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ASYMMETRIC DISPLACEMENTS OF A RIGID DISC INCLUSION EMBEDDED IN A TRANSVERSELY ISOTROPIC ELASTIC MEDIUM OF INFINITE EXTENT

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Abstract—Asymmetric problems related to a penny-shaped rigid inclusion embedded in bonded contact with a transversely isotropic elastic medium are investigated. The asymmetric displacements of the rigid circular inclusion correspond to a rotation about a diametral axis and an in-plane lateral translation. These problems are formulated in terms of Hankel integral transforms and reduced to systems of dual integral equations. The rotational and translational stiffnesses for the embedded rigid circular disc inclusion are obtained in exact closed forms.

1. INTRODUCTION

THE CATEGORY of problems which examines the behaviour of disc-shaped inclusions embedded in elastic media has received considerable attention. The solution to the problem of a penny-shaped rigid inclusion embedded in bonded contact with an isotropic elastic medium and subjected to a constant displacement normal to its plane was examined by Collins[1]. Keer[2] considered the problem related to the in-plane translation of the disc inclusion. Kassir and Sih[3] subsequently extended these investigations to include the behaviour of elliptical disc inclusions. Solutions to disc inclusion problems can also be recovered as limiting cases of results derived for ellipsoidal and spheroidal rigid inclusion problems [4–6]. A variety of analytical techniques have been utilized in the above investigations; these include the complex potential function approach of Green[7], singularity methods and direct spheroidal and ellipsoidal harmonic function techniques.

This paper examines two asymmetric problems related to a rigid circular disc inclusion embedded in bonded contact with a transversely isotropic elastic medium. The asymmetric deformations in the transversely isotropic elastic medium are due to the rotation of the circular inclusion about a diametral axis and an in-plane lateral translation. The rigid circular disc inclusion is embedded in such a way that the axis of elastic symmetry of the transversely isotropic elastic medium is normal to the plane of the disc inclusion. The analysis of these problems can be approached by making use of the technique which extends the complex potential function approach to transversely isotropic elastic materials (see, e.g. Shield[8]). An alternative procedure would include a direct spheroidal harmonic function approach related to the analysis of a rigid spheroidal inclusion embedded in a transversely isotropic elastic medium. In this paper, however, we analyse the stated problems by appeal to a Hankel integral transform formulation. The use of such a procedure is facilitated by the antisymmetry or symmetry that the problems exhibit about the plane $z=0$ (Fig. 1). By invoking the relevant symmetry properties, these disc inclusion problems can be reduced to mixed boundary value problems associated with a half-space region. The mixed boundary value problems are further reduced to two sets of dual integral equations. These sets of dual integral equations correspond to the states of asymmetry induced by the rotation and in-plane translation of the disc inclusion. The solution of the dual systems is readily obtained from the generalized results given by Titchmarsh[9] and Sneddon[10, 11]. The results of primary interest to engineering applications, namely, the rotational and in-plane translational stiffnesses of the disc inclusion embedded in a transversely isotropic elastic medium are obtained in exact closed form.

2. GOVERNING EQUATIONS

The problem of determining the distribution of displacement and stress fields in a medium with transverse isotropy has been considered by Lekhnitskii[12], Elliott[13, 14], Shield[8], Payne[15] *et al.* These formulations are based on the use of potential functions. These potential functions are harmonic in spaces different from the physical space. Except for the state of axisymmetric torsion, at least two potential functions are needed for the solution of a purely axisymmetric problem related to a transversely isotropic elastic medium. The solution of

