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The Boussinesq–Mindlin problem for a non-homogeneous elastic halfspace

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Dedicated to Professor J. M. Hill on the occasion of his 70th Birthday.

Abstract. Boussinesq's problem for the indentation of an isotropic, homogeneous elastic halfspace by a rigid circular punch constitutes a seminal problem in the theory of contact mechanics as does Mindlin's problem for the action of a concentrated force at the interior of an isotropic, homogeneous elastic halfspace. The combined action of the surface indentation in the presence of the interior loading is referred to as the Boussinesq–Mindlin problem, which has important applications in the area of geomechanics. The Boussinesq–Mindlin problem, which represents a self-stressing loading configuration, serves as a useful model for interpreting the mechanics of indentation of geologic media for purposes of estimating their bulk elasticity properties. In this paper, the analysis of the problem is extended to include an exponential variation in the linear elastic shear modulus of the halfspace region.

Mathematics Subject Classification. 74B05 · 74E05.

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1. Introduction

The problem of the indentation of an isotropic elastic halfspace by a rigid circular indenter with a frictionless flat base is a seminal problem in contact mechanics that has applications in a variety of areas in the engineering sciences ranging from materials engineering to the earth sciences. The solution to the problem was obtained by Boussinesq [1] using results of potential theory, and subsequently it was solved by Harding and Sneddon [2] using the theory of dual integral equations. Since these developments, the area of contact mechanics has been extensively and exhaustively examined by researchers to the point that an extensive list of articles cannot be documented within the context of a regular article. Some of the landmark studies related to elastostatic contact mechanics in general are due to Hertz [3], Shtaerman [4], Galin [5], Ufliand [6] and Lur'e [7], and references to studies in contact mechanics are also given by Korenev [8], Ling [9], Goodman [10], Selvadurai [11–14], Gladwell [15], Johnson [16], Curnier [17] and Selvadurai and Atluri [18]. The attraction of the solution rests on the fact that the load–displacement relationship for the rigid circular indenter can be obtained in exact closed form. This provides a basis for the estimation of deformability characteristics of the elastic material. The facility provided by the analytical solution has unfortunately been the *object of abuse* in material science literature through indiscriminate applications. For example, consider the application of micro-indentation tests advocated in recent literature as a technique for estimating the elasticity properties of metallic and geological materials. At the scale of such a test configuration, the indenter can interact with a polycrystalline fabric material and could interact with either a single crystal or multiple crystals or boundaries of crystals. The use of the classical results for indentation mechanics based on isotropic elasticity to estimate the

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deformability characteristics will lead to incorrect interpretations of the bulk elasticity properties of the polycrystalline material.

The geomechanical properties of rocks in particular can be determined through testing of core samples recovered from boreholes or from in situ field tests. The former approach is convenient but provides only a point estimate of the properties, and extensive testing is necessary to develop a spatial distribution of the properties relevant to a representative volume element of the rock mass. Field testing directly provides an estimate of the geomechanical properties applicable to a representative volume element of the geological medium, which can be readily used in engineering calculations. The results for indentation tests have been successfully used in geomechanics, particularly to estimate the deformability characteristics of rock masses when sample recovery for laboratory testing is not feasible [19–22]. In order to conduct plate loading tests on geological media, substantially large loads need to be applied and these loads are normally provided through reaction against a static weight [23]. In situations where the plate load tests are carried out in tunnels or adits, the test loads can be provided through reaction against the tunnel wall [24]. Alternative arrangements for the application of test loads include reaction piles, which require cumbersome testing arrangements that are non-routine [25]. The *Cable Jacking Test* was developed by Zienkiewicz and Stagg [26] in an attempt to minimize the complex test load application methods associated with the previous test arrangements. The method involves the installation of a rock anchor remote from the test plate. The axis of the test plate can be aligned in the orientation of the anchor to provide an axisymmetric anchoring load. The implicit assumption in the *Cable Jacking Test* is that the anchor should be located remote from test plate in order to minimize the influence of the reaction load on the displacements of the tests plate. This aspect, however, was not examined in detail in the study by Zienkiewicz and Stagg [26]. A theoretical estimate of the influence of a concentrated anchor load (Fig. 1) on the deflection of the test plate was first provided by Selvadurai [27]. It was shown that the concentrated anchor load should be located at a depth of at least ten diameters of the test plate in order to minimize its influence on the test plate deflection by 5 %. The analysis of the Cable Jacking Test has been extended to include distributed anchor loads [28], flexibility of the test plate [29,30], transverse isotropy of the rock mass [31] and creep effects of the rock mass [32]. The mathematical analysis of the Cable Jacking Test developed by Selvadurai [27] makes use of integral equation techniques for the analysis of contact problems as do the highlighted volumes on contact mechanics cited previously (and elegantly summarized in the article by Sneddon [33]) and employs Mindlin's solution for the axisymmetric problem related to the action of a concentrated axial force within a halfspace region void of surface tractions. It has been shown [34–39] that results related to interaction between indenters, inclusions and externally placed localized loads can also be obtained quite conveniently by making use of Betti's reciprocal theorem.

