Dynamic response of an infinite beam supported by a saturated poroelastic halfspace and subjected to a concentrated load moving at a constant velocity

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The steady-state displacements and moments in a Bernoulli–Euler beam of finite width and infinite extent, resting on a poroelastic halfspace and subjected to a concentrated load moving at a constant velocity, were investigated using the concept of the equivalent stiffness of the halfspace. Expressions for the equivalent stiffness of the saturated poroelastic halfspace interacting with the infinite beam of finite width were derived analytically using a contour integration procedure. The influence of adhesion and drainage effects between the beam and the halfspace surface is accounted for by considering “bounding techniques” for prescribing the boundary conditions at the interface. Comparisons have been made between situations for the elastic and poroelastic halfspace with regard to their equivalent stiffness and the dynamic responses of the beam for different velocities of the moving load.

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1. Introduction

The analysis of the dynamic interaction between an elastic beam and ground is fundamental to the understanding of the dynamic behavior of railway tracks with the supporting subsoil under the action of high speed trains. This topic has attracted significant research effort during the past decades because of the ever-growing public concern over the noise and vibration pollution. Literature in rail track-ground dynamics can be divided into two categories: elastic soils and saturated poroelastic soils, according to whether the pore fluid is considered in the soil model. Filippov (1961) pioneered the research on vibrations of an infinite beam resting on an elastic halfspace subjected to a moving point load. Later, Labra (1975) investigated the effect of the axial compressive force on the critical velocity of the beam. By taking the contribution of each sleeper of the track into account, Krylov (1995, 1996) investigated the ground vibrations generated by high speed trains. Kaynia et al. (2000) developed a numerical model to predict the vibrations induced in the railway embankment and the layered elastic ground by high speed trains. Sheng et al. (1999a,b, 2003, 2004a,b) have also conducted a series of theoretical investigations on the coupled vibrations of a layered elastic ground and the rail track, which is modeled as a layered beam structure, using the analytical solutions of the wave equations. Using the same approach, more specialized models have been proposed and solved by Takemiya and Bian (2005) to include the sleeper passing effect, by Karlstrom and Bostrom (2006) to consider the rectangular embankment and by Xia et al. (2010) to investigate the contribution of the vehicle components. Numerical techniques, such as the finite element method (Hall, 2003), the boundary element method (Galvin and Dominguez, 2007; Celebi and Schmid, 2005), or a combination of them (Auersch, 2005a,b), have also been frequently employed to obtain the elastic ground vibrations due to the passage of a high speed train. For more articles on this topic, one is referred to the comprehensive review by Lombaert et al. (2015).

When there is ground water present, the pores between the soil skeleton can be completely saturated with water. A quasi-static theory of poroelasticity was developed by Biot (1941) (see also Selvadurai, 1996, 2007; Schanz, 2009; Cheng, 2015) and further improved by himself (Biot, 1956) to take into consideration the dynamic effects of the soil-skeleton and pore-fluid phases. Using the dynamic theory of poroelasticity (Biot, 1956), Cai et al. (2007, 2008a,b, 2010) made significant contributions to the coupled vibrations of the railway track and the saturated poroelastic halfspace under the action of a high speed train. The dynamic response of saturated ground was found to be significantly different from that of the elastic ground when the train speed increased to greater than the critical velocity of the ground.

The theoretical/numerical investigations cited above and the field experiments (Madshus and Kaynia, 2000; Lombaert et al., 2006; Lombaert and Degrange, 2009) have revealed that large
dynamic amplifications appear in the track and ground as the train speed approaches an apparently critical value. This critical velocity of the track-ground system has been mathematically demonstrated by Dieterman and Metrikine (1996; 1997) using a highly-idealized model: the track structure is simplified as a Bernoulli–Euler beam with finite width and infinite extent, the ground is modeled as a homogeneous elastic halfspace and the train loading is represented by a concentrated load of constant magnitude and velocity. The equivalent stiffness of the elastic halfspace was first evaluated using the contour integration method and then substituted into the governing equations of the beam, whose responses were obtained by a numerical Fourier inversion. Two critical velocities were found to exist in this model: the first is equal to the Rayleigh wave velocity of the halfspace, at which the equivalent stiffness of the halfspace becomes zero; the second is slightly smaller, which is generated due to dynamic interaction between the beam and the halfspace. By following the same procedure, these results were extended by Kononov and Wolfert (2000) to take into account the viscous properties of the elastic halfspace. Due to the energy dissipation caused by the viscosity, the equivalent stiffness is complex at the Rayleigh wave velocity; in this case, only the second critical velocity exists.

