SELVADURAI, A. P. S. (1973). *Géotechnique* 23, No. 3, 407–421.

Bending of an infinite beam resting on a porous elastic medium

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The theories of classical continuum mechanics ignore the microstructure a material may possess in its elemental volume. Results obtained from such a theory for the forces, displacements, stresses, etc., are representative of average values over regions, the dimensions of which are large on the scale of the internal structure. It is then foreseeable that for materials with a dominant internal structure such as porous, granular, fibrous or laminated media under certain loading conditions, the classical theories will fail to give meaningful results.

In recent years, numerous concerted efforts have been directed towards the development of continuum theories which describe the role of microstructure in material behaviour. In the simplest form of a micromorphic elastic material a consideration of its microstructure requires the introduction of two new material constants, in addition to the classical Lamé constants, one of which has dimensions of length. The presence of this new material constant makes it possible to introduce the effects of the internal structure or size effects which are unrecognized in the classical theory. It is shown that the response of the micromorphic elastic continuum closely resembles that of an idealized porous elastic medium. A solution is presented to the problem of an infinite beam resting on such a medium.

Les théories classiques de la mécanique du milieu continu ne tiennent pas compte de la microstructure des matériaux à l'échelle de l'élément de volume. Les résultats que l'on obtient par l'application de ces théories au calcul des forces, déplacements, contraintes etc., représentent des valeurs moyennes sur des régions de grande dimension par rapport à l'échelle de la structure interne. On peut donc prévoir que ces théories classiques ne pourront donner de résultats valables sous certaines conditions de charge dans le cas des matériaux ayant une structure interne préponderante, tels que les milieux poreux, granulaires, fibreux ou à lamelles.

Au cours des dernières années, de nombreuses recherches ont été consacrées au développement de théories du milieu continu décrivant le rôle de la microstructure dans le comportement des matériaux. Sous la forme la plus simple d'un matériau élastique à structure fine, pour tenir compte de sa microstructure, il faut introduire en plus des constantes classiques de Lamé deux nouvelles constantes dont l'une a la dimension d'une longueur. La présence de cette nouvelle caractéristique de matériau rend possible l'introduction des effets de structure interne ou d'effets de dimension qui sont ignorés dans la théorie classique. On montre que la réponse du milieu continu élastique à structure fine est similaire à celle d'un milieu élastique poreux idéal. On présente une solution du problème d'une poutre infinie reposante sur un tel milieu.

In classical continuum mechanics the fundamental assumption is made that the transmission of loads on both sides of a surface element, in a material volume, can be described completely in terms of the field of stress vectors defined on this surface. These stress vectors represent the force per unit area transmitted across the surface. The concept of including 'couple stress vectors' in addition to these conventional stress vectors, representing the couple per unit area transmitted across the surface, was originally introduced by Voigt (1887) and later expanded by the Cosserats (1909). Such an assumption seems appropriate for materials possessing a fibrous, granular, porous or layered internal structure, where interaction between adjacent material elements may introduce internal moments. The development of theories of microcontinua capable of describing materials which possess an internal structure has recently become a very active field of research. Comprehensive expositions and surveys of the subject are given by Truesdell and Toupin (1960), Grioli (1960), Aero and Kuvshinski (1960), Toupin (1962), Mindlin and Tiersten (1962), Mindlin (1962), Koiter (1964), Mindlin (1964), Green and Rivlin (1964), Eringen and Suhubi (1964), Eringen (1969), Kröner (1969), Hermann (1972) and others. In the present Paper, however, attention will be limited to the theories of elastic microcontinua proposed by Mindlin and Tiersten (1962) and Eringen and

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Suhubi (1964). These theories are generally referred to as the 'indeterminate' couple stress theory (Cosserat continua) and the micromorphic theory (micromorphic continua) respectively. In particular, the linear theory of couple stress elasticity for a centrosymmetric homogeneous isotropic elastic material was formulated by Mindlin and Tiersten (1962). The couple stress theory derived by Mindlin and Tiersten may be regarded as a generalization of the classical theory of elasticity although it differs from the classical theory in several significant aspects. First, the modified strain energy density function from which the constitutive equations are derived contains not only terms involving the usual infinitesimal strains but also the gradients of the local rigid rotation (macrorotation) or curvature. Second, and the most significant feature of the theory, is that the generalized constitutive equations for an isotropic elastic material contain, in addition to the classical Lamé constants, two new elastic constants, one of which can be expressed in terms of a material parameter l which has dimensions of length. The presence of a characteristic length in the couple stress theory makes it possible to introduce the effects of internal structure or size effects, which are unrecognized in the classical theory. As this characteristic length tends to zero, so both the new elastic constants approach zero and the couple stress theory converges to the classical theory.

The influence of the characteristic length parameter l on stress concentration phenomena in elastic solids has been investigated by Mindlin and Tiersten (1962), Mindlin (1962), Muki and Sternberg (1965) and others. The plane strain problem of a cylindrical cavity in an infinite body subjected to a homogeneous field of stress at infinity has been studied by Mindlin (1962). The solutions obtained by Mindlin indicate that the stress concentration at the cavity surface is no longer independent of the radius a of the cavity, as it is in the case of the classical elasticity theory, but depends upon the ratio l/a. As l/a tends to zero the stress concentration factors for the couple stress solution approach their classical values. As the factor l/a becomes large the stress concentration predicted by the couple stress theory becomes lower than those encountered in the classical theory.

It may be further noted that in the presence of couple stresses, the conventional stress tensor is no longer symmetric and that generally the antisymmetric part of this stress tensor and the isotropic part of the couple stress tensor are indeterminate. However, imposing the condition that the conventional stress tensor of the couple stress theory converges to its equivalent in the classical theory, as $l \to 0$, for every choice of the isotropic couple stress component, it can be shown that in the absence of body couples the solution of a boundary value problem in the couple stress theory of elasticity is unique to within an arbitrary isotropic couple stress field (Muki and Sternberg, 1965).

The couple stress theory proposed by Mindlin and Tiersten (1962) may offer only a partial representation of all possible microstructural effects. However, it provides a simple model for the investigation of certain kinds of phenomena which may be attributed to the microstructure of a material. For example it offers a simple approach for the treatment of polycrystalline plastic materials (Lippman, 1968), internal buckling of laminated media (Biot, 1965) and several other problems of engineering interest.

