3}}\frac{1+4N}{4N}\frac{3\gamma^2}{\alpha^3}d\gamma. \tag{4.5}$$

The derived condition (4.4) can be treated as the condition of zero change of the fluid content in the composite sphere.

The final solution of the problem can be obtained by integrating equations written in incremental form. We assume that the increment of the applied pressure dq is given and our task is to find dp and dU_{rb}.

The yield condition (3.14) can be written in the incremental form as

$$dp = dq - \chi\frac{2}{\sqrt{3}}\left(1 - \frac{1}{\gamma^3}\right)3\gamma^2 d\gamma \tag{4.6}$$

By combining the Eqs. (4.5) and (4.6) and excluding dp from them, we can find the increment of the plastic parameter $d\gamma$ in terms of the given increment of the applied pressure dq:

$$d\gamma = \frac{dq\left(1 - \frac{K_s}{K_f}\right)}{\chi\frac{2}{\sqrt{3}}3\gamma^2\left[\frac{1+4N}{4N}\frac{1}{\alpha^3} - \left(\frac{1}{\gamma^3} - 1\right)\left(1 - \frac{K_s}{K_f}\right)\right]}. \tag{4.7}$$

Once we know $d\gamma$, using (4.5), we can find the increment of the fluid pressure

$$dp = \frac{dq\left(\frac{1+4N}{4N}\right)}{\frac{1+4N}{4N} - \alpha^3\left(\frac{1}{\gamma^3} - 1\right)\left(1 - \frac{K_s}{K_f}\right)}. \tag{4.8}$$

The increment of the radial displacement dU_{rb} can be computed, for example, using the stress-strain relationship (3.18)

$$\frac{1 + \frac{1}{4N}}{\frac{1}{\gamma^3} + \frac{1}{4N}}dp = dq + \frac{\frac{1}{\gamma^3} - 1}{\frac{1}{\gamma^3} + \frac{1}{4N}}\frac{1}{N}dU_{rb}$$

or (3.17_1)

$$dU_{rb} = -Ndq + N\chi\frac{2}{\sqrt{3}}\left(\frac{1}{4N} + 1\right)3\gamma^2 d\gamma.$$

We now have to find the applied external pressure q_0, fluid pressure p_0 and displacement U_{rb}^0 at the onset of yielding.

From the condition of zero change in the fluid content (4.4) and by setting $\gamma = \alpha$, the fluid pressure is

$$p_0 = \frac{\chi\frac{2}{\sqrt{3}}\frac{1+4N}{4N}}{1 - \frac{K_s}{K_f}}. \tag{4.9}$$

If $K_s/K_f > 1$, then $\chi = -1$ and the fluid pressure is positive.

The external pressure at the onset of yielding can be found from the yield condition (3.14),

$$q_0 = p_0 - \chi\frac{2}{\sqrt{3}}(1 - \alpha^3). \tag{4.10}$$

We note that here $\chi = -1$ and thus $q_0 > p_0$.

Consequently, the displacement at the onset of yielding can be computed using the elastic stress-strain relationship (3.29)

$$U_{rb}^0 = \frac{\frac{1}{\alpha^3} + \frac{1}{4N}}{1 - \frac{1}{\alpha^3}}Nq_0 - \frac{1 + \frac{1}{4N}}{1 - \frac{1}{\alpha^3}}Np_0 = N(q_0 - B_{el}p_0)\frac{\frac{1}{\alpha^3} + \frac{1}{4N}}{1 - \frac{1}{\alpha^3}}. \tag{4.11}$$

In Suvorov and Selvadurai (2010) we derived the void occupancy equation, which in the undrained state means that there is zero change in the fluid content. Using the normalizations and notation from this paper, this void occupancy equation for the undrained condition can be written (in incremental form) as

$$B_{el}dU_{rb} + N\left(B_{el} + \alpha^3\left(\frac{K_s}{K_f} - 1\right)\right)dp = (B_{el} - 1)dU_{rb}^{pl}. \tag{4.12}$$

