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ON THE DISPLACEMENT INDUCED IN A RIGID CIRCULAR PUNCH ON AN ELASTIC HALFSPACE DUE TO AN EXTERNAL FORCE

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Introduction

This paper examines the problem relating to the interaction between a rigid circular punch resting in smooth contact with an isotropic elastic halfspace and a concentrated force located exterior to the punch. The results for the displacement and rotation of the punch due to the external load are evaluated in exact closed form. Furthermore, it is shown that these results may also be recovered by appeal to Betti's reciprocal theorem.

Analysis

The class of problems which deal with the indentation of a halfspace region by a rigid punch has received considerable attention (see, e.g., Sneddon [1], Uflyand [2], Galin [3], de Pater and Kalker [4] and Gladwell [5]). These investigations concentrate on indentation problems which involve directly loaded indentors with arbitrary profiles, varying degrees of interface friction and variable contact regions. The category of problems wherein the interaction between the indenting punch and the elastic halfspace is perturbed by externally placed loads has received little attention (see, e.g., Galin [3]). In this note we examine the problem of the interaction between a rigid circular punch with a flat base resting in smooth contact with an elastic halfspace and an externally located normal concentrated force (Fig. 1). The analysis of the problem is facilitated by employing certain solutions developed for an associated external crack problem (see, e.g., Uflyand [2] and Sneddon and Lowengrub [6]). In addition, it is shown that the resultant displacements of the indenting punch due to the externally placed load can be obtained by appeal to Betti's [7] reciprocal theorem. The auxiliary solution required for the application of Betti's theorem is obtained from the analysis of the directly loaded punch.

Results for the directly loaded rigid punch

Before examining the title problem we shall record here some salient results related to the directly loaded rigid punch. We consider the problem of a rigid circular punch resting in smooth contact with an isotropic elastic halfspace. Furthermore we shall assume that the plane contact region $(r \le a,\ z = 0)$ is capable of sustaining tensile surface tractions. The rigid punch is subjected to an eccentric load Q at the location $r = \zeta a,\ \theta = 0$. The solution to this problem can be obtained by combining the separate solutions developed for the rigid circular punch subjected to a central force (Boussinesq [8], Sneddon [1]) and a moment about a horizontal axis (Bycroft [9], Florence [10]). For completeness we note that a Hankel transform development of the two separate problems yields two sets of dual integral equations of the type

$$\begin{aligned} H_0\{\xi^{-1}A_1(\xi)\ ;\ r\} &= w_0 \quad ; \quad 0 \le r \le a \\ H_0\{\ A_1(\xi)\ ;\ r\} &= 0 \quad ; \quad a < r < \infty \end{aligned} \tag{1}$$

and

$$\begin{aligned} H_1\{\xi^{-1}A_2(\xi)\ ;\ r\} &= \phi_0 r \quad ; \quad 0 \le r \le a \\ H_1\{\ A_2(\xi)\ ;\ r\} &= 0 \quad ; \quad 0 < r < \infty \end{aligned} \tag{2}$$

for the unknown functions $A_i(\xi)$ $(i = 1,2)$. In (1) and (2) w_0 and ϕ_0 are the rigid displacement and rotation of the circular punch and H_n is the Hankel operator given by

$$H_n\{g(\xi)\ ;\ r\} = \int_0^\infty \xi\ g(\xi) J_n(\xi r/a)\ d\xi \tag{3}$$

The results of particular interest to this paper are the axial surface displacements of the halfspace region due to the eccentrically loaded punch. We have

$$u_z(r,\theta,0) = \frac{Q(1-\nu^2)}{2Ea}\left\{1 + \frac{3\zeta\rho\cos\theta}{2}\right\} \quad ; \quad 0 \le \rho \le 1 \tag{4}$$

$$u_z(r,\theta,0) = \frac{Q(1-\nu^2)}{2Ea}\left(\frac{2}{\pi}\sin^{-1}\left(\frac{1}{\rho}\right) + \frac{3\zeta\rho\cos\theta}{2}\left\{1 - \frac{2}{\pi}\tan^{-1}\sqrt{\rho^2-1} - \frac{2}{\pi}\frac{\sqrt{\rho^2-1}}{\rho^2}\right\}\right) \quad ; \quad 1 \le \rho \le \infty \tag{5}$$

where $\rho = r/a$.