A general non-linear theory for simple microelastic solids was developed by Eringen and Suhubi (1964). A special case of the theory proposed by Eringen and Suhubi is the theory of micropolar elasticity. A full account of the linear theory of micropolar elasticity is given by Eringen (1966, 1969). The deformation of a micropolar elastic medium is fully described by a displacement vector and an independent rotation vector (microrotation). This independent rotation vector contributes to the basic difference between micropolar elasticity and couple stress elasticity (and of course the classical theory of elasticity). From a physical point of view micropolar elasticity is concerned with materials whose constituents are dumb-bell shaped molecules which are allowed to rotate independently without any stretch. In the special case where the microrotation becomes equal to the macrorotation the micropolar theory of elasticity converges to the indeterminate couple stress theory. Therefore for future reference the infinite beam problem will be formulated in relation to the linear theory of micropolar elasticity. Despite the many significant advances in the theoretical aspects of the mechanics of materials with internal structure, the experimental verification of the applicability of these theories to real materials has received only limited attention. Koiter (1964) has suggested a test procedure for the determination of the characteristic length parameter l. Tests include simple torsion of cylindrical specimens and bending of plate elements. Tests carried out by Schijve (1966) and Ellis and Smith (1967) indicate that for certain metallic materials, within the range of testing considered, couple stresses have no significant effect. Brown and Evans (1972) have analysed the propagation of twist in a granular medium consisting of spherical elastic particles and conjecture that couple stresses are less significant in a granular body than in metallic materials. On the other hand experiments conducted by Hoppmann and Shahwan (1965) on a 'grid model' of a continuum indicate the possibility of couple stress effects being present in materials with a fully continuous internal structure. In this article it will be shown that the mathematical model of the micropolar elastic continuum bears a close formal resemblance to an elastic medium which exhibits an idealized porous microstructure and that the characteristic length parameters (b, c) of the mathematical model (and hence the characteristic length parameter l of the indeterminate couple stress theory) bear a direct relation to the pore size of such a medium. However, it must be emphasized that the micropolar theory of elasticity is a rational continuum theory based on proper postulates, thermodynamics and uniqueness theorems. Its interpretation as an idealized porous elastic medium is only motivational to exhibit its physical basis.

LINEAR MICROPOLAR ELASTICITY

The theory of linear micropolar elasticity is given by Eringen (1966, 1969) and only the relevant results will be briefly reviewed. Attention will be confined to a body in a state of plane micropolar strain, the deformation of which is completely determined by the displacement components $u(x, y)$ and $v(x, y)$ in the rectangular Cartesian co-ordinate directions x, y respectively and the microrotation $\Phi(x, y)$.

The components of the linearized micropolar strain tensor can be expressed in terms of u, v and Φ in the form

$$\left.\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} & \epsilon_{yy} &= \frac{\partial v}{\partial y} \\ \epsilon_{xy} &= \frac{\partial v}{\partial x} - \Phi & \epsilon_{yx} &= \frac{\partial u}{\partial y} + \Phi \end{aligned}\right\} \quad \ldots \ldots \quad (1)$$

In plane strain the constitutive equations for the stress tensor **t** and the couple stress tensor **m** for a linear micropolar elastic material are

$$\left.\begin{aligned} t_{xx} &= \lambda\epsilon + (2\mu+\kappa)\epsilon_{xx} & t_{xy} &= (\mu+\kappa)\epsilon_{xy} + \mu\epsilon_{yx} \\ t_{yy} &= \lambda\epsilon + (2\mu+\kappa)\epsilon_{yy} & t_{yx} &= (\mu+\kappa)\epsilon_{yx} + \mu\epsilon_{xy} \\ t_{zz} &= \lambda\epsilon \end{aligned}\right\} \quad \ldots \quad (2)$$

and

$$m_{xz} = \gamma\frac{\partial\Phi}{\partial x} \qquad m_{yz} = \gamma\frac{\partial\Phi}{\partial y} \qquad (3)$$

respectively, where $\epsilon = \epsilon_{xx} + \epsilon_{yy}$.

Note that λ and μ are the classical Lamé constants and γ and κ are the two additional constants which arise due to the consideration of the microstructure of the medium. It can be shown that the following thermodynamical restrictions on λ, μ, κ and γ are necessary and sufficient to ensure the internal energy to be non-negative (i.e. if the internal energy is non-negative, energy is only stored during a deformation and no energy is produced).

$$3\lambda + 2\mu + \kappa \geqslant 0 \qquad \mu \geqslant 0 \qquad \kappa \geqslant 0 \qquad \gamma \geqslant 0 \qquad (4)$$

In the absence of body forces and body couples the static equations of force equilibrium and moment equilibrium are

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{yx}}{\partial y} = 0 \qquad \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} = 0 \qquad (5)$$

and

$$\frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + t_{xy} - t_{yx} = 0 \qquad (6)$$

respectively. The components of surface traction $\mathbf{T}$, in the x and y directions on a surface $F(x, y) = 0$ are

$$T_x = n_x t_{xx} + n_y t_{yx} \qquad T_y = n_x t_{xy} + n_y t_{yy} \qquad (7)$$

where n_x and n_y are components of the outward unit normal to the surface and

$$n_x^2 + n_y^2 = 1 \qquad \frac{n_x}{n_y} = \frac{\partial F/\partial x}{\partial F/\partial y} \qquad (8)$$

Similarly the boundary condition for the surface couple is

$$M_z = m_{xz} n_x + m_{yz} n_y \qquad (9)$$

Stress functions

Stress functions $\phi(x, y)$ and $\Psi(x, y)$ are introduced, consistent with equations (5) and (6), such that

$$\left.\begin{aligned} t_{xx} &= \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\Psi}{\partial x\partial y} & t_{yy} &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\Psi}{\partial x\partial y} \\ t_{xy} &= -\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\Psi}{\partial y^2} & t_{yx} &= -\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial^2\Psi}{\partial x^2} \\ m_{xz} &= \frac{\partial\Psi}{\partial x} & m_{yz} &= \frac{\partial\Psi}{\partial y} \end{aligned}\right\} \qquad (10)$$

The compatibility conditions necessary and sufficient for the integrability of the kinematic system in a simply connected domain are

$$\frac{\partial\epsilon_{xx}}{\partial y} - \frac{\partial\epsilon_{yx}}{\partial x} + \frac{\partial\Phi}{\partial x} = 0 \qquad \frac{\partial\epsilon_{xy}}{\partial y} - \frac{\partial\epsilon_{yy}}{\partial x} + \frac{\partial\Phi}{\partial y} = 0 \qquad \frac{\partial m_{xz}}{\partial y} = \frac{\partial m_{yz}}{\partial x} \qquad (11a)$$