This equation was derived in Suvorov and Selvadurai (2010) under the assumption that the porosity is constant. It can be shown that when there is zero change in the fluid content, (4.5)

$$dp\left(1-\frac{K_s}{K_f}\right)=\chi\frac{2}{\sqrt{3}}\frac{1+4N}{4N}\frac{3\gamma^2}{\alpha^3}d\gamma,$$

derived here for composite sphere assemblage, is a special case of a more general Eq. (4.12) provided that (1) the yield condition is given by (4.6)

$$dp=dq-\chi\frac{2}{\sqrt{3}}\left(1-\frac{1}{\gamma^3}\right)3\gamma^2 d\gamma,$$

(2) the increment of the plastic displacement is given by (3.31)

$$dU_{rb}^{pl}=N\chi\frac{2}{\sqrt{3}}\left(\frac{1}{4N}+1\right)\frac{1-\frac{\alpha^3}{\gamma^3}}{1-\alpha^3}3\gamma^2 d\gamma,$$

and (3) the total displacement increment is found from (3.17_1)

$$dU_{rb}+Ndq=N\chi\frac{2}{\sqrt{3}}\left(\frac{1}{4N}+1\right)3\gamma^2 d\gamma.$$

5. Numerical example

Consider the numerical example in which the composite sphere has a porosity $\varphi=0.1$. The composite sphere is loaded rapidly by the external pressure q (normalized) and thus the undrained condition prevails. The objective is to compute both the radial displacement U_{rb} (normalized) of the external surface and the fluid pressure p (normalized). We assume that Poisson's ratio of the solid phase is $\nu_s=0.48$, which gives $B_{el}=0.6842$, $N=0.0135$.

We compare the exact analytical solution for the composite sphere assemblage, presented previously, with the finite element solution, obtained using ABAQUS™ program. The ABAQUS™ solution will be obtained for an homogenized porous fluid-saturated material, for which the exact microgeometry of the composite sphere assemblage is not modeled. Only overall properties of the homogenized material, consistent with those for the composite sphere assemblage, will be used in the ABAQUS™ simulations. Since the stresses are uniform for the isotropic compression of the homogenized material, a mesh consisting of only one finite element is sufficient for solving this problem.

The following parameters are supplied to the ABAQUS™ finite element code. Suppose, for simplicity, that the initial yield stress in shear $k=1$. Also, the bulk modulus of the fluid phase is set to $K_f=20$. Specify the bulk modulus of the solid phase K_s, e.g., $K_s=600$. With the knowledge that the elastic Biot coefficient $B_{el}=0.6842$, which was chosen beforehand, we can find the elastic bulk modulus of the drained medium as $K_D=(1-B_{el})K_s=189.48$. By choosing Poisson's ratio of the drained medium ν_D more or less arbitrary but smaller than ν_s, e.g. $\nu_D=0.2$, we can find the elastic Young's modulus of the drained medium as $E_D=3K_D(1-2\nu_D)=341.1$. Parameters E_D, ν_D, K_s, K_f need to be supplied to the ABAQUS™ finite element code.

If the Cam-Clay plasticity model (Selvadurai & Suvorov, 2012, 2014) is used for the homogenized material, we must also input the evolution of the effective yield pressure (effective pressure at yield) versus the volumetric plastic strain. This data is taken from the solution of the composite sphere assemblage problem. The plastic part of the normalized displacement for the composite sphere assemblage is given by (3.30)

$$U_{rb}^{pl}=N\chi\frac{2}{\sqrt{3}}\left(\frac{1}{4N}+1\right)\left[\gamma^3-\left(1-\gamma^3+\ln\frac{\gamma^3}{\alpha^3}\right)\frac{\alpha^3}{1-\alpha^3}\right].$$

This expression can be converted to the volumetric plastic strain, using the normalization rule (3.9) and definition of the parameter N (3.6)

$$\varepsilon_V^{pl}=\frac{kU_{rb}^{pl}}{K_sN}=\frac{k}{K_s}\chi\frac{2}{\sqrt{3}}\left(\frac{1}{4N}+1\right)\left[\gamma^3-\left(1-\gamma^3+\ln\frac{\gamma^3}{\alpha^3}\right)\frac{\alpha^3}{1-\alpha^3}\right] \tag{5.1}$$