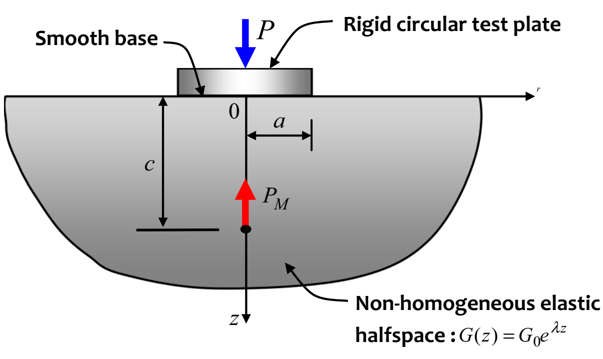

Fig. 1. Schematic view of the "Cable Jacking Test"

In this paper, we examine the problem of the Cable Jacking Test related to an isotropic non-homogeneous elastic halfspace region where the linear elastic shear modulus varies exponentially with depth. The class of contact problems related to non-homogeneous elastic media has been examined quite extensively in the literature, and comprehensive accounts of developments in this area are given in [11,14,15]. The axisymmetric contact problems related to the indentation of an elastic halfspace region where the linear elastic shear modulus has an exponential variation in the axial direction were examined by Korenev [8], Mossakovskii [40], Kassir and Chuapresert [41] and Muravskii [42]. The studies by Gibson [43], Awojobi and Gibson [44] and Rajapakse and Selvadurai [45] examine problems where the linear elastic shear modulus of an incompressible elastic material varies linearly with depth. The exponential and linear variations implemented in these studies lead to unbounded values of the shear modulus as the axial coordinate $z \to \infty$. To overcome this limitation, Selvadurai et al. [46] proposed an exponential variation of the linear elastic shear modulus that has a bounded variation within the halfspace region and examined the Reissner–Sagoci problem, which involves the axisymmetric torsional indentation of a halfspace region by a rigid circular disc. Other indentation problems involving bounded variations in the linear elastic shear modulus are given in [47–50]. The exponential variation in the linear elastic shear modulus of an *incompressible* elastic halfspace has recently been used to examine boundary value problems related to surface loading, internal loading and adhesive indentation by a rigid circular punch. In this paper, we examine the Cable Jacking Problem related to the interaction of an indenting rigid circular punch with a smooth base and an internally placed concentrated Mindlin force (Fig. 1). The resulting integral equation, however, cannot be solved with conventional analytical procedures that have been used to examine contact problems. In the current study, a discretization procedure [11,14,15,51–54] is used to develop an approximate solution. The results obtained using this technique are compared with results developed using a computational approach.

2. Governing equations

We restrict attention to axisymmetric problems in the theory of linear elasticity, referred to the cylindrical polar coordinate system (r, θ, z);the displacement vector is defined by

$$\mathbf{u} = \{u_r(r,z), 0, u_z(r,z)\} \tag{1}$$

and the strain tensor $\boldsymbol{\varepsilon}$ is defined by

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \\ 0 & \frac{u_r}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) & 0 & \frac{\partial u_z}{\partial z} \end{pmatrix} \tag{2}$$

The corresponding nonzero components of the Cauchy stress tensor $\boldsymbol{\sigma}$ are

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{rz} & 0 & \sigma_{zz} \end{pmatrix} \tag{3}$$

Considering the type of elastic non-homogeneity, which is restricted to a depth-dependent variation of shear modulus $G(z)$ and a constant Poisson's ratio ν, Hooke's law can be written as:

$$\boldsymbol{\sigma} = 2G(z)[\alpha e\mathbf{I} + \boldsymbol{\varepsilon}] \tag{4}$$

where $\mathbf{I}$ is the unit matrix and

$$e = \mathrm{tr}(\boldsymbol{\varepsilon}); \quad \alpha = \frac{\nu}{(1-2\nu)} \tag{5}$$

In the absence of body forces, the axial symmetric forms of the equations of equilibrium reduce to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{6}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{7}$$

These can be expressed in terms of the displacements as follows:

$$\nabla^2 u_r + (1+2\alpha)\frac{\partial e}{\partial r} - \frac{u_r}{r^2} + g(z)\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = 0 \tag{8}$$

$$\nabla^2 u_z + (1+2\alpha)\frac{\partial e}{\partial z} + 2g(z)\left(\alpha e + \frac{\partial u_z}{\partial z}\right) = 0 \tag{9}$$

where ∇^2 is the axisymmetric form of Laplace's operator given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{10}$$

with

$$g(z) = \frac{1}{G}\frac{\partial G}{\partial z} \tag{11}$$

Attention is restricted to inhomogeneous elastic materials where the linear elastic shear modulus varies as

$$G(r,z) = G_0 e^{\lambda z}; \quad r \in (0,\infty); \quad z \in (0,\infty) \tag{12}$$

and

$$\nu(r,z) = \nu = \text{Constant}; \quad r \in (0,\infty); \quad z \in (0,\infty) \tag{13}$$

where G_0 is the shear modulus at the surface of the halfspace and the non-dimensional parameter $\lambda \geq 0$ characterizes the depth-dependent variation. We also introduce a non-dimensional parameter $\bar{\lambda} = \lambda/a$, where a is a length parameter associated with the problem (e.g. the radius of the rigid indenter). The positive definiteness of the parameter λ is necessary to ensure that the shear modulus does not reduce to zero as $z \to \infty$.