By making use of equations (2), (3) and (10) the compatibility conditions can be reduced to the form

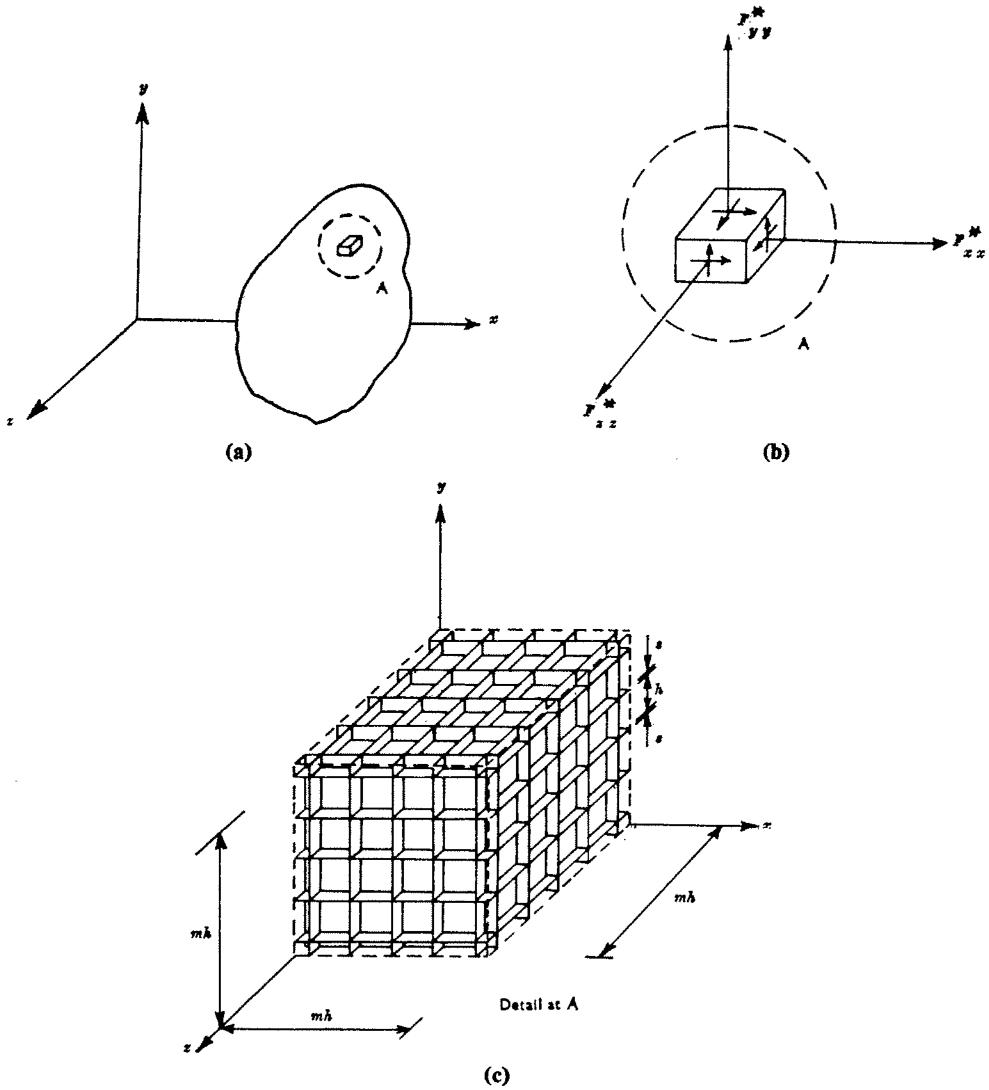

Fig. 1. The porous medium

$$\left.\begin{aligned}\frac{\partial}{\partial x}(\Psi - c^2\nabla^2\Psi) &= -2(1-\nu)b^2\frac{\partial}{\partial y}\nabla^2\phi \\ \frac{\partial}{\partial y}(\Psi - c^2\nabla^2\Psi) &= 2(1-\nu)b^2\frac{\partial}{\partial x}\nabla^2\phi\end{aligned}\right\} \quad \ldots\ldots \quad (11b)$$

where

$$c^2 = \frac{\lambda(\mu+\kappa)}{\kappa(2\mu+\kappa)} \qquad b^2 = \frac{\gamma}{2(2\mu+\kappa)} \qquad \nu = \frac{\lambda}{(2\lambda+2\mu+\kappa)} \quad \ldots \quad (12)$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad \ldots\ldots\ldots \quad (13)$$

Eliminating Ψ and ϕ in turn from equation (11b) gives

$$\left.\begin{aligned}\nabla^4\phi &= 0\\ \nabla^2(\Psi - c^2\nabla^2\Psi) &= 0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (14)$$

The solution of the plane strain problem in linear micropolar elasticity is reduced to the solution of the linear partial differential equations (14) subject to boundary conditions (equations (7) and (9)).

AN IDEALIZED POROUS ELASTIC MEDIUM

The mathematical theory of the micropolar elastic medium as presented in the previous section embodies purely abstract concepts of microstructure. In deriving the constitutive equations (2) for such a medium we have not taken into account any specific form of an internal structure. It is therefore pertinent to inquire whether in fact such a theory could in any way model the behaviour of a material with a specific internal structure.

Several researchers have attempted to correlate microcontinua to specific structural models of continua where beam-like elements composing the internal structure of the material are not only capable of sustaining traction forces but also traction couples. Details of such treatments are given by Adomeit (1967), Askar and Cakmak (1968), Banks and Sokolowski (1968), Bažant and Christensen (1972) and Flügge (1972).

The simplest model of an idealized porous elastic material consisting of plate-like elements arranged as shown in Fig. 1(c) is considered. These elements possess perfect elastic properties (no couple stress effects) which are characterized by elastic constants E_0 and ν_0, the Young's modulus and Poisson's ratio respectively. The spacings of the plate elements h are assumed to be large compared to their thicknesses $s(h \gg s)$. These plate elements are rigidly connected to each other such that the relative rotations at the joints are zero. Other elaborate forms of microstructure (including models containing joints capable of restrained rotations) can also be considered (Fig. 2) but this particular model presents a case which is simple enough to permit analysis of the overall response of the model by conventional elasticity (Selvadurai, 1973). By subjecting the model of the porous elastic medium to simple states of stress such as uniform compression, tension with restricted lateral contractions, shear, bending and torsion, expressions can be obtained for the micropolar elastic constants λ, μ, κ, γ in terms of E_0, ν_0, s and h.