The effective yield pressure is obtained from the yield condition for the composite sphere assemblage (3.14)

$$q-p=-\chi\frac{2}{\sqrt{3}}\left(1-\gamma^3+\ln\frac{\gamma^3}{\alpha^3}\right)>0. \tag{5.2}$$

Both these quantities are functions of the parameter γ, which changes within the known limits, $\alpha\leq\gamma\leq 1$. Thus, by varying the parameter γ, we can evaluate both the yield pressure (5.2) and the corresponding plastic volumetric strain (5.1) and supply these quantities to ABAQUS™.

Fig. 3 shows the evolution of the plastic volumetric strain (in absolute value), $|\varepsilon_V^{pl}|$, versus the effective pressure at yield, $q-p$. Note that for larger values of the plastic strain, even a small change in the effective pressure results in a noticeable increment of the plastic strain.

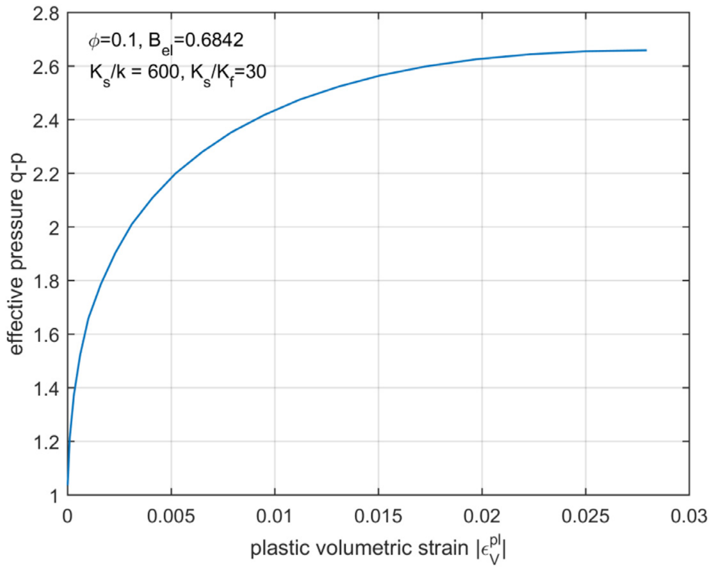

Fig. 3. Plastic volumetric strain (absolute value) versus effective pressure at yield: data supplied to ABAQUS^{TM} for the Cam-Clay plasticity model.

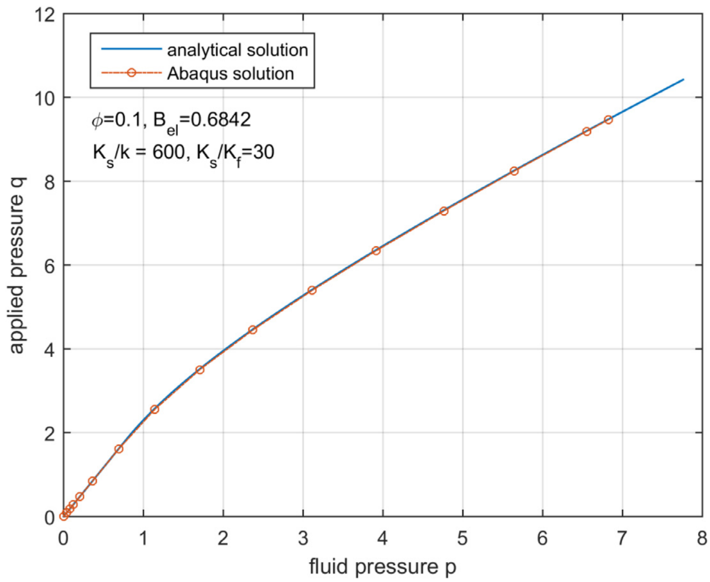

Fig. 4. Dependence of the fluid pressure on the applied pressure for the composite sphere assemblage.