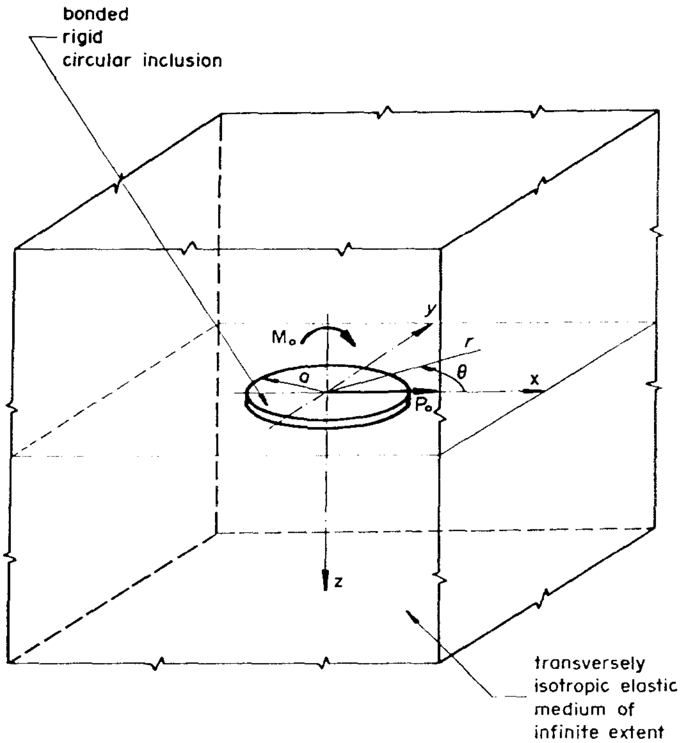

Fig. 1. Geometry of the disc inclusion embedded in a transversely isotropic elastic medium.

asymmetric problems, however, requires the use of three such potential functions. Complete accounts of these developments, together with references to further work involving transversely isotropic elastic media, are given by Green and Zerna[16], Kassir and Sih[17] and Chen[18].

It can be shown that, in the absence of body forces, the displacement and stress fields in a transversely isotropic linear elastic material, subjected to a state of asymmetric deformation about the axis of elastic symmetry, can be expressed in terms of three potential functions $\varphi_1(r, \theta, z)$, $\varphi_2(r, \theta, z)$ and $\psi(r, \theta, z)$ which are solutions of

$$\left\{\nabla_1^2 + \frac{\partial^2}{\partial z_i^2}\right\} \varphi_i(r, \theta, z) = 0, \quad i = 1, 2 \tag{1}$$

and

$$\left\{\nabla_1^2 + \frac{\partial^2}{\partial z_3^2}\right\} \psi(r, \theta, z) = 0 \tag{2}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \theta^2}; \quad z_i = \frac{z}{\sqrt{\nu_i}}; \quad (i = 1, 2, 3) \tag{3}$$

$$\nu_3 = \frac{2c_{11}}{(c_{11} - c_{44})} \tag{4}$$

and ν_1 and ν_2 are the roots of the equation

$$c_{11}c_{44}\nu^2 + [c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}]\nu + c_{33}c_{44} = 0. \tag{5}$$

The cylindrical polar coordinate system (r, θ, z) is chosen such that the z-axis is parallel to the material axis of symmetry. The roots ν_1 and ν_2 may be real or complex depending upon the elastic constants c_{11}, c_{12}, c_{13}, c_{33} and c_{44} (see e.g. [16]). The component ν_3 is always real and positive. We specify that $\sqrt{\nu_1}$, $\sqrt{\nu_2}$ and $\sqrt{\nu_3}$ always have positive real parts. The displacement and stress components referred to a cylindrical polar coordinate system are given by

$$\begin{aligned} u_r &= \frac{\partial}{\partial r}(\varphi_1+\varphi_2)+\frac{1}{r}\frac{\partial\psi}{\partial\theta} \\ u_\theta &= \frac{1}{r}\frac{\partial}{\partial\theta}(\varphi_1+\varphi_2)-\frac{\partial\psi}{\partial r} \\ u_z &= \frac{\partial}{\partial z}(k_1\varphi_1+k_2\varphi_2) \end{aligned} \tag{6a–c}$$

and

$$\frac{\sigma_{rr}}{c_{44}} = -\frac{(1+k_1)}{\nu_1}\frac{\partial^2\varphi_1}{\partial z_1^2}-\frac{(1+k_2)}{\nu_2}\frac{\partial^2\varphi_2}{\partial z_2^2}+\frac{2}{\nu_3}\left[\left\{\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right\}(\varphi_1+\varphi_2)-\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right\}\right] \tag{7a}$$