The external load - rigid punch interaction

We now consider the problem of the rigid circular punch resting in smooth contact with an isotropic elastic halfspace and subjected to an external concentrated force Q^* at the location $(\lambda a,0,0)$. It is further assumed that the punch is subjected to a sufficiently large axisymmetric force P to maintain the contact stress at the smooth interface compressive for all choices of Q^* and λ. In the ensuing we shall disregard, without comment, the effect of P and restrict our attention to the interaction between the rigid punch and the perturbing force Q^*. To examine this problem, we superpose on the rigid punch a central force $(\bar{Q})$ and a moment about the y-axis $(\bar{M})$ such that the rigid punch experiences zero displacement in the contact region (Fig. 2). The contact region is now subject to the boundary conditions

$$u_z(r,\theta,0) = 0 \quad ; \quad 0 \le \theta \le 2\pi \quad ; \quad 0 \le r \le a \tag{6}$$

$$\sigma_{zz}(r,\theta,0) = -p(r,\theta); \quad 0 \le \theta \le 2\pi \quad ; \quad a < r < \infty \tag{7}$$

$$\sigma_{rz}(r,\theta,0) = 0 \quad ; \quad 0 \le \theta \le 2\pi \quad ; \quad 0 \le r < \infty \tag{8}$$

where $p(r,\theta)$ is an even function of θ. Following the Hankel transform development of the equations of elastic equilibrium (see, e.g., Muki [11]) it can be shown that when the condition $\sigma_{rz}(r,\theta,0) = 0$ is satisfied, the appropriate expressions for u_z and σ_{zz} reduce to the following on the plane $z = 0$:

$$u_z(r,\theta,0) = 2(1-\nu)\sum_{m=0}^{\infty} H_m[\xi^{-2}\Psi_m(\xi) \; ; \; r] \cos m\theta \tag{9}$$

$$\sigma_{zz}(r,\theta,0) = -\frac{E}{(1+\nu)}\sum_{m=0}^{\infty} H_m[\xi^{-1}\Psi_m(\xi)\ ;\ r]\cos m\theta \tag{10}$$

where $\Psi_m(\xi)$ are unknown functions to be determined by satisfying the mixed boundary conditions (6) and (7). Assuming that $p(r,\theta)$ can be expressed in the form

$$p(r,\theta) = \frac{E}{(1+\nu)}\sum_{m=0}^{\infty} g_m(r)\cos m\theta \tag{11}$$

the boundary conditions (6) and (7) can be reduced to the set of dual integral equations

$$\begin{aligned} H_m[\xi^{-2}\Psi(\xi)\ ;\ r] &= 0 \quad ; \quad 0 \le r \le a \\ H_m[\xi^{-1}\Psi(\xi)\ ;\ r] &= g_m(r) \quad ; \quad a < r < \infty \end{aligned} \tag{12}$$

The solution of the dual system (12) is given by several authors including Noble [12] and Sneddon and Lowengrub [6] and the details will not be pursued here. It is sufficient to note that for the concentrated external loading of the halfspace

$$[g_0(r)\ ;\ g_m(r)] = \frac{Q^*(1+\nu)a\delta(r-\ell)[1:2]}{4\pi^2 Er} \tag{13}$$

where $\delta(r-\ell)$ is the Dirac delta function, and the contact stress distribution at the interface region $r \le a$ is given by