Fig. 4 shows dependence of the fluid pressure p on the external pressure q, applied to the composite sphere assemblage. The exact analytical solution is shown with a solid line and the ABAQUS^{TM} solution is shown with small circles. The agreement between two solutions is very good. Note that the fluid pressure is always smaller than the applied pressure but as the plastic flow progresses the relative fraction of the total stress supported by the fluid gets larger. This is due to the fact that the fluid is capable of supporting larger stresses as the solid phase gets weaker in plastic flow.

Finally, Fig. 5 shows how the total normalized displacement of the external surface U_{rb} depends on the external pressure q. As we already noted in (5.1), due to our normalizations, the total normalized displacement can be related to the overall volumetric strain of the composite sphere assemblage as

$$\varepsilon_V = \frac{kU_{rb}}{K_s N}. \tag{5.3}$$

Since $NK_s = G_s/3$, we have from (5.3)

$$U_{rb}\frac{K_s}{G_s} = \varepsilon_V \frac{K_s}{3k}. \tag{5.4}$$

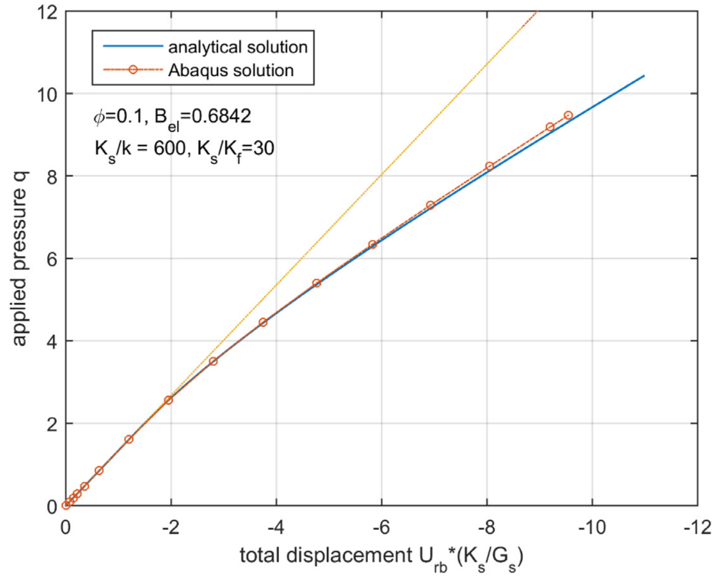

Fig. 5. Total displacement (normalized) versus the applied pressure for the composite sphere assemblage.

On the horizontal axis of Fig. 5 it is more convenient to use the quantity $U_{rb}(K_s/G_s)$ instead of U_{rb} because of its simpler relationship to the bulk modulus of the solid phase K_s. Again, the exact analytical solution is shown with a solid line and the solution obtained using ABAQUS^{TM} is shown with small circles. (Straight line corresponds to linear elasticity solution.) The agreement between the analytical and numerical solutions is acceptable although not perfect for larger values of the total displacement (larger values of plastic strain).

This small difference in displacements can be explained by the fact that the ABAQUS^{TM} finite element code takes into account the change in the void ratio (or porosity) and these changes can be significant in plastic flow. However, our exact analytical solution assumes that the porosity is constant. For this specific example, *the strains are small* and the change in the void ratio, computed by ABAQUS^{TM}, is relatively small. However, if we had taken smaller values of the bulk modulus K_s, e.g., $K_s/k = 60$ instead of $K_s/k = 600$, then we would have ended up with *larger strains* and consequently a larger difference between ABAQUS^{TM} and analytical solutions because of the substantial reduction of the void ratio during plastic flow. It is interesting to note that even in the case of a softer material ($K_s/k = 60$), the fluid pressure predictions will not be too different, which suggests that the fluid pressure magnitude is not significantly influenced by the porosity change. But the plastic strain, and thus total displacement, will be affected more substantially by the change in the void ratio (in fact, the total displacement will be smaller if porosity reduction is taken into account). This can be explained by referring to the results of Fig. 3 which show that even a small change in the fluid pressure (and thus, the effective pressure) results in a large change in the volumetric plastic strain during the later stages of plastic flow.