The methods of solution of the equations governing problems in non-homogeneous elastic media are many and varied, and accounts of developments in this area are summarized by Selvadurai [11,14], Gladwell [15] and Aleynikov [54]. A particularly effective method for the solution of the exponential non-homogeneity problem applicable to a compressible elastic solid was proposed by Ter-Mkrtich'ian [55], which involves the introduction of an auxiliary function $\kappa(r,z)$ that is used to express the radial displacement as

$$u_r(r,z) = \frac{\partial \kappa}{\partial r} \tag{14}$$

Using this substitution in (8) and (9) and eliminating the axial displacement $u_z(r,z)$, we obtain the following equation for the function $\kappa(r,z)$; i.e.

$$\nabla^2\nabla^2\kappa - \alpha\nabla^2\kappa + 2\lambda\frac{\partial}{\partial z}\nabla^2\kappa + \beta\frac{\partial^2\kappa}{\partial z^2} = 0 \tag{15}$$

where ∇^2 is the axisymmetric form of Laplace's operator given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{16}$$

and

$$\alpha = \frac{\lambda^2\nu}{1-\nu}, \quad \beta = \frac{\lambda^2}{1-\nu} \tag{17}$$

We note that an identical formulation of the exponential inhomogeneity has been presented in the literature [56], but unfortunately, without a proper acknowledgement of the original contribution of Ter-Mkrtich'ian [55].

In the case of a homogeneous elastic material $\lambda \equiv 0$, and (15) reduces to the biharmonic equation for $\kappa(r,z)$; this function can be identified as Love's strain potential [57,58]. Following [55], the solution of (15) applicable to a halfspace region can be written as

$$\kappa(r,z) = \int_0^\infty \mathrm{e}^{-tz}[F_1(t)\cos pz + F_2(t)\sin pz]J_0(qr)\mathrm{d}t \tag{18}$$

where $F_1(t)$ and $F_2(t)$ are arbitrary functions and the parameters p and q can be defined as

$$q = (2t+\lambda)\sqrt{\frac{t(t+\lambda)}{\alpha+(2t+\lambda)^2}}, \quad p = \sqrt{\frac{\alpha t(t+\lambda)}{\alpha+(2t+\lambda)^2}} \tag{19}$$

By substituting Eq. (13) into Eq. (10) and using the equilibrium equations, both components of the displacement vector can be expressed as ($z \geq 0$)

$$u_r(r,z) = -\int_0^\infty \mathrm{e}^{-tz}q[F_1(t)\cos pz + F_2(t)\sin pz]J_1(qr)\mathrm{d}t \tag{20}$$

and

$$u_z(r,z) = \int_0^\infty \mathrm{e}^{-tz}[\psi_1(t)\cos pz + \psi_2(t)\sin pz]J_0(qr)\mathrm{d}t \tag{21}$$

where

$$\psi_1(t) = \frac{1}{p^2+\eta^2}[\eta\phi_1(r,t) + p\phi_2(r,t)] \tag{22}$$

$$\psi_2(t) = \frac{1}{p^2+\eta^2}[\eta\phi_2(r,t) + p\phi_1(r,t)] \tag{23}$$

$$\phi_1(r,t) = \left[q^2F_1(t) - (1-2\nu)\sqrt{\alpha}qF_2(t)\right]J_0(qr) \tag{24}$$

$$\phi_2(r,t) = \left[q^2F_2(t) - (1-2\nu)\sqrt{\alpha}qF_1(t)\right]J_0(qr) \tag{25}$$

and

$$\eta = t + \lambda(1-2\nu) \tag{26}$$

This formally completes the formulation of the equations governing the elastostatic boundary value problem related to the compressible elastic medium with an exponential variation in the linear elastic shear modulus.

3. The normal indentation of the non-homogeneous elastic halfspace

The focus of the paper is on the analysis of the axisymmetric indentation of the non-homogeneous elastic halfspace by a smooth rigid indenter of radius a. The problem can be formulated as an integral equation governing the unknown normal contact stress distribution $q(r,\theta)$ complemented by the equation of equilibrium for the indenter. Unlike in the case of the indentation of a homogeneous elastic medium, where the solution can be found in exact closed form, in the case of the indentation of a non-homogeneous elastic halfspace, the integral equation cannot be solved in a convenient way. The solution to the axisymmetric normal loading over a circular region of the non-homogeneous elastic halfspace with zero shear tractions was presented by Ter-Mkrtich'ian [55]. This solution can also be used to develop, as a limiting case, the equivalent Boussinesq solution for the halfspace. Alternative developments are presented in [56]. Consider the problem of the loading of the non-homogeneous elastic halfspace by a concentrated normal load P.