As an example of such a calculation consider the case of the cubical element (Fig. 1(b)) subjected to a strain $\epsilon^*_{xx} = \epsilon_{xx}$ (the terms indicated with an asterisk refer to the components of the stress and strain in the plate-like elemetns) in the x-direction. We note that each of these plate elements is subjected to a state of plane stress such that the stresses t^*_{xx}, t^*_{yy}, t^*_{xy} (where $t^*_{xy} = t^*_{yx}$) in a plate element parallel to the xy plane are given by

$$\left.\begin{aligned}t^*_{xx} &= \frac{E_0}{(1-\nu_0{}^2)}(\epsilon^*_{xx} + \nu_0\epsilon^*_{yy}) \qquad t^*_{yy} = \frac{E_0}{(1-\nu_0{}^2)}(\epsilon^*_{yy} + \nu_0\epsilon^*_{xx})\\ t^*_{xy} &= \frac{E_0}{(1+\nu_0)}\epsilon^*_{xy}\end{aligned}\right\} \quad . \quad (15)$$

where ϵ^*_{xx}, ϵ^*_{yy}, ϵ^*_{xy} are the components of the classical strain tensor (obtained by substituting $\Phi = 0$ in equation (1)) in the plate elements. The condition of restricted lateral contraction implies that $\epsilon^*_{zz} = \epsilon^*_{yy} = 0$. The shear strains are also equal to zero.

On any plane $x =$ constant the applied force is distributed over both horizontal and vertical plate elements, the material area of which is $2m^2sh$. The total force in the x-direction,

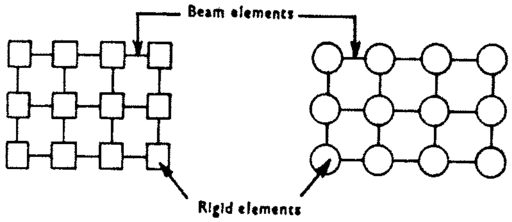

Fig. 2. Idealized models of porous media

F^*_{xx} (Fig. 1(b)) is

$$F^*_{xx} = 2m^2sht^*_{xx} = 2m^2sh\frac{E_0}{(1-\nu_0^2)}\epsilon_{xx} \quad . \quad . \quad . \quad . \quad (16)$$

Due to the applied state of strain and by definition, the vertical plate elements which lie normal to the x-axis are subjected to zero stress and zero strain. On the planes y=constant, the force necessary to maintain such a state of strain is

$$F^*_{yy} = m^2sht^*_{yy} = m^2sh\frac{E_0\nu_0}{(1-\nu_0^2)}\epsilon_{xx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

A similar force will be necessary on planes z=constant, or

$$F^*_{zz} = m^2sh\frac{E_0\nu_0}{(1-\nu_0^2)}\epsilon_{xx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When these forces are divided by the gross area over which they act the gross stress for the porous medium is obtained. Hence

$$\left.\begin{aligned} t_{xx} &= \frac{2E_0}{(1-\nu_0^2)}\frac{s}{h}\epsilon_{xx} \\ t_{yy} = t_{zz} &= \frac{E_0\nu_0}{(1-\nu_0^2)}\frac{s}{h}\epsilon_{xx} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

When the porous medium is subjected to the three normal strain components ϵ_{xx}, ϵ_{yy} and ϵ_{zz}, by superposition

$$t_{xx} = \frac{s}{h}\frac{E_0}{(1-\nu_0^2)}[2\epsilon_{xx}+\nu_0(\epsilon_{yy}+\epsilon_{zz})] \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

In general, for a linear micropolar elastic medium

$$t_{xx} = (\lambda+2\mu+\kappa)\epsilon_{xx}+\lambda(\epsilon_{yy}+\epsilon_{zz}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

By comparing equations (20) and (21) it is noted that

$$\left.\begin{aligned} (\lambda+2\mu+\kappa) &= \frac{s}{h}\frac{2E_0}{(1-\nu_0^2)} \\ \lambda &= \frac{s}{h}\frac{E_0\nu_0}{(1-\nu_0^2)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Similarly by subjecting the model of the porous medium to other simple states of stress as described earlier it can be shown that the elastic constants λ, μ, κ, γ of the micropolar elastic medium are related to E_0, ν_0, s and h by the expressions

$$\left.\begin{aligned} \lambda &= \frac{E_0\nu_0}{(1-\nu_0^2)}\frac{s}{h} & \gamma &= \frac{E_0}{6(1-\nu_0^2)}sh \\ 2\mu &= \frac{E_0}{(1+\nu_0)}\frac{s}{h} & \kappa &= \frac{E_0}{(1-\nu_0^2)}\frac{s}{h} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

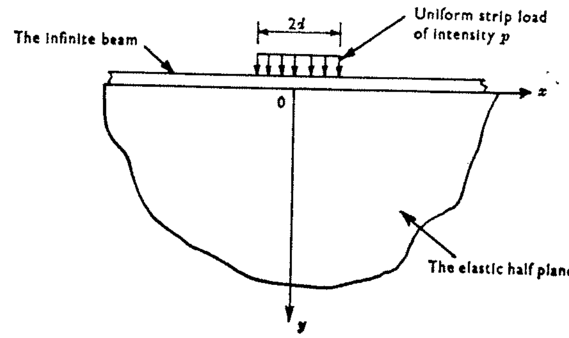

Fig. 3. **The infinite beam subjected to a uniform strip load**

It should be noted that although the thicknesses s and the spacings h of the plate elements are the same in all three co-ordinate directions the model itself is not isotropic. This may be easily verified by rotating the reference co-ordinate system x, y, z. The co-ordinate system as chosen coincides with the principal directions of the model.

THE INFINITE BEAM PROBLEM

Consider the plane strain problem of an infinite beam resting on a micropolar half plane. We assume smooth contact at the beam–elastic half plane interface. The beam is subjected to a uniform strip load of intensity p and width $2d$ (Fig. 3). First consider the problem in which the surface of the micropolar elastic half plane, $y=0$, is subjected to a sinusoidal normal stress.