6. Concluding remarks

In this paper, we studied a composite sphere assemblage as a model for predicting the behavior of a porous fluid-saturated medium. The inner core (pore space) of the composite sphere is filled with fluid and the outer shell of the sphere represents the solid phase. Both fluid and solid phases are assumed compressible and, in addition, the material of the solid phase is assumed to be elasto-plastic of the von-Mises type. The elasto-plastic response of such a composite sphere assemblage in isotropic compression was the focus of this paper.

The analytical solution for this problem, derived by Nadai (1930), Sokolovsky (1946), and Chu and Hashin (1971), was adjusted here to obtain the constitutive equations for the composite sphere assemblage in a form that emphasizes the effect of the fluid pressure in the pore space. As is well known, the fluid pressure enters the stress-strain constitutive equation for the porous medium with a multiplier equal to the Biot coefficient. From the analytical solution for the composite sphere, we found that the tangent Biot coefficient changes its value during the plastic flow from an initial (elastic) value to a maximum value of 1. This result is at odds with another accepted form of the constitutive equation for the elasto-plastic porous medium in which the tangent Biot coefficient remains constant and equal to its initial (elastic) value (see Pariseau,1999; Khalili & Loret, 2001). In addition, we established that the tangent Biot coefficient is related to the tangent bulk modulus in the same way as the elastic Biot coefficient is related to the elastic bulk modulus of the drained medium.

We also showed that the yield condition for the composite sphere assemblage depends on Terzaghi's effective stress, in which the fluid pressure has a multiplier equal to 1. Thus, the plastic strains derived from the yield function also depend on

Terzaghi's effective stress (and not on Biot's effective stress) and this fact is consistent with Rice's observation (Rice, 1977) made for fissured rock masses.

Since in a porous fluid-saturated medium the effect of the fluid pressure is strongest during the undrained response, this response was studied in more detail. The equation that represents zero change of the fluid content was identified from the analytical solution and it was found that this equation is a special case of the more general void occupancy equation derived for inelastic systems by Suvorov and Selvadurai (2010).

The analytical solution for the elasto-plastic composite sphere assemblage in the undrained condition was then compared with the results of finite element simulations obtained using the ABAQUS™ code. At this point it is important to mention that the overall response of the elasto-plastic composite sphere assemblage will not be of von-Mises type since there will be a volumetric plastic strain caused by isotropic compression. Thus, for the finite element simulations we selected the Cam-Clay plasticity model available in ABAQUS™ since this model allows for volumetric plastic strain in isotropic compression. To simulate the *overall* behavior of the elasto-plastic porous medium in isotropic compression, it is sufficient to use only one finite element in the model since the stresses and strains are uniform.

The overall elastic bulk modulus and the bulk moduli of the solid and fluid phases were supplied to the ABAQUS™ program and were set equal to the properties of the composite sphere assemblage used to construct the analytical solution. In addition, the dependence of the volumetric plastic strain on the effective pressure, derived from the analytical solution, was inputted to the ABAQUS™ code.

The dependence of the fluid pressure and overall volumetric strain on the applied pressure was studied in detail for the undrained condition. Good agreement between the ABAQUS™ solution and the analytical solution was obtained if the strains were small enough (on the order of 0.01).

The magnitude of strains can be controlled by changing the ratio of the overall bulk modulus to the yield stress of the solid phase (the higher this ratio, the smaller the strains are). Once the magnitude of the overall strain increases, the agreement between the analytical solution and the ABAQUS™ results deteriorates (more so for the overall volumetric strain, and to a lesser extent for the fluid pressure). This can be explained by the fact that the ABAQUS™ code takes into account the changes in the void ratio or porosity that are typical for large deformations but the analytical solution used here does not account for these changes.