The analytical solution for the axial surface displacement resulting from the action of the concentrated force can be obtained in the form [56]

$$u_z(r) = \frac{P}{4\pi G_0} F(r) \tag{27}$$

where

$$F(r) = \int_0^\infty q \frac{\chi_1}{\chi_2} J_0(qr) \mathrm{d}q \tag{28}$$

in which

$$\begin{aligned}
\chi_1 &= 2(1-\nu)(\gamma_1 + \lambda/2)q^2 - (1-\nu)(1-2\nu)\lambda q^2 + (1-2\nu)\nu\lambda \left[(\gamma_1+\lambda/2)^2 + \gamma_2^2\right] \\
\chi_2 &= \nu(1-\nu)\left[(\gamma_1+\lambda/2)^2+\gamma_2^2\right]^2 + \{(1-\nu)^2\left[3(\gamma_1+\lambda/2)^2 - \gamma_2^2\right] \\
&\quad +(2-\nu)\nu\left[(\gamma_1+\lambda/2)^2+\gamma_2^2\right] - 2(1-\nu)\lambda(\gamma_1+\lambda/2)\}q^2 - (1-\nu)(2-\nu)q^4 \\
\gamma_1^2 &= \frac{1}{8}\left[\sqrt{\left(\lambda^2+4q^2\right)^2 + (4\lambda\vartheta q)^2} + \left(\lambda^2+4q^2\right)\right] \\
\gamma_2^2 &= \frac{1}{2}\left[\sqrt{\left(\lambda^2+4q^2\right)^2 + (4\lambda\vartheta q)^2} - \left(\lambda^2+4q^2\right)\right] \\
\vartheta^2 &= \frac{\nu}{1-\nu}
\end{aligned} \tag{29}$$

The distribution of contact normal stress at the smooth interface at an arbitrary point located within the contact zone is denoted by $\sigma(r,\theta)$. Integrating the result for the concentrated normal force (27) over the contact region, we obtain the following integral equation for determining the contact stress $\sigma(r,\theta)$:

$$u_z(r,\theta) = \frac{1}{4\pi G_0}\int_0^{2\pi}\int_0^a \sigma(\rho,\varphi)F(R)\rho \mathrm{d}\rho \mathrm{d}\varphi = \varDelta = \text{const.} \tag{30}$$

where

$$R = \left[r^2 + \rho^2 - 2r\rho\cos(\theta-\varphi)\right]^{1/2} \tag{31}$$

The contact stresses should also satisfy the equation of equilibrium for the indenter: i.e.

$$\int_0^{2\pi}\int_0^a \sigma(r,\theta) r \mathrm{d}r \mathrm{d}\theta + P = 0 \tag{32}$$

where P is the external load applied to the indenter. The result (30) can be further reduced by imposing conditions of axial symmetry, although the resulting equations cannot be solved in a convenient way using techniques developed for the solution of integral equations [59,60]. The approach adopted in this paper is to use an approximate technique where the contact stress distribution is represented as a discrete set of uniform annular contact stress regions. This approach has been extensively applied to examine contact problems involving indenters and halfspace regions [52,53,61] and has recently been applied to examine bonded contact problems for a non-homogeneous elastic halfspace and a rigid circular indenter [62].

4. A discretization approach for the analysis of the contact problem

A fundamental result that is needed to perform a discretization analysis of the contact problem relates to the uniform normal loading of the non-homogeneous elastic halfspace by a normal loading of intensity p_0 and radius a.

The boundary conditions pertaining to this problem are as follows:

$$\sigma_{zz}(r,0) = \begin{cases} p_0; & 0 \le r < a \\ 0; & a < r < \infty \end{cases} \tag{33}$$

$$\sigma_{rz}(r,0) = 0; \quad 0 \le r < \infty \tag{34}$$

The functions $F_1(t)$ and $F_2(t)$ introduced in the expressions (20) and (21) can be determined from the above boundary conditions.

In addition to these boundary conditions, it is assumed that the displacements and stresses should satisfy the appropriate regularity conditions in the half space region as $(r, z) \to \infty$. Consistent with the regularity condition at infinity, the function $F_1(t)$ and $F_2(t)$ can be obtained as follows

$$F_1(t) = \frac{(1-2\nu)}{2G}\frac{p_0}{q}J_1(q)\frac{Q(t)}{S(t)-P(t)T(t)} \tag{35}$$

$$F_1(t) = P(t)F_1(t) \tag{36}$$

$$Q(t) = 2\sqrt{\frac{t(t+\lambda)}{\alpha+(2t+\lambda)^2}} + \frac{(2t+\lambda)^2(\alpha+\lambda^2)}{2\sqrt{(t^2+\lambda t)}\left[\alpha+(2t+\lambda^2)\right]^3} \tag{37}$$

$$P(t) = \frac{q^2\eta + qp\sqrt{\alpha}(1-2\nu) + t\left(\eta^2+p^2\right)}{q^2p - q\sqrt{\alpha}\eta(1-2\nu) + p\left(\eta^2+p^2\right)} \tag{38}$$

$$\begin{aligned} S(t) = \frac{1}{\eta^2+p^2}&\left[-\nu q^2\left(\eta^2+p^2\right) + (1-\nu)tq^2\eta + (1-\nu)(1-2\nu)tpq\sqrt{\alpha}\right.\\ &\left.+(1-\nu)q^2p^2 + (1-\nu)(1-2\nu)qp\sqrt{\alpha}\eta\right] \end{aligned} \tag{39}$$

$$\begin{aligned} T(t) = \frac{1}{\eta^2+p^2}&\left[2q^2pt(1-\nu) - qt\sqrt{\alpha}\eta(1-\nu)(1-2\nu) + (1-\nu)(1-2\nu)q^2p\lambda\right.\\ &\left.-(1-\nu)(1-2\nu)qp^2\sqrt{\alpha}\right] \end{aligned} \tag{40}$$

First we consider the indentation of a non-homogeneous compressible elastic halfspace by a rigid circular indenter of radius a with smooth contact subjected to an axial load P. We assume that the contact stresses at the smooth/frictionless interface are compressive, and there is no separation at the contact region (i.e. smooth bilateral contact). The method of solution assumes that the contact region of the indenter can be discretized into annular regions of equal area and that the contact stress within each annular area is uniform (Fig. 2).