The boundary conditions (7) and (9) of the micropolar elastic problem reduce to

$$\left.\begin{aligned} t_{yy}(x,0) &= -Q_0\cos\alpha x \\ t_{yx}(x,0) &= 0 \\ m_{yz}(x,0) &= 0 \end{aligned}\right\} \qquad (24)$$

and in addition the stress components $\mathbf{t}$ and the couple stress components $\mathbf{m}$ should tend to zero as $y \to \infty$.

It can be verified that the complete solution of this problem is given by the two stress functions

$$\phi(x,y) = \frac{1}{\alpha^2}(c_1 e^{-\alpha y} + c_2\alpha y e^{-\alpha y})\cos\alpha x \qquad (25)$$

and

$$\Psi(x,y) = (c_3 e^{-\delta y} + c_4 e^{-\alpha y})\sin\alpha x \qquad (26)$$

where

$$\left.\begin{aligned} c_1 = Q_0 \qquad c_2 = \frac{Q_0}{\eta} \qquad \delta^2 = \alpha^2 + \frac{1}{c^2} \\ c_3 = 4Q_0(1-\nu)\frac{b^2\alpha}{\delta\eta} \qquad c_4 = -4Q_0(1-\nu)\frac{b^2}{\eta} \end{aligned}\right\} \qquad (27)$$

The deflexion $v(x, 0)$ of the surface of the half plane produced by the sinusoidal load $Q_0\cos\alpha x$ can be shown to be equal to (within a rigid body displacement)

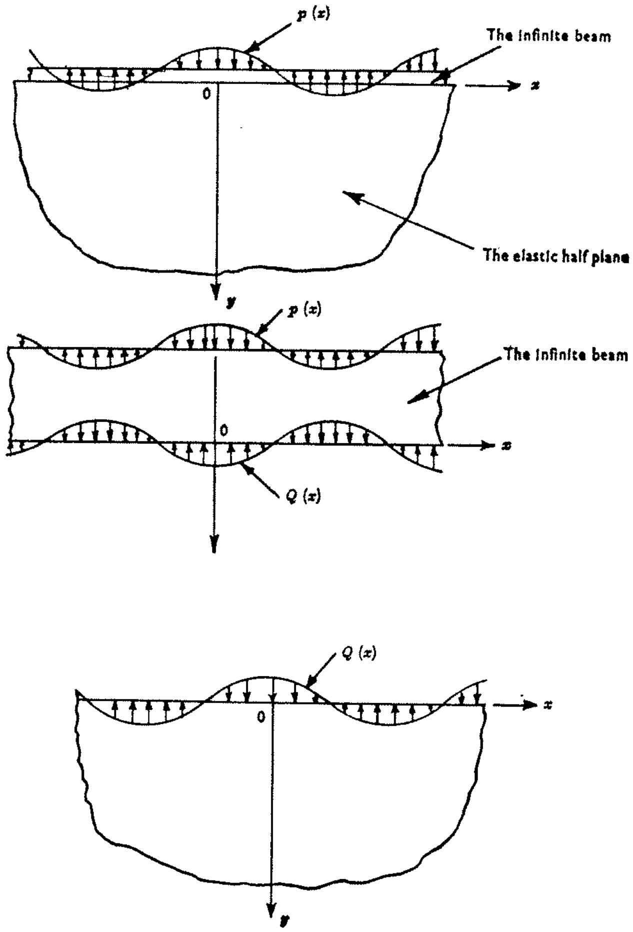

Fig. 4. The infinite beam subjected to a sinusoidal load

$$v(x, 0) = \frac{2Q_0\chi}{(2\mu+\kappa)(\chi+\lambda)}\left[\frac{1}{\alpha}+\frac{\beta}{\eta}(\alpha-\delta)\right]\cos\alpha x \quad . \quad . \quad . \quad . \quad (28)$$

where

$$\chi = (\lambda+2\mu+\kappa) \qquad \beta = 4(1-\nu)\frac{b^2\alpha}{\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Consider the bending of the infinite elastic beam. It is assumed that the beam is subjected to an external sinusoidal load distribution

$$p(x) = p_0 \cos\alpha x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

and that the contact pressure distribution at the interface is $Q(x)$ (Fig. 4(c)). The differential equation for the deflexion of the beam, $w(x)$ is

$$\tilde{E}I\frac{d^4w}{dx^4}+Q(x) = p(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

where $\tilde{E}I$ is the rigidity of the beam.

It is noted that

$$t_{yy}(x, 0) = Q(x) \qquad v(x, 0) = w(x) \qquad (32)$$

and from (24) and (28) we have

$$Q(x) = \frac{(2\mu+\kappa)(\chi+\lambda)}{2\chi}\alpha\eta w(x) \qquad (33)$$

From equations (30), (31) and (33)

$$w(x) = \frac{2\chi p_0 \cos \alpha x}{[2\chi \bar{E}I\alpha^4 + (2\mu+\kappa)(\chi+\lambda)\alpha\eta]} \qquad (34)$$

Consider the case of the infinite elastic beam subjected to the uniform strip load as shown in Fig. 3. Any arbitrary load $p(x)$ symmetrical about the y-axis can be represented in the form of a Fourier integral

$$p(x) = \int_0^\infty \bar{p}(\alpha) \cos \alpha x \, d\alpha \qquad (35)$$

where

$$\bar{p}(\alpha) = \int_0^\infty p(\zeta) \cos \alpha\zeta \, d\zeta \qquad (36)$$

From equations (34) and (35) it may be concluded that the deflexion of the infinite beam due to the uniform strip load is given by

$$w(x) = \frac{2pa^4}{\pi \bar{E}I}\int_0^\infty \frac{\sin(\xi d/a)\cos(\xi x/a)}{\xi\{\xi^4+\xi[1+\Omega(\xi)]\}} d\xi \qquad (37)$$

where

$$\left.\begin{aligned} a^3 &= \frac{2\bar{E}I\chi}{(2\mu+\kappa)(\chi+\lambda)} \qquad \xi = \alpha a \\ \Omega(\xi) &= 4(1-\nu)(\bar{b})^2\xi^2\left\{1-\frac{\xi}{[(\bar{c})^2+\xi^2]^{\frac{1}{2}}}\right\} \\ \bar{b} &= b/a \qquad \bar{c} = a/c \end{aligned}\right\} \qquad (38)$$