Since the agreement between the analytical and finite element solutions is good for the small-strain case (for which the changes in the void ratio are insignificant), we can conclude that the ABAQUS™ code uses the same yield function and the same tangent Biot coefficient as those in the analytical solution. Since the Terzaghi's effective stress is used in the yield function of the analytical solution, the same definition of the effective stress is exploited in the yield function by the ABAQUS™ code. It should also be noted that the results presented in this paper are relevant to situations involving monotonic loading of the fluid-saturated system up to and beyond the attainment of failure. If unloading is involved, an appropriate unloading analysis should be incorporated to examine the residual effects.

Finally, we note that (i) von-Mises plasticity theory for modeling solid phase of rocks and (ii) composite sphere assemblage model for representing microstructure of rocks may not be physically appropriate. However, these simplifications of the model did not prevent us from getting at least one consistent result analytically (for a realizable fluid-saturated structure) and making a conclusion on the appropriateness of using Terzaghi's definition of effective stress in the yield function and for derivation of plastic strains. It was also shown that Terzaghi's definition of effective stress leads naturally to evolution of tangent Biot coefficient during plastic flow.

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References

Biot, M. A. (1941). General theory of three-dimensional consolidation. *Journal of Applied Physics, 12*, 155–164.
Chadwick, P. (1959). The quasi-static expansion of a spherical cavity in metals and ideal soils. *Quarterly Journal of Mechanics and Applied Mathematics, 12*, 52–71.
Chakrabarty, J. (1998). *Theory of plasticity*. New York: McGraw-Hill.
Cheng, A. H.-D. (2015). *Poroelasticity*. Berlin: Springer-Verlag.
Christensen, R. M. (1979). *Mechanics of composite materials*. New York: John Wiley and Sons.
Chu, T. Y., & Hashin, Z. (1971). Plastic behavior of composites and porous media under isotropic stress. *International Journal of Engineering Science, 9*, 971–994.
Cowin, S. C. (2001). *Bone mechanics handbook*. New York: Taylor and Francis.
Davis, R. O., & Selvadurai, A. P. S. (2002). *Plasticity and geomechanics*. Cambridge: Cambridge University Press.
de Boer, R. (Ed.). (1999). *Porous media: Theory and experiments*. Dordrecht, the Netherlands: Kluwer Academic Publishers.
de Boer, R. (2000). *Theory of porous media. Highlights in historical development and current state*. Berlin, Germany: Springer-Verlag.
Drucker, D. C. (1967). *Introduction to mechanics of deformable solids*. New York: McGraw-Hil.
Fillunger, P. (1936). *Erdbaumechanik?*. Wien, Germany: Selbstverlag des Verfassers.
Hopkins, H. G. (1960). Dynamic expansion of spherical cavities in metals, Chapter 3. In I. N. Sneddon, & R. Hill (Eds.). In *Progress in solid mechanics: 1* (pp. 83–164). North-Holland Publ. Co., Amstredam.
Kachanov, L. M. (1974). *Fundamentals of the theory of plasticity*. Moscow: Mir Publishers (Translated by M. Konyaeva).