Considering the annular regions of uniform normal stress intensity $\sigma_1, \sigma_2, \ldots, \sigma_n$ acting within the annular region of internal radii $0, r_1, r_2, \ldots, r_{n-1}$ and external radii $r_1, r_2, \ldots, r_n$, respectively, the surface displacement $w_1, w_2, w_3, \ldots, w_n$ at the mid-radii position of the annular regions due to normal surface

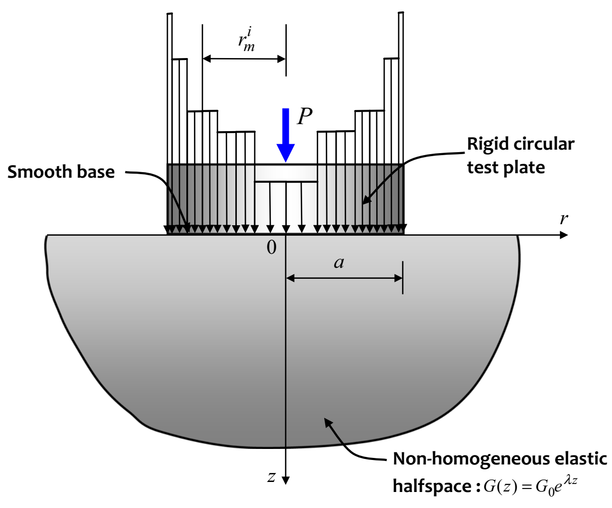

Fig. 2. Representation of the normal contact stress distribution as equi-areal annular regions of uniform stress

tractions can be obtained by superposition of the solutions obtained using (21). The compatibility between the displacements of the non-homogenous compressible elastic halfspace and settlement of the indenter, $\varDelta$, is then established at the mid-point location of each annular region. The physical domain of interest is taken to be a non-homogeneous compressible elastic halfspace in which the shear modulus has an exponential variation over the entire depth of the halfspace (12).

In order to assign equal annular areas, the dimensions of r_i take the following forms:

$$r_i = \left(\frac{i}{n}\right)^{\frac{1}{2}} a; \quad i = 1, 2, 3, \ldots, n \tag{41}$$

Similarly for the mid-locations of the annular regions,

$$r_{m1} = 0, \quad r_{mi} = \left(\frac{r_{i-1} + r_i}{2}\right); \quad i = 2, 3, \ldots, n \tag{42}$$

We further assume that the uniform normal stress elements σ_i can be represented as multiples of the average contact stress p_0 exerted by the rigid indenter; i.e.

$$\sigma_i = \tilde{\sigma}_i p_0 \quad \text{where } i = 1, 2, 3, \ldots, n \tag{43}$$

where

$$p_0 = \frac{P}{\pi a^2} \tag{44}$$

Using the above developments, it is possible to express the surface displacements w_i due to normal contact stresses $\tilde{\sigma}_i$ in the form of the matrix relation

$$\{\mathbf{w}\} = \frac{p_0 a}{G_0}[\mathbf{C}]\{\tilde{\boldsymbol{\sigma}}\} \tag{45}$$

where $\{\mathbf{w}\}$ and $\{\tilde{\boldsymbol{\sigma}}\}$ are column vectors and the coefficients of the square matrix $[\mathbf{C}]$ are as follows:

$$[\mathbf{C}] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \cdots & w_{1j} \\ w_{21} & w_{22} & w_{23} & \cdots & w_{2j} \\ w_{31} & w_{32} & w_{33} & \cdots & w_{3j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & w_{i2} & \cdots & w_{ij} \end{bmatrix} \tag{46}$$

where w_{ij} is the surface displacement at the ith annulus mid-point location due to all the normal contact stresses σ_j.

We shall now focus attention on the rigid indenter problem. For compatibility at the soil–indenter interface, w_i will have the same value as the rigid displacement of the indenter $\varDelta$; this constitutes an additional unknown of the problem. The remaining equation required for the solution of Eq. (45) is furnished by the equilibrium equation for the entire indenter, i.e.

$$\sum_{i=1}^{n} \tilde{\sigma}_i = n \tag{47}$$

The matrix formed by combining Eqs. (45) and (47) can be inverted to determine the non-dimensional contact stresses $\tilde{\sigma}_i$ and the non-dimensional indenter displacement

$$\varDelta^* = \frac{G_0 \varDelta}{p_0 a} \tag{48}$$

In this paper, the rigid indenter is discretized into 15 equal annular areas (i.e. $n = 15$). The accuracy of solution depends on the number of annular regions. However, the number of annular regions cannot be increased indefinitely; such a procedure usually results in an ill-conditioned set of equations. These difficulties stem from the singular behaviour of the contact stress at the indenter boundary (see e.g. [11,62])