Similarly expressions for the bending moment $M(x)$ shearing force $V(x)$ in the beam and the contact pressure $Q(x)$ at the interface are

$$M(x) = -\frac{2pa^2}{\pi}\int_0^\infty \frac{\sin(\xi d/a)\cos(\xi x/a)}{[1+\xi^3+\Omega(\xi)]} d\xi \qquad (39)$$

$$V(x) = \frac{2pa}{\pi}\int_0^\infty \frac{\xi \sin(\xi d/a)\sin(\xi x/a)}{[1+\xi^3+\Omega(\xi)]} d\xi \qquad (40)$$

$$Q(x) = \frac{2p}{\pi}\int_0^\infty \frac{[1+\Omega(\xi)]\sin(\xi d/a)\cos(\xi x/a)}{\xi[1+\xi^3+\Omega(\xi)]} d\xi \qquad (41)$$

It is noted that the expressions thus obtained for the deflexion, bending moment, shearing force etc. (equations (37), (39), (40) and (41)) for the micropolar elastic problem are very similar in character to those obtained by Biot (1937) for the classical elastic problem and they are dependent on the two new material parameters κ and γ. It has also been assumed that the infinite beam–elastic half plane interface is capable of transmitting tensile stresses and that there is no loss of contact at the interface. Alternatively a uniform normal load may be imposed throughout the length of the infinite beam. A normal load of this type, by virtue of its symmetry, will give a constant vertical displacement on the surface of the elastic half plane. The infinite beam is then subjected to a rigid body translation in the vertical direction and the flexure of the infinite beam is thus unaffected by the additional superimposed loading.

LIMITING CONDITIONS

First, the micropolar elastic solution to the problem of an infinite beam loaded by a concentrated force P can be obtained as a limiting case of the uniform strip load problem. As the width of the loaded region $d \to 0$, the total load $2pd \to P$ and considering that

$$\xi d \overset{\mathrm{Lt}}{\to} 0 \qquad \frac{\sin \xi d}{\xi d} = 1$$

the expressions (37), (39), (40) and (41) reduce to

$$w(x) = \frac{Pa^3}{\pi \bar{E} I} \int_0^\infty \frac{\cos(\xi x/a)}{\{\xi^4 + \xi[1+\Omega(\xi)]\}} \mathrm{d}\xi \qquad \text{(42a)}$$

$$M(x) = -\frac{Pa}{\pi} \int_0^\infty \frac{\xi \cos(\xi x/a)}{[1+\xi^3+\Omega(\xi)]} \mathrm{d}\xi \qquad \text{(42b)}$$

$$V(x) = \frac{P}{\pi} \int_0^\infty \frac{\xi^2 \sin(\xi x/a)}{[1+\xi^3+\Omega(\xi)]} \mathrm{d}\xi \qquad \text{(42c)}$$

$$Q(x) = \frac{P}{\pi a} \int_0^\infty \frac{[1+\Omega(\xi)] \cos(\xi x/a)}{[1+\xi^3+\Omega(\xi)]} \mathrm{d}\xi \qquad \text{(42d)}$$

The solution to the concentrated force problem is of fundamental importance in the treatment of beams of finite length. By superposing two concentrated force solutions the solution to the problem of an infinite beam loaded by a concentrated moment can be obtained. A successive superposition, similar to that adopted by Drapkin (1955) for the linear elastic case, can then be used to obtain solutions for beams of finite length.

Second, the additional micropolar elastic constants κ and γ are contained in the function $\Omega(\xi)$. As these constants tend to zero $\Omega(\xi) \to 0$, and the micropolar elastic solution converges to the classical elastic solution. In the case of an infinite beam loaded by a concentrated force the expressions (42) converge to Biot's solution as $\Omega(\xi) \to 0$.

Third, by replacing $(2\mu+\kappa)$ by 2μ, γ by 4η and subsequently letting $\kappa \to \infty$ the couple stress solution to the infinite beam problem is obtained on the basis of the indeterminate couple stress theory of Mindlin and Tiersten (1964). The equivalent solutions for the deflexion, bending moment etc., are obtained by replacing $\Omega(\xi)$ (equation (38)) in the equations (37) to (41) by $\tilde{\Omega}(\xi)$ where

$$\tilde{\Omega}(\xi) = 4(1-\nu)\xi^2/g^2\left[1 - \frac{\xi}{(g^2+\xi^2)^{\frac{1}{2}}}\right]$$

where

$$g = \frac{a}{l} \qquad l^2 = \frac{\eta}{\mu} = \frac{h^2}{12(1-\nu_0)}$$

and η is the single new parameter necessary to describe couple stress effects for plane strain deformations.

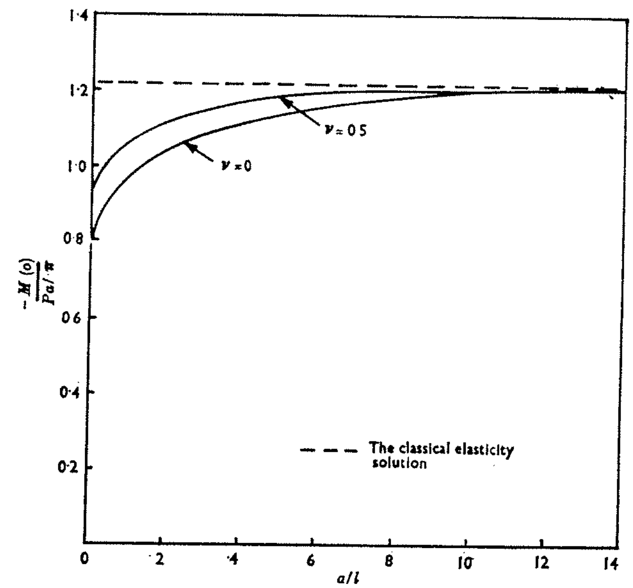

Fig. 5. The variation of $M(o)$ with a/l for the concentrated force

NUMERICAL RESULTS

The infinite integrals of the type (37) and (39) to (41) can be evaluated numerically, by employing the methods outlined by Biot (1937), Drapkin (1955) and Vésic (1961). The integration procedure can be further improved by replacing the integration parameter ξ by $e^{in\xi}$. The Author gives numerical results for the infinite beam problem derived from the indeterminate couple stress theory.