Khalili, N., & Loret, B. (2001). An elasto-plastic model for non-isothermal analysis of flow and deformation in unsaturated porous media: Formulation. *International Journal of Solids and Structures, 38*, 8305–8330.
Makhnenko, R. Y., & Labuz, J. F. (2016). Elastic and inelastic deformation of fluid-saturated rock. *Philosophical Transactions of Royal Society A, 374*, 20150422.
Nadai, A. (1930). *Plasticity*. McGraw-Hill, New York: Translation from German.
Narasimhan, R. (2004). Analysis of indentation of pressure sensitive plastic solids using the expanding cavity model. *Mechanics of Materials, 26*, 633–645.
Pariseau, W. G. (1999). Poroelastic-plastic consolidation – analytical solution. *International Journal for Numerical and Analytical Methods in Geomechanics, 23*, 577–594.
Quang, H. L., & He, Q.-C. (2008). Effective pressure-sensitive elastoplastic behavior of particle-reinforced composites and porous media under isotropic loading. *International Journal of Plasticity*, *24*, 343–370.
Rice, J. R. (1977). Pore pressure effects in inelastic constitutive formulations for fissured rock masses. In *Advances in Civil Engineering through Engineering Mechanics* (pp. 360–363). New York, NY: ASCE.
Rice, J. R., & Cleary, M. P. (1976). Some basic stress diffusion solutions for fluid-saturated elastic porous media with compressible constituents. *Reviews of Geophysics and Space Phys, 14*, 227–241.
Selvadurai, A. P. S. (Ed.). (1996). *Mechanics of poroelastic media*. The Netherlands: Kluwer Academic Publishers p. 302.
Selvadurai, A. P. S. (2004). Stationary damage modelling of poroelastic contact. *International Journal of Solids and Structures, 41*, 2043–2064.
Selvadurai, A. P. S. (2007). The analytical method in geomechanics. *Applied Mechanics Reviews, 60*, 87–106.
Selvadurai, A. P. S. (2018). The Biot coefficient for a low permeability heterogeneous limestone. *Continuum Mechanics and Thermodynamics*. doi:10.1007/s00161-018-0653-7.
Selvadurai, A. P. S., & Shirazi, A. (2004). Mandel-Cryer effects in fluid inclusions in damage susceptible poroelastic media. *Computers and Geotechnics, 37*, 285–300.
Selvadurai, A. P. S., & Shirazi, A. (2005). An elliptical disc anchor in a damage-susceptible poroelastic medium. *International Journal of Numerical Methods in Engineering, 16*, 2017–2039.
Selvadurai, A. P. S., & Suvorov, A. P. (2012). Boundary heating of poroelastic and poro-elastoplastic spheres. *Proceedings of the Royal Society, Mathematical and Physical Sciences Series A*. doi:10.1098/rspa.2012.0035.
Selvadurai, A. P. S., & Suvorov, A. P. (2014). Thermo-poromechanics of a fluid-filled cavity in a fluid-saturated porous geomaterial. *Proceedings of the Royal Society, Mathematical and Physical Sciences Series A, 470*, 20130634. http://dx.doi.org/10.1098/rspa.2013.0634.
Selvadurai, A. P. S., & Suvorov, A. P. (2016). Coupled hydro-mechanical effects in a poro-hyperelastic material. *Journal of the Mechanics and Physics of Solids, 91*, 311–333.
Selvadurai, A. P. S., & Suvorov, A. P. (2017). On the inflation of poro-hyperelastic annuli. *Journal of the Mechanics and Physics of Solids, 107*, 229–252.
Selvadurai, A. P. S., & Suvorov, A. P. (2018). On the development of instabilities in an annulus and a shell composed of a poro-hyperelastic material. *Proceedings of the Royal Society, Mathematical and Physical Sciences Series A*. https://dx.doi.org/10.6084/m9.figshare.c.4271114.
Sokolovsky, V. V. (1946). *Theory of plasticity* (p. 306). Publishing House “Science Academy of USSR”.
Suvorov, A. P., & Selvadurai, A. P. S. (2010). Macroscopic constitutive equations of thermo-poroviscoelasticity derived using eigenstrains. *Journal of the Mechanics and Physics of Solids, 58*, 1461–1473.
Suvorov, A. P., & Selvadurai, A. P. S. (2016). On poro-hyperelastic shear. *Journal of the Mechanics and Physics of Solids, 96*, 445–459.
Terzaghi, K. (1925). *Erdbaumechanik auf bodenphysikalischer grundlage*. Vienna, Austria: Franz Deuticke.
Verruijt, A. (2015). *Theory and problems of poroelasticity*. The Netherlands: Delft University of Technology.
Zimmerman, R. W. (2000). Micromechanics of poroelastic rocks. In K. Markov, & L. Preziosi (Eds.), *Heterogeneous media* (pp. 411–469). Boston: Birkhäuser. Chapter 5.