$$\frac{\sigma_{\theta\theta}}{c_{44}} = -\frac{(1+k_1)}{\nu_1}\frac{\partial^2\varphi_1}{\partial z_1^2}+\frac{(1+k_2)}{\nu_2}\frac{\partial^2\varphi_2}{\partial z_2^2}-\frac{2}{\nu_3}\left[\frac{\partial^2}{\partial r^2}(\varphi_1+\varphi_2)+\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right\}\right] \tag{7b}$$

$$\frac{\sigma_{zz}}{c_{44}} = \frac{\{k_1c_{33}-\nu_1c_{13}\}}{\nu_1c_{44}}\frac{\partial^2\varphi_1}{\partial z_1^2}+\frac{\{k_2c_{33}-\nu_2c_{13}\}}{\nu_2c_{44}}\frac{\partial^2\varphi_2}{\partial z_2^2} \tag{7c}$$

$$\frac{\sigma_{r\theta}}{c_{44}} = \frac{2}{\nu_3}\left[\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta}\right](\varphi_1+\varphi_2)+\frac{1}{\nu_3}\left[\frac{2}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z_3^2}+\frac{2}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\right] \tag{7d}$$

$$\frac{\sigma_{\theta z}}{c_{44}} = \frac{(1+k_1)}{\sqrt{\nu_1}}\frac{1}{r}\frac{\partial^2\varphi_1}{\partial\theta\partial z_1}+\frac{(1+k_2)}{\sqrt{\nu_2}}\frac{1}{r}\frac{\partial^2\varphi_2}{\partial\theta\partial z_2}-\frac{1}{\sqrt{\nu_3}}\frac{\partial^2\psi}{\partial r\partial z_3} \tag{7e}$$

$$\frac{\sigma_{rz}}{c_{44}} = \frac{(1+k_1)}{\sqrt{\nu_1}}\frac{\partial^2\varphi_1}{\partial r\partial z_1}+\frac{(1+k_2)}{\sqrt{\nu_2}}\frac{\partial^2\varphi_2}{\partial r\partial z_2}+\frac{1}{\sqrt{\nu_3}}\frac{1}{r}\frac{\partial^2\psi}{\partial\theta\partial z_3} \tag{7f}$$

respectively. In eqns (6) and (7), the parameters k_1 and k_2 are given by

$$k_i = \left[\frac{c_{11}\nu_i-c_{44}}{c_{13}+c_{44}}\right]; \quad i=1,2. \tag{8}$$

3. THE DISC INCLUSION PROBLEM

We consider the asymmetric problem related to a penny-shaped rigid inclusion, of radius a, embedded in bonded contact with a transversely isotropic elastic medium of infinite extent. The rigid inclusion is subjected to a couple M_0 which acts about the y-axis and a force P_0 directed along the x-axis. Due to the action of the couple and the force, the rigid inclusion experiences a rotation Ω about the diametral axis at $\theta=\pi/2$ and a lateral translation δ in the x–y plane. It is evident that the problem related to the rotation of the embedded disc inclusion exhibits a state of antisymmetry (in σ_{zz}, u_r and u_θ) about the plane $z=0$, whereas the problem related to the translation of the disc inclusion exhibits a state of symmetry (in u_z and in the traction component in the y-direction) about the plane $z=0$. Therefore, we may restrict the analysis to a single half-space region of the transversely isotropic elastic medium in which the plane $z=0$ is subjected to boundary conditions which reflect the state of antisymmetry or symmetry of the problem.