5. Interaction of the rigid test plate and the Mindlin force

As indicated previously, the action of the concentrated Mindlin force P_{M}, shown in Fig. 1, will induce an additional displacement of the rigid test plate. The induced rigid displacement at the indenter can be incorporated into the calculations shown in Sect. 4. This, however, entails developing a solution to the problem of a Mindlin force that acts at a point at the interior of the halfspace region with an exponential non-homogeneity in the linear elastic modulus. This analysis is not trivial and requires the study of the interaction between a non-homogeneous elastic layer and a non-homogeneous elastic halfspace region with exponential variations in the non-homogeneity in the shear modulus. This problem was solved by Selvadurai and Katebi [63] for the case of an incompressible elastic halfspace with an exponential variation in the shear modulus. (Also, in the case of the isotropic homogeneous problem, Mindlin's solution for the surface displacement of the halfspace can be obtained in a relatively simple form.) We assume that, due to the combined action of the indentation load P and the internal Mindlin force, the contact stresses remain compressive, and there is no loss of contact at the unilateral interface. For the application of Betti's reciprocal theorem, we directly make use of the discretized equi-areal annular uniform region representation of the contact normal stress determined in Sect. 4 and evaluate the axial displacement $\varDelta_{\mathrm{M}}^{P}$ at the point of application of the Mindlin force P_{M} due to the contact stresses σ_i induced by P. The uniform displacement (unknown) of the rigid test plate due to the Mindlin load applied at the interior of the non-homogeneous elastic halfspace is denoted by $\varDelta_{P}^{P_{\mathrm{M}}}$. By applying the Betti reciprocal theorem, we have:

$$\varDelta_{P_{\mathrm{M}}}^{P} P_{\mathrm{M}} = \varDelta_{P}^{P_{\mathrm{M}}} P \tag{49}$$

Therefore, the net displacement $\tilde{\varDelta}$ (vertical displacement at $z = 0, r = 0$ due to the combined action of the forces P and P_{M}) can be written as:

$$\tilde{\varDelta} = \varDelta - \frac{P_{\mathrm{M}}}{P} \varDelta_{P_{\mathrm{M}}}^{P} \tag{50}$$

where $\varDelta$ is defined by Eq. (48).

6. Numerical results

The procedure outlined in first section leads to explicit infinite integral expressions for the displacement and stress fields within the non-homogeneous elastic half-space under a circular load (Eqs. 20, 21). The integrands of these integrals cannot be expressed in explicit forms. Consequently, results of interest for practical applications can only be developed through a numerical integration of the infinite integrals. Due to the oscillatory and singular nature of these integrands, an adaptive numerical procedure is used to enhance the accuracy of the numerical results; examples of such an application are given by Katebi et al. [64] and Katebi and Selvadurai [65]. For numerical evaluation of the integrals, the upper limit of integration is replaced by a finite value ξ_0; this limit is increased until a convergent result is obtained.

Figure 3 shows the normalized displacement of a contact problem (without the internal Mindlin Force) over the classic Boussinesq [1] results for different $\tilde{\lambda}$ (which is indicative of the depth-dependent variation in the shear modulus) and different Poisson's ratio. As expected, the displacement decreases as the shear modulus of elasticity increases. The influence of Poisson's ratio is strong, and it increases as the rigidity of the medium increases.

Figures 4, 5 and 6 show the relative displacement of the rigid disc indenter due to the combined action of the external force P and Mindlin force P_{M} for different locations of the Mindlin force. The results show the influence of the non-homogeneity parameter ($\tilde{\lambda}$) on the relative displacement of the rigid disc for different values of Poisson's ratio (ν). It can be seen from the figures that the effect of the internal Mindlin force decreases as the depth of the loading increases. It is also evident that the presence of an

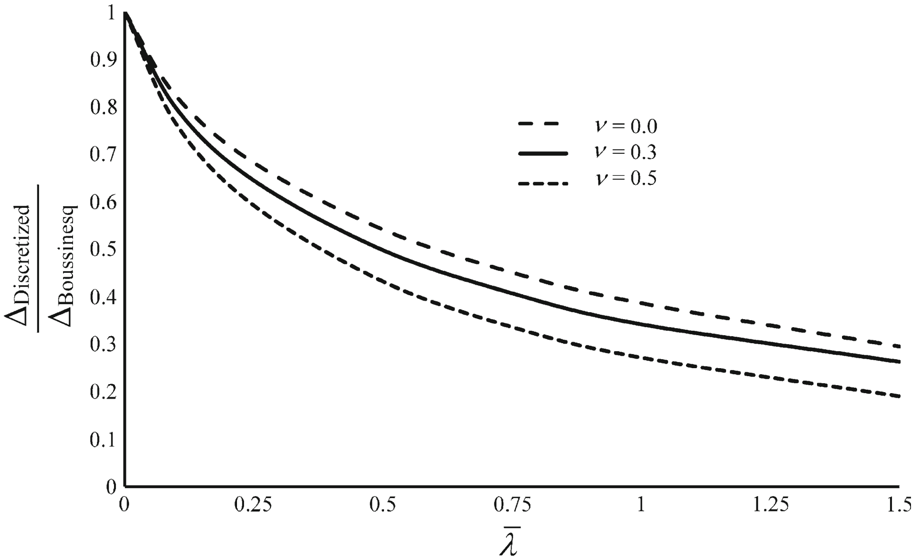

Fig. 3. Ratio of displacement of the rigid disc on a non-homogeneous medium (numerical discretized solution) to homogeneous medium [1] for different $\tilde{\lambda}$; comparison between different Poisson's ratio