First consider results for the infinite beam loaded by a concentrated force P. The variation of the values of the bending moment at $x=0$, $M(0)$, computed from equation (42b) for various values of a/l and for values of $\nu=0$ and 0·5 are shown in Fig. 5. The variation of the bending moment along the length of the infinite beam for values of $a/l=0{\cdot}01$, 1·0 and $\nu=0$, 0·5 is shown in Fig. 6. The corresponding curves for the classical elastic case, $a/l \to \infty$, given by Biot (1937) are also presented in these figures for comparison. We observe that as the characteristic length parameter l of the supporting medium becomes larger, the maximum value of both the positive and negative bending moment is significantly reduced. The contact pressure distribution at the infinite beam–elastic half plane interface, derived from equation (42d), is shown in Fig. 7. As the characteristic length parameter l increases the maximum value of the contact pressure also increases.

The variation of $M(0)$ with a/l for the case of the infinite beam loaded with a uniform strip load of width $2a$ is shown in Fig. 8.

CONCLUSIONS

In classical continuum mechanics the existence of any internal structure in the material is ignored. Therefore, the results obtained from the classical theory for the forces, stresses, displacements etc., are representative of the average values over regions the dimensions of which are large on the scale of the internal structure. It is then foreseeable that for materials

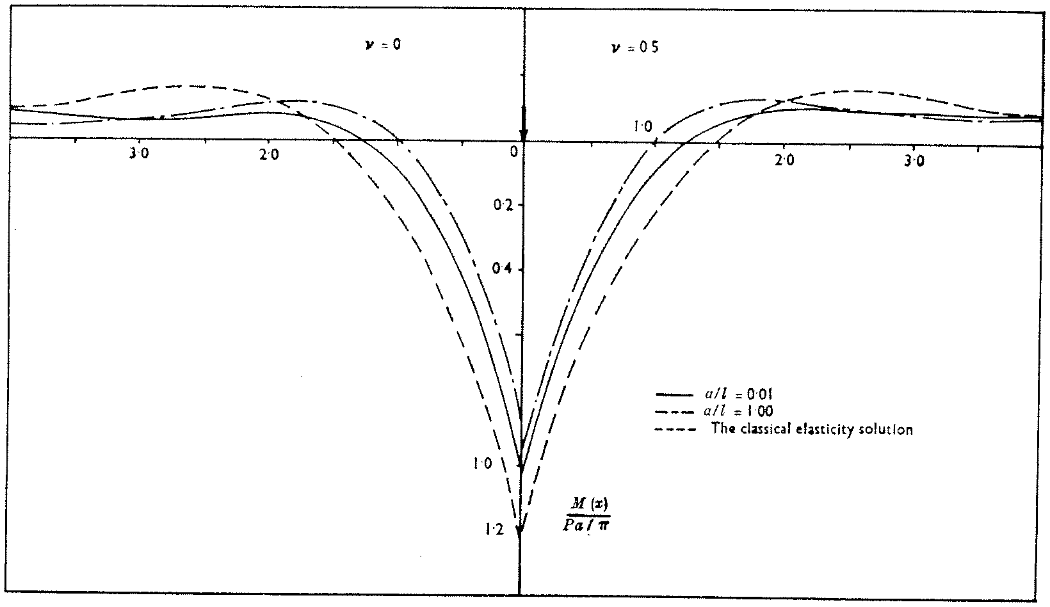

Fig. 6. Concentrated force problem: variation of bending moment

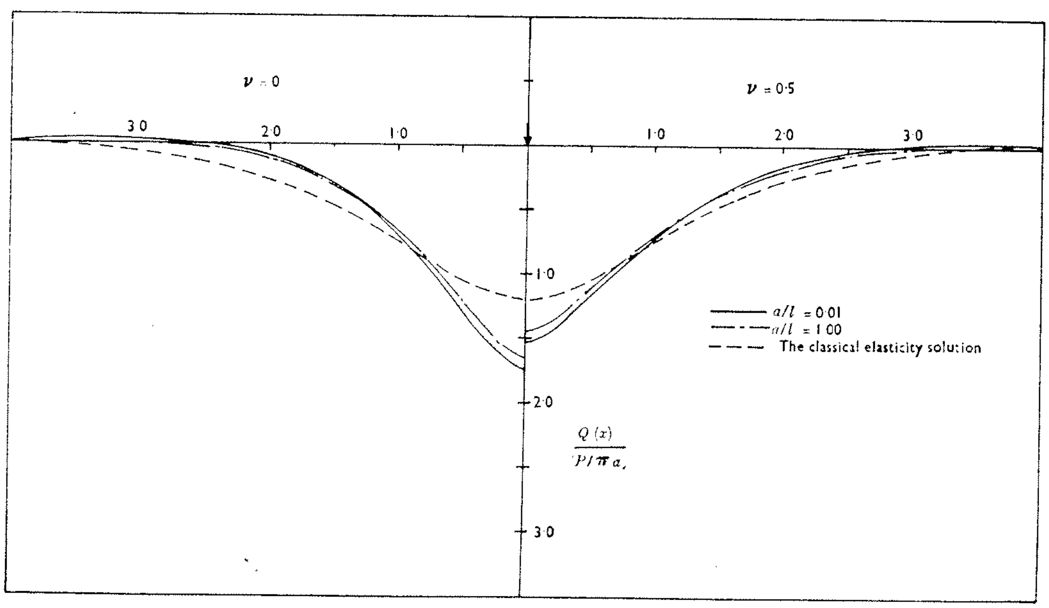

Fig. 7. Concentrated force problem: distribution of contact pressure at the beam–elastic medium interface

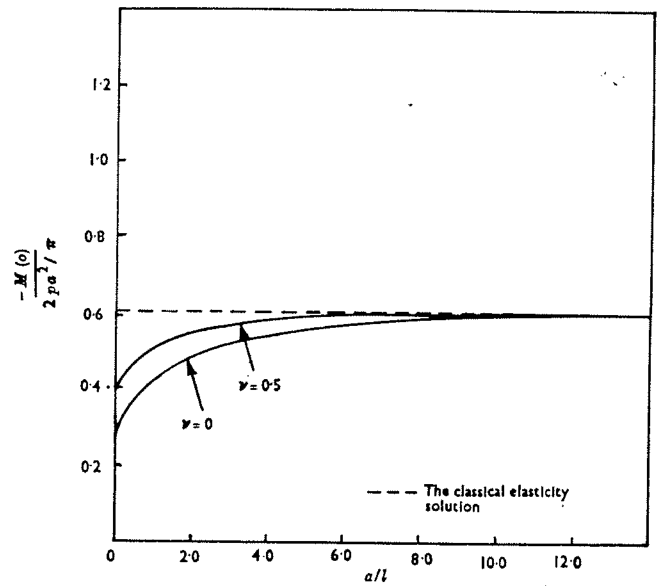

Fig. 8. The variation of $M(o)$ with a/l for the uniform strip load of width $2a$

with a dominant internal structure, such as poroelastic media, under certain conditions of loading, the classical theory will fail to give meaningful results. In these circumstances it becomes necessary to take into account the effects of force systems which vary significantly in distances comparable with the lengths that characterize the internal structure. The theories in continuum mechanics which employ the concepts of couple stresses therefore serve to form a basis for the treatment of materials with internal structure.