(i) For the rotation of the disc inclusion, the plane $z=0$ is subjected to the following mixed boundary conditions

$$u_r(r,\theta,0)=u_\theta(r,\theta,0)=0; \qquad r\ge 0 \tag{9a}$$

$$u_z(r,\theta,0)=\Omega r\cos\theta; \qquad 0\le r\le a \tag{9b}$$

$$\sigma_{zz}(r,\theta,0)=0; \qquad a<r<\infty. \tag{9c}$$

(ii) For the translation of the disc inclusion, the plane $z=0$ is subjected to the following mixed boundary conditions

$$u_z(r,\theta,0)=0 \qquad r\geq 0 \tag{10a}$$

$$\sigma_{rz}\sin\theta+\sigma_{\theta z}\cos\theta=0 \qquad r\geq 0 \tag{10b}$$

$$u_r(r,\theta,0)=\delta\cos\theta \qquad 0\leq r\leq a \tag{10c}$$

$$u_\theta(r,\theta,0)=-\delta\sin\theta \qquad 0\leq r\leq a \tag{10d}$$

$$\sigma_{rz}\cos\theta-\sigma_{\theta z}\sin\theta=0 \qquad a\leq r\leq\infty. \tag{10e}$$

For the solution of these asymmetric problems we introduce the nth order Hankel transform of $\varphi(r,\theta,z)$ as follows

$$\bar{\varphi}^n(\xi,\theta,z)=\mathcal{H}_n\{\varphi(r,\theta,z);\xi\}=\int_0^\infty r\varphi(r,\theta,z)J_n(\xi r/a)\,\mathrm{d}r. \tag{11a}$$

The appropriate Hankel inversion theorem is

$$\varphi(r,\theta,z)=\mathcal{H}_n^{-1}\{\bar{\varphi}^n(\xi,\theta,z);r\}=\frac{1}{a^2}\int_0^\infty \xi\bar{\varphi}^n(\xi,\theta,z)J_n(\xi r/a)\,\mathrm{d}\xi. \tag{11b}$$

4. ROTATION OF THE RIGID DISC INCLUSION

We examine the problem related to the rotation of the embedded inclusion and restrict our attention to the analysis of the half-space region $z\geq 0$. In this region, the displacement and stress fields derived from φ_i and ψ should reduce to zero as $(r^2+z^2)^{1/2}\to\infty$. The solutions of (1) and (2) appropriate for the half-space region $z\geq 0$ are

$$\varphi_i(r,\theta,z)=\left[\frac{1}{a^2}\int_0^\infty \xi A_i(\xi)\,\mathrm{e}^{-\lambda_i z}J_1(\xi r/a)\,\mathrm{d}\xi\right]\cos\theta; \quad (i=1,2) \tag{12}$$

and

$$\psi(r,\theta,z)=\left[\frac{1}{a^2}\int_0^\infty \xi A_3(\xi)\,\mathrm{e}^{-\lambda_3 z}J_1(\xi r/a)\,\mathrm{d}\xi\right]\sin\theta \tag{13}$$

where $A_i(\xi)$ $(i=1,2,3)$ are arbitrary functions and $\lambda_i=\xi/a\sqrt{\nu_i}$. Using (12) and (13) in the general expressions for the displacement components u_r and u_θ given by (6a and b) we obtain

$$u_r(r,\theta,0)=\left[\frac{1}{a^2}\int_0^\infty \xi\{-A_1(\xi)-A_2(\xi)+A_3(\xi)\}\frac{J_1(\xi r/a)}{r}\,\mathrm{d}\xi \right.$$
$$\left. +\frac{1}{a^3}\int_0^\infty \xi^2\{A_1(\xi)+A_2(\xi)\}J_0(\xi r/a)\,\mathrm{d}\xi\right]\cos\theta \tag{14}$$

$$u_\theta(r,\theta,0)=\left[\frac{1}{a^2}\int_0^\infty \xi\{-A_1(\xi)-A_2(\xi)+A_3(\xi)\}\frac{J_1(\xi r/a)}{r}\,\mathrm{d}\xi \right.$$
$$\left. +\frac{1}{a^3}\int_0^\infty \xi^2 A_3(\xi)J_0(\xi r/a)\,\mathrm{d}\xi\right]\sin\theta. \tag{15}$$