$$\sigma_{zz}(r,\theta,0) = \frac{2Q^*}{\pi^2}\sum_{m=0}^{\infty}\left(\frac{r}{\ell}\right)^m \cos m\theta \int_a^{\infty} \frac{t\,H(\ell-t)\,dt}{\sqrt{\ell^2-t^2}\,(t^2-r^2)^{3/2}} \tag{14}$$

and $H(\ell-t)$ is the Heaviside step function.
This expression can be reduced to the explicit form

$$\sigma_{zz}(r,\theta,0) = \frac{Q^*\sqrt{\lambda^2-1}}{\pi^2\sqrt{\rho^2-1}\,(\lambda^2+\rho^2-2\rho\lambda\cos\theta)} \quad ; \quad 0 \le \rho < 1 \tag{15}$$

where $\rho = r/a$ and $\lambda = \ell/a$. Furthermore, the displacement and stress fields corresponding to the problem posed above tend to zero as $r,z \to \infty$. It is of interest to note that the distribution of normal stress beneath the punch due to the external force (as defined by (15)) has a form identical (except for a multiplicative constant) to that of the distribution of charge density on a thin conducting disc of radius a due to a point charge located at a distance ℓ (see,e.g., Jeans [13]). This is similar to the long established analogy between the free distribution of charge on a thin conducting disc and the distribution of contact stresses beneath a centrally loaded rigid circular punch indenting an isotropic homogeneous elastic halfspace (see,e.g., Boussinesq [8], Copson [14], Sneddon [1]).

The force resultant $\bar{P}$ required to maintain zero axial displacement (u_z) in the contact region $(r \leq a)$ is given by

$$\bar{P} = 2\int_0^a \int_0^\pi \sigma_{zz}(r,\theta,0)\ r dr d\theta = \frac{2Q^*}{\pi} \sin^{-1}\left(\frac{a}{\ell}\right) \tag{16}$$

Similarly, the moment $\bar{M}$ required to maintain zero displacement in the contact region is given by

$$\bar{M} = 2\int_0 \int_0^\pi \sigma_{zz}(r,\theta,0) r^2 \cos\theta dr d\theta = Q^*\ell \left\{1-\frac{2}{\pi}\tan^{-1}\sqrt{\lambda^2-1} - \frac{2}{\pi}\frac{\sqrt{\lambda^2-1}}{\lambda^2}\right\} \tag{17}$$

By subjecting the rigid punch to force and moment resultants in the opposite sense (to $\bar{P}$ and $\bar{M}$) we can render the rigid punch free from external force (except ofcourse for the central force P ; Fig.2). The displacement of the rigid punch at an arbitrary location $(\rho a,\theta,0)$ within the contact region due to the external force Q^* can be obtained by making use of (4), (5), (16) and (17). We have for $0\leq\rho\leq 1$

$$u_z(r,\theta,0) = \frac{Q^*(1-\nu^2)}{2aE}\left\{\frac{2}{\pi}\sin^{-1}\left(\frac{1}{\lambda}\right) + \frac{3}{2}\lambda\rho\cos\theta \times \left\{1 - \frac{2}{\pi}\tan^{-1}\sqrt{\lambda^2-1} - \frac{2}{\pi}\frac{\sqrt{\lambda^2-1}}{\lambda^2}\right\}\right\} \tag{18}$$

Consider the displacement w^* of the rigid punch at the location $(\zeta a, 0, 0)$ due to the externally placed load Q^* applied at $(\lambda a, 0, 0)$ (Fig. 3a); we have

$$w^* = \frac{Q^*(1-\nu^2)}{2aE} \left[\frac{2}{\pi} \sin^{-1}\left(\frac{1}{\lambda}\right) + \frac{3\lambda\zeta}{2} \left\{ 1 - \frac{2}{\pi} \tan^{-1}\sqrt{\lambda^2-1} - \frac{2}{\pi} \frac{\sqrt{\lambda^2-1}}{\lambda^2} \right\} \right] \tag{19}$$