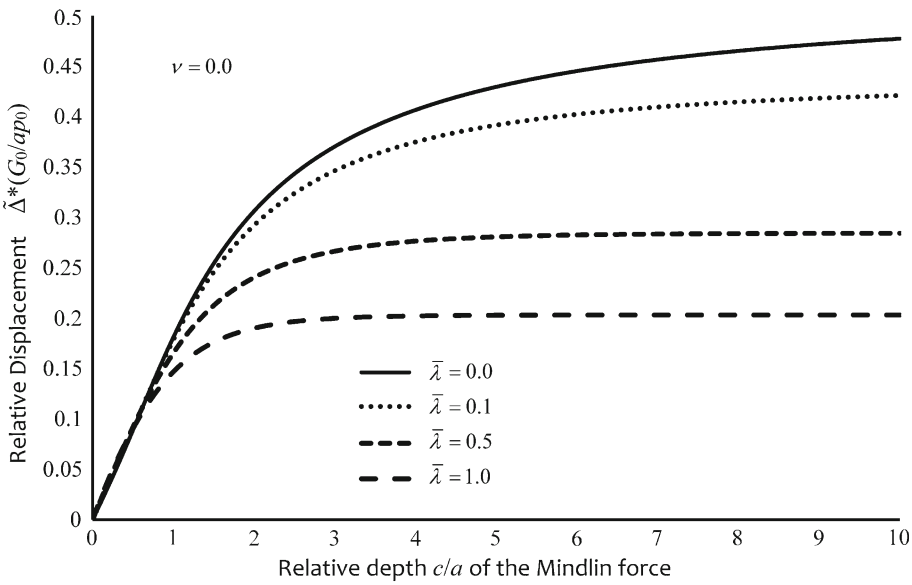

Fig. 4. Relative displacement of the rigid circular indenter due to the combined action of the external force P and Mindlin force $P_{\mathrm{M}}(P_{\mathrm{M}} = P)$ for different locations of the Mindlin force, examining the effect of shear modulus for the case where $\nu = 0.0$

elastic non-homogeneity condition has a significant effect on the relative displacement of the rigid disc indenter.

Figure 7 shows the relative displacement of the rigid disc indenter due to the combined action of the external force P and Mindlin force P_{M} for different locations of the Mindlin force. The results show the

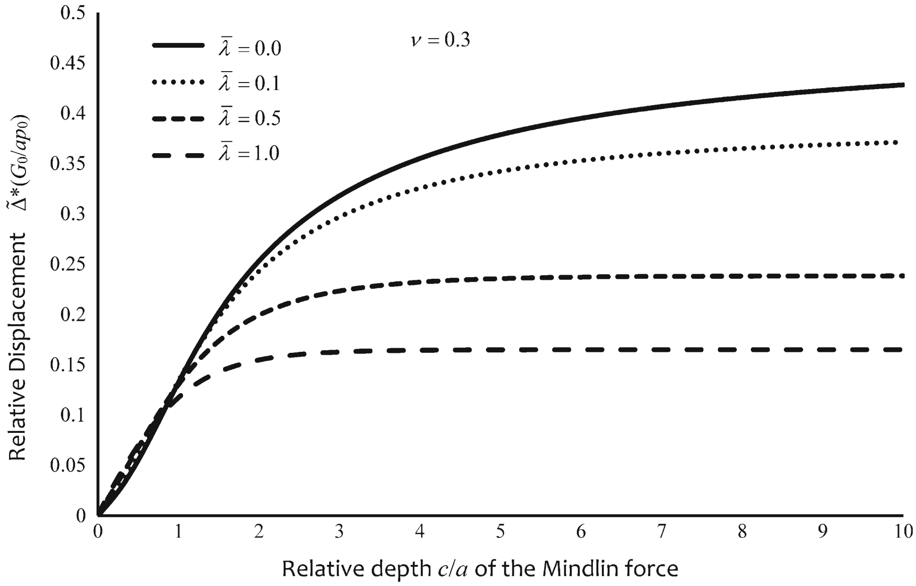

FIG. 5. Relative displacement of the rigid circular indenter due to the combined action of the external force P and Mindlin force $P_M(P_M = P)$ for different locations of the Mindlin force, examining the effect of shear modulus for the case where $\nu = 0.3$