From (14) and (15) it is evident that the boundary conditions (9a) will be satisfied for all r and θ provided

$$A_1(\xi)=-A_2(\xi) \quad \text{and} \quad A_3(\xi)=0. \tag{16}$$

By using the above results and the expressions for u_z and σ_{zz} given by (6c) and (7c), respectively, it can be shown that the boundary conditions (9b and c) are equivalent to

$$\left[\frac{\{k_2\sqrt{\nu_1}-k_1\sqrt{\nu_2}\}}{a^3\sqrt{(\nu_1\nu_2)}}\int_0^\infty \xi^2 A_1(\xi)J_1(\xi r/a)\,d\xi\right]\cos\theta=\Omega r\cos\theta;\quad 0\le r\le a$$
$$\frac{c_{33}\{k_1\nu_2-k_2\nu_1\}}{\nu_1\nu_2 a^4}\left[\int_0^\infty \xi^3 A_1(\xi)J_1(\xi r/a)\,d\xi\right]\cos\theta=0;\quad a<r<\infty. \tag{17}$$

Further, by using the substitutions

$$\frac{\Omega a^3\sqrt{(\nu_1\nu_2)}}{\{k_2\sqrt{\nu_1}-k_1\sqrt{\nu_2}\}}=\Omega_0;\quad \zeta=\frac{r}{a};\quad \xi^2A_1(\xi)=M(\xi) \tag{18}$$

the dual system (17) can be reduced to

$$\mathcal{H}_1\{\xi^{-1}M(\xi);\ \zeta\}=\Omega_0\zeta;\qquad 0\le\zeta\le 1$$
$$\mathcal{H}_1\{M(\xi);\ \zeta\}=0;\qquad 1<\zeta<\infty. \tag{19}$$

The general solution of this system of dual integral equations is given by Titchmarsh[9] and also by Copson[19]. The final form of the solution given by Titchmarsh occurs in a rather complicated form. Sneddon[20] subsequently presented a simpler result in which the dual system (19) is reduced to an integral equation of the Abel type. This solution gives

$$M(\xi)=\frac{2}{\pi a}\int_0^a\left\{\frac{1}{t}\frac{d}{dt}\int_0^t\frac{r^2f(r)\,dr}{\sqrt{(t^2-r^2)}}\right\}\sin(\xi t/a)\,dt \tag{20}$$

where

$$f(r)=\frac{\Omega a^3\sqrt{(\nu_1\nu_2)}r}{\{k_2\sqrt{\nu_1}-k_1\sqrt{\nu_2}\}}. \tag{21}$$

From (20) and (21) we have

$$A_1(\xi)=\frac{4\Omega a^4\sqrt{(\nu_1\nu_2)}\{\sin\xi-\xi\cos\xi\}}{\pi\xi^4\{k_2\sqrt{\nu_1}-k_1\sqrt{\nu_2}\}}. \tag{22}$$

Formal integral expressions for the displacement and stress fields in the half-space region $z\ge 0$ can be determined by making use of the relevant general results presented above. To determine the rotational stiffness of the disc inclusion embedded in the transversely isotropic elastic material it is necessary to evaluate the normal contact stress distributions at the bonded interfaces $z=0^+$ and $z=0^-$. We observe that, due to the antisymmetry of the deformation $\sigma_{zz}(r,\theta,0^+)=-\sigma_{zz}(r,\theta,0^-)$. The integral expression for $\sigma_{zz}(r,\theta,0^+)$ is

$$\sigma_{zz}(r,\theta,0^+)=\frac{4c_{33}\Omega\{k_1\nu_2-k_2\nu_1\}\cos\theta}{\pi\sqrt{(\nu_1\nu_2)}\{k_2\sqrt{\nu_1}-k_1\sqrt{\nu_2}\}}\int_0^\infty\left\{\frac{\sin\xi}{\xi}-\cos\xi\right\}J_1(\xi r/a)\,d\xi. \tag{23}$$

Integrating (23) (see, e.g. [21]) we have

$$\sigma_{zz}(r,\theta,0^+)=\begin{cases}\dfrac{4c_{33}\Omega\{k_1\nu_2-k_2\nu_1\}r\cos\theta}{\pi\sqrt{(\nu_1\nu_2)}\{k_2\sqrt{\nu_1}-k_1\sqrt{\nu_2}\}\sqrt{(a^2-r^2)}}; & 0\le r<a\\ 0\qquad ; & a<r<\infty.\end{cases} \tag{24}$$