Similarly, the surface displacement of the elastic halfspace at the exterior point $(\lambda a, 0, 0)$ due to the loading of the rigid punch by a concentrated force Q applied at $(\zeta a, 0, 0)$ is denoted by w (Fig. 3b). The value of w can be obtained from result (5). By comparing this expression with (19) it is evident that the two states satisfy Betti's reciprocal relationship

$$Q^* w = Q w^* \tag{20}$$

Conclusions

In this paper we have examined the interaction problem for a rigid circular foundation resting in smooth contact with an isotropic elastic halfspace and subjected to an external concentrated force. The results for the displacement and rotation of the rigid punch induced by the external concentrated force can be evaluated in exact closed form. Furthermore, it is shown that the solution to the problem of an eccentrically loaded rigid circular punch serves as an auxiliary solution which enables the application of Betti's reciprocal theorem.

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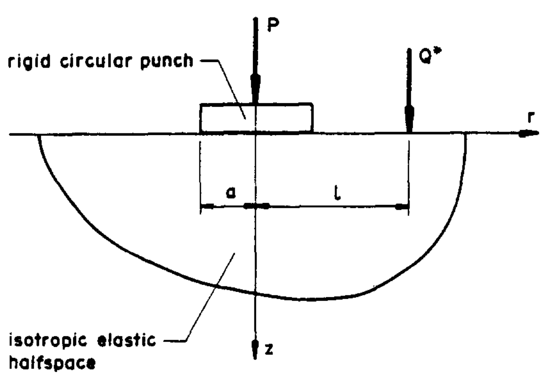

Fig. 1. Geometry of the rigid circular punch on an elastic halfspace and the external loading.

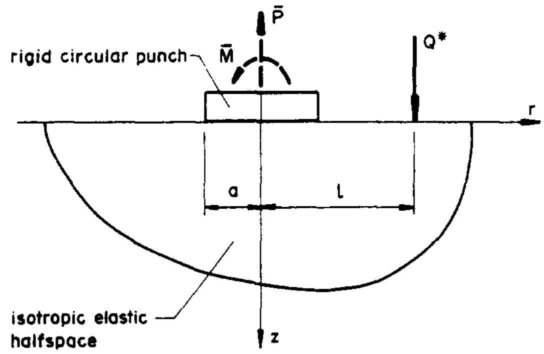

Fig. 2. Corrective force resultants

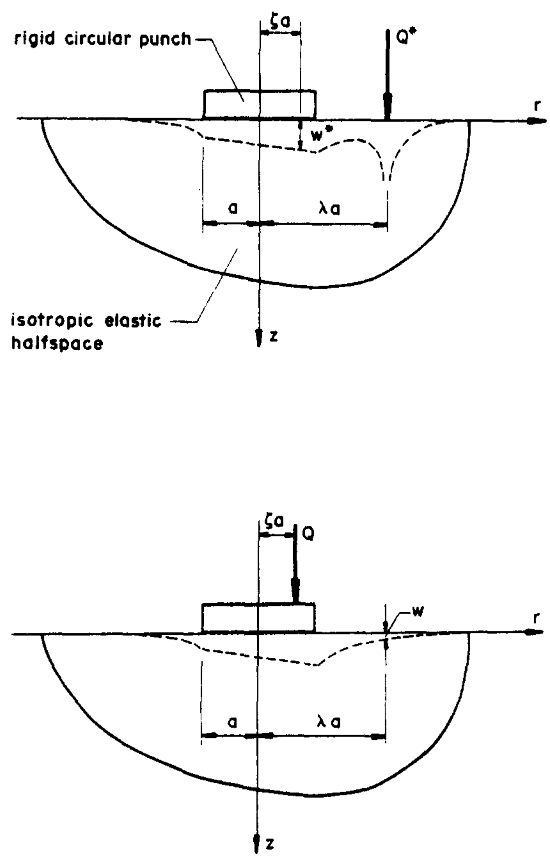

Fig. 3. Reciprocal states.