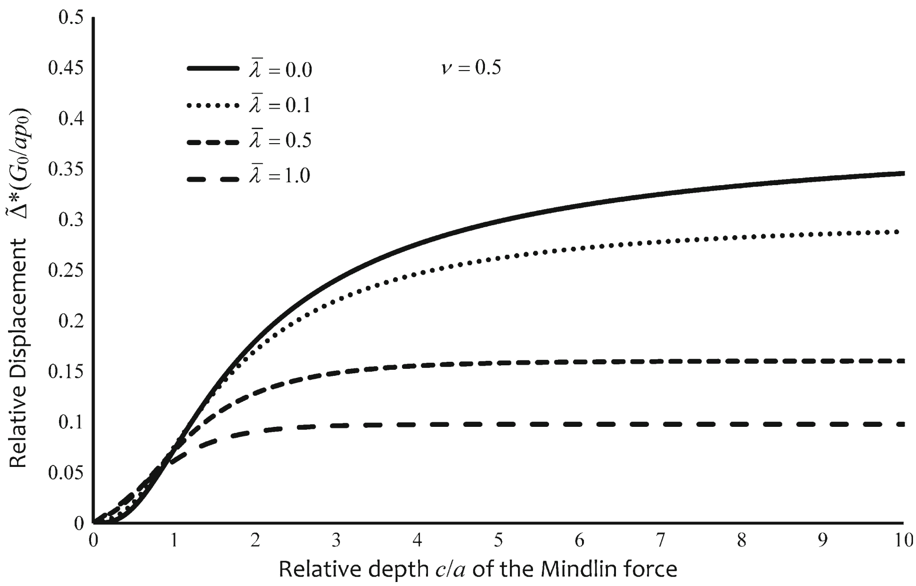

FIG. 6. Relative displacement of the rigid circular indenter due to the combined action of the external force P and Mindlin force $P_M(P_M = P)$ for different locations of the Mindlin force, examining the effect of shear modulus for the case where $\nu = 0.5$

influence of the Poisson's ratio parameter (ν) on the relative displacement of the rigid disc for shear modulus parameter ($\bar{\lambda} = 0.1$). It can be seen from the figure that an increase in Poisson's ratio has a significant effect on the relative displacement of the rigid disc.

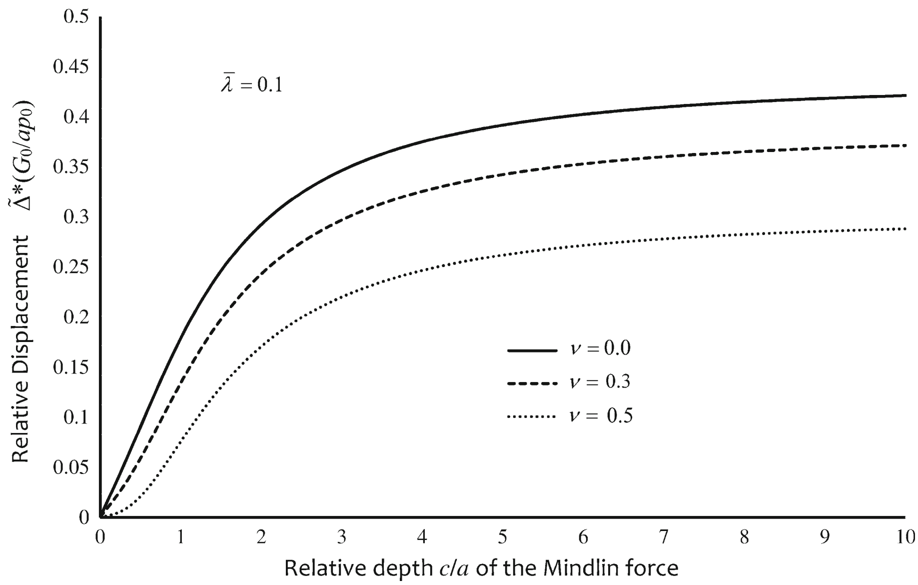

FIG. 7. Relative displacement of the rigid circular indenter due to the combined action of the external force P and Mindlin force $P_{\mathrm{M}}(P_{\mathrm{M}} = P)$ for different locations of the Mindlin force, examining the effect of Poisson's ratio for the case where $\bar{\lambda} = 0.1$

7. Concluding remarks

The present work provides the solutions for the case of a Cable Jacking Test related to an isotropic non-homogeneous elastic halfspace region where the linear elastic shear modulus varies exponentially with distance normal to the indented surface. The resulting integral equations, however, cannot be solved using the conventional analytical procedures that have been used to examine contact problems. In this paper, a discretization procedure [11,53,61,62] is used to develop an approximate solution. The results developed using this technique are presented to examine the effect of the Poisson's ratio, variation of shear modulus with depth and the depth of location of the internal Mindlin-type point load. It should be noted that the basic objective of the *Cable Jacking Test* is to interpret the geomechanical properties of the tested rock mass through an interpretation of the load-displacement relationship for the test plate. Even for the situation where the tests are carried out on homogeneous geologic media, the test stiffness provides only information on the combination of the elasticity parameters as identified by the linear elastic shear modulus and Poisson's ratio. In the case of an inhomogeneous geological formation, the specific depth variation of the elastic inhomogeneity can only be assessed through additional test data involving depth-dependent plate load tests. The results for Cable Jacking Tests conducted on plates located on the surface of a rock mass provides estimates for the bulk deformability parameters that takes into account the influence of both the depth-dependent elastic inhomogeneity and Poisson's ratio.

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