The couple exerted by the transversely isotropic elastic medium on the rigid disc inclusion is

$$M_0=\int_0^a\int_{-\pi}^{\pi}[\sigma_{zz}(r,\theta,0^+)-\sigma_{zz}(r,\theta,0^-)]r^2\cos\theta\,dr\,d\theta. \tag{25}$$

Evaluating (25) we have

$$M_0 = \frac{16c_{33}\Omega a^3\{k_1\nu_2 - k_2\nu_1\}}{3\{k_2\sqrt{\nu_1} - k_1\sqrt{\nu_2}\}\sqrt{(\nu_1\nu_2)}}. \tag{26}$$

In the limiting case of isotropic elastic behaviour of the medium we note that $\nu_i(i = 1, 2, 3) \to 1$ and

$$\left\{\frac{k_1\nu_2 - k_2\nu_1}{k_2\sqrt{\nu_1} - k_1\sqrt{\nu_2}}\right\} = -\frac{2c_{44}}{(c_{11} + c_{44})}. \tag{27}$$

In (27)

$$c_{11} = c_{33} = (\lambda + 2\mu); \qquad c_{44} = \mu \tag{28}$$

where λ and μ are Lame's constants for the isotropic elastic material. Using these results we obtain the following expression for the rotational stiffness of a circular rigid inclusion embedded in bonded contact with an isotropic elastic solid

$$M_0 = -\frac{64\mu\Omega a^3(1-\nu)}{3(3-4\nu)}. \tag{29}$$

This result is in agreement with the expression for the rotational stiffness of an embedded disc inclusion derived by Kanwal and Sharma [4] using the method of singularities.

5. TRANSLATION OF THE RIGID DISC INCLUSION

In order to examine the problem related to the in-plane translation of the rigid disc inclusion we restrict the analysis to the half-space region $z \geq 0$. The solutions of (1) and (2) appropriate for the half-space region are given by (12) and (13). To satisfy the boundary condition (10a) we require

$$A_2(\xi) = -\frac{k_1}{k_2}\sqrt{\left(\frac{\nu_2}{\nu_1}\right)}\, A_1(\xi). \tag{30}$$

Similarly, by invoking the boundary condition (10b) related to the traction vector in the y-direction we have

$$A_3(\xi) = \frac{\sqrt{\nu_3}(k_2 - k_1)}{k_2\sqrt{\nu_1}}\, A_1(\xi). \tag{31}$$

The remaining boundary conditions (10c–e) yield the following results

$$\left[\frac{1}{a^2}\int_0^\infty \xi A_1(\xi)\left\{-1 + \frac{k_1\sqrt{\nu_2}}{k_2\sqrt{\nu_1}} + \frac{\sqrt{\nu_3}(k_2-k_1)}{k_2\sqrt{\nu_1}}\right\}\frac{J_1(\xi r/a)}{r}\,d\xi + \frac{1}{a^3}\int_0^\infty \xi^2 A_1(\xi)\left\{1 - \frac{k_1\sqrt{\nu_2}}{k_2\sqrt{\nu_1}}\right\}J_0(\xi r/a)\,d\xi\right]\cos\theta = \delta\cos\theta; \quad 0 \leq r \leq a. \tag{32a}$$

$$\left[\frac{1}{a^2}\int_0^\infty \xi A_1(\xi)\left\{-1 + \frac{k_1\sqrt{\nu_2}}{k_2\sqrt{\nu_1}} + \frac{\sqrt{\nu_3}(k_2-k_1)}{k_2\sqrt{\nu_1}}\right\}\frac{J_1(\xi r/a)}{r}\,d\xi + \frac{1}{a^3}\int_0^\infty \xi^2 A_1(\xi)\left\{\frac{\sqrt{\nu_3}(k_2-k_1)}{k_2\sqrt{\nu_1}}\right\}J_0(\xi r/a)\,d\xi\right]\sin\theta = -\delta\sin\theta; \quad 0 \leq r \leq a \tag{32b}$$

$$-\frac{c_{44}(k_2-k_1)}{a^4 k_2\sqrt{\nu_1}}\int_0^\infty \xi^3 A_1(\xi)J_0(\xi r/a)\,d\xi = 0; \qquad a \leq r \leq \infty. \tag{32c}$$