The Author has considered here the bending of an infinite beam resting on an elastic half plane exhibiting couple stresses. The numerical results presented indicate that the characteristic length parameter l of the supporting medium has a significant influence on the magnitude and distribution of the bending moment in the infinite beam and on the contact pressure at the infinite beam–elastic half plane interface. It has been shown that the couple stress parameter l bears a direct relation to the physical dimensions which characterize the internal structure of an idealized poroelastic medium. The reliability of such an idealization of a porous elastic medium can only be verified by careful experimental investigation.

ACKNOWLEDGEMENT

The Author is grateful to Mr H. Atefi for assistance with the numerical computations.

REFERENCES

Adomeit, G. (1967). Determination of elastic constants of a structured material. *Proc. IUTAM Symposium Freudenstadt and Stuttgart.* New York: Springer.

Aero, E. L. & Kuvshinski, E.V. (1960). Fundamental equations of the theory of elastic media with rotationally interacting particles. *Fizika Tverdogo Tela* **2**, 1399. English Translation in *Soviet Physics Solid State* (1961) **2**, 1272–1281.

Askar, A. & Cakmak, A. S. (1968). A structural model of a micropolar continuum. *Int. Jnl Engng Sci.* **6**, 583–589.

Banks, C. B. & Sokolowski, M. (1968). On certain two dimensional applications of the couple stress theory. *Int. Jnl Solids and Structures* **4**, 15–29.
Bažant, Z. P. & Christensen, M. (1972). Analogy between micropolar continuum and grid frameworks under initial stress. *Int. Jnl Solids and Structures* **8**, 327–346.
Biot, M. A. (1937). Bending of an infinite beam of an elastic foundation. *Jnl Appl. Mech.* **4**, A1–A7.
Biot, M. A. (1965). Further developments of the theory of internal buckling of multilayers. *Bull. Geol. Soc. Am.* **76**, 833–840.
Brown, C. B. & Evans, R. J. (1972). On the application of couple-stress theories to granular media. *Géotechnique* **22**, No. 2, 356–361.
Cosserat, E. & Cosserat, F. (1909). *Théorie des corps déformables.* Paris: A. Herman et Fils.
Drapkin, B. (1955). Grillage beams on elastic foundation. *Proc. Am. Soc. Civ. Engrs* **81**, No. 771, 1–19.
Ellis, R. W. & Smith, C. W. (1967). A thin-plate analysis and experimental evaluation of couple-stress effects. *Experimental Mechanics* **7**, 372–380.
Eringen, A. C. (1966). Linear theory of micropolar elasticity. *Jnl Math. Mech.* **15**, 909–923.
Eringen, A. C. (1969). Theory of micropolar elasticity. *Fracture—an advanced treatise* **2**. New York: Academic Press.
Eringen, A. C. & Suhubi, E. S. (1964). Nonlinear theory of simple micro-elastic solids I and II. *Int. Jnl Engng Sci.* **2**, 189–203 and 389–404.
Flugge, W. (1972). *Tensor analysis and continuum mechanics.* Berlin: Springer-Verlag.
Green, A. E. & Rivlin, R. S. (1964). Multipolar continuum mechanics. *Archs Ration. Mech. Analysis* **17**, 113–147.
Grioli, G. (1960). Elastica asimmetrica. *Annali Mat. pura appl.* Ser. IV, **50**, 389–417.
Herrmann, G. (1972). Some applications of micromechanics. *Experimental Mechanics* **13**, 235–238.
Hoppmann, W. H. & Shahwan, F. O. F. (1965). Physical model of a three-constant isotropic elastic material. *Jnl Appl. Mech.* **32**, 837–841.
Koiter, W. T. (1964). Couple stresses in the theory of elasticity. *Proc. Koninkl, Neder. Akad. Wetenschappen—Amsterdam* Ser. B, **67**, 17–44.
Kröner, E. (1968). Mechanics of generalized continua. *Proc. IUTAM Symposium Freudenstadt and Stuttgart.* New York: Springer.
Lippmann, H. (1968). A Cosserat-type theory of plastic flow. *Proc. 12th Int. Congr. Appl. Mech.*, Stanford University, California.
Mindlin, R. (1962). Influence of couple-stresses on stress concentrations. *Experimental Mechanics* **3**, 1–7.
Mindlin, R. (1964). Microstructure in linear elasticity. *Archs Ration. Mech. Analysis* **16**, 51–78.
Mindlin, R. & Tiersten, H. F. (1962). Effects of couple-stresses in linear elasticity. *Archs Ration. Mech. Analysis* **11**, 415–448.
Muki, R. & Sternberg, E. (1965). The influence of couple-stresses on singular stress concentrations in elastic solids. *Z. Angew. Math. Phys.* **16**, 611–648.
Schijve, J. (1966). Note on couple stresses. *Jnl Mech. Phys. Solids* **14**, 113–120.
Selvadurai, A. P. S. (1973). *Physical and mathematical models of poroelastic media.* Unpublished work.
Toupin, R. A. (1962). Elastic materials with couple stresses. *Archs Ration. Mech. Analysis* **11**, 385–414.
Truesdell, C. & Toupin, R. A. (1960). The classical field theories. *Encyclopædia of Physics* **III/1**, Sections 200, 203, 205. Berlin: Springer.
Vésic, A. (1961). Bending of beams resting on an isotropic elastic solid. *Jnl Eng. Mech. Div. Am. Soc. Civ. Engrs* **87**, EM2, 1, 35–53.
Voigt, W. (1887). Theoretische Studien über die Elastizitätsverhältnisse der Kristalle. *Abh. Ges. Wiss. Göttingen* **34**.