We note that the eqns (32a and b) can be combined to form a single nonhomogeneous integral equation. Further, by introducing the substitutions

$$\frac{2\delta a^3 k_2 \sqrt{\nu_1}}{\{k_2\sqrt{\nu_1} - k_1\sqrt{\nu_2} + \sqrt{\nu_3}(k_2 - k_1)\}} = \delta_0; \quad \zeta = \frac{r}{a}; \quad \xi^2 A_1(\xi) = P(\xi) \tag{33}$$

the system (32) can be reduced to the following pair of dual integral equations

$$\begin{aligned} &\mathcal{H}_0\{\xi^{-1}P(\xi); \zeta\} = \delta_0; \qquad 0 \le \zeta \le 1 \\ &\mathcal{H}_0\{P(\xi); \zeta\} = 0 \qquad 1 \le \zeta < \infty. \end{aligned} \tag{34}$$

The general solution of this system of dual integral equations is given by Sneddon[10, 11]. Using these general results we have

$$P(\xi) = \frac{2\delta_0 \sin \xi}{\pi \xi} = \xi^2 A_1(\xi). \tag{35}$$

Again, the formal integral expressions for the displacement and stress fields can be determined from the preceding results. To determine the in-plane translational stiffness of the rigid circular inclusion embedded in a transversely isotropic elastic medium it is necessary to determine the resultant traction ($T_x(r, \theta, z)$) in the x-direction, acting at the bonded interfaces $z = 0^+$ and $z = 0^-$. Also from the symmetry of the problem about the plane $z = 0$ we have $T_x(r, \theta, 0^+) = T_x(r, \theta, 0^-)$. Evaluating the traction vector, we have

$$T_x(r, \theta, 0^+) = -\frac{4c_{44}\delta(k_2 - k_1)}{\pi a\{k_2\sqrt{\nu_1} - k_1\sqrt{\nu_2} + \sqrt{\nu_3}(k_2 - k_1)\}} \int_0^\infty \sin \xi J_0(\xi r/a)\, \mathrm{d}\xi. \tag{36}$$

Evaluating the integral expression in (36) we obtain

$$T_x(r, \theta, 0^+) = \begin{cases} -\dfrac{4c_{44}\delta(k_2 - k_1)}{\pi\{k_2\sqrt{\nu_1} - k_1\sqrt{\nu_2} + \sqrt{\nu_3}(k_2 - k_1)\}\sqrt{(a^2 - r^2)}}; & 0 \le r < a \\ 0 & ; \quad a < r < \infty. \end{cases} \tag{37}$$

The total force in the x-direction (P_0) exerted by the transversely isotropic elastic medium on the displaced inclusion is given by

$$P_0 = \int_0^a \int_{-\pi}^{\pi} [T_x(r, \theta, 0^+) + T_x(r, \theta, 0^-)] r \, \mathrm{d}r \, \mathrm{d}\theta. \tag{38}$$

Evaluating (38) we have

$$P_0 = -\frac{16c_{44}\delta a(k_2 - k_1)}{\{k_2\sqrt{\nu_1} - k_1\sqrt{\nu_2} + \sqrt{\nu_3}(k_2 - k_1)\}}. \tag{39}$$

In the limit as $\nu_i \to 1$ ($i = 1, 2, 3$) we obtain from (39) the in-plane translational stiffness of a disc inclusion embedded in bonded contact with an isotropic elastic medium, i.e.

$$P_0 = \frac{-64\mu\delta a(1 - \nu)}{(7 - 8\nu)}. \tag{40}$$

This result is in agreement with the expression for the translational stiffness that may be derived with the aid of the results given by Keer[2].

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