A similar model has been employed by Jin (2004) and Xu et al. (2007). After replacing the elastic halfspace by a saturated poroelastic one, they studied the displacement responses of the beam under different load velocities. The equivalent stiffness of the poroelastic halfspace was evaluated using the numerical Fourier inversion procedure along the real axis of the wavenumber, under the assumption that no Rayleigh poles or branch points of the integrand are encountered. This assumption holds true when the soil permeability is low, since the high viscous coupling between the soil skeleton and the pore water renders the branch points and the Rayleigh pole complex-valued, and thus far away from the real axis of the wavenumber. However, when the permeability is high or in the extreme condition of infinite permeability, the Rayleigh pole and branch points would move closer to or be situated directly on the real axis of the wavenumber. In this case the summations involved in the numerical inversions may contain singularities, which would cause substantial oscillations to the resulting equivalent stiffness and thus make any evaluation of the beam response unreliable. Furthermore, the equivalent stiffness needs to be evaluated in a more rigorous fashion, for example, by using the contour integration method, so that the characteristics of the halfspace dynamics can be established mathematically.

In this study we present a mathematical formulation for the dynamic interaction problem of an infinite beam of finite width that is resting on a saturated poroelastic halfspace of infinite permeability and subjected to a concentrated load moving at a constant velocity. Four sets of boundary conditions, i.e., free draining-frictionless (Case A), impervious-frictionless (Case B), free draining-inextensible (Case C) and impervious-inextensible (Case D) boundary conditions, were prescribed over the entire surface of the poroelastic halfspace, respectively, to make the problem analytically tractable. Firstly, the equivalent stiffness of the poroelastic halfspace was derived for the four cases using the method of contour integration. Then, the equivalent stiffness is substituted into the equilibrium equation of the Bernoulli–Euler beam to obtain the displacement and moment responses using the numerical Fourier inversion. Detailed comparisons were performed for the four cases and between the elastic- and poroelastic-halfspace solutions with regard to the equivalent stiffness of the halfspace and the dynamic response of the beam for different load velocities. It is mathematically demonstrated that the first critical velocity, at which the equivalent stiffness vanishes, equals the Rayleigh wave velocity for Cases A and B, while it changes to the shear wave velocity for Cases C and D. The second critical velocity, which is due to the mechanical coupling of the beam and the halfspace, is slightly smaller than the corresponding first critical velocity in each case. The critical velocities of the beam-elastic halfspace system are found to be smaller than those of the beam-poroelastic halfspace system.

2. Problem formulation

Fig. 1 shows the analysis model that consists of an infinite beam of finite width (2a) resting on the surface of a homogeneous poroelastic halfspace of infinite permeability. A moving constant load of amplitude $F_0$ and velocity $c$ is applied to the center-line of the beam and acts vertical to the halfspace surface. The model is at rest initially and reaches a steady state when the load has been moving along the beam for a long time.

2.1. Governing equations

The beam experiences flexure only in the longitudinal direction and its flexural response is described by the Bernoulli–Euler beam theory

$$EI \frac{\partial^4 w_b}{\partial x^4} + m_b \frac{\partial^2 w_b}{\partial t^2} + \delta_b \frac{\partial w_b}{\partial t} + q_b(x,t) = F_0 \delta(x - ct)$$

(1)

where $w_b$ is the beam deflection; $EI$ is the bending rigidity of the beam section; $m_b$ is the mass of the beam per unit length; $\delta_b$ is the viscosity coefficient of the beam; $q_b$ is the unknown contact line force at the beam-halfspace interface acting along the center line of the beam, which has the dimension of $[M T^{-2}]$; $\delta(x - ct)$ is the Dirac delta function and $\delta(x - ct)$ has the dimension of $[T^{-1}]$. The bending moment $M_b$ of the beam is determined by $M_b = -EI (\partial^2 w_b/\partial x^2)$.

The dynamics of the saturated poroelastic halfspace governed by Biot’s theory (Biot, 1956) take the forms

$$\mu u_{i,j} + (\lambda + \alpha^2 M + \mu) u_{i,j} + \alpha M w_{j,i} = \rho u_{i} + \rho \dot{w}_i$$

(2)

$$\alpha M u_{j,i} + M w_{j,i} = \rho \ddot{u}_i + m \ddot{w}_i + b w_i$$

(3)

The constitutive equations are

$$\sigma_{ij} = \lambda \delta_{ij} \theta + \mu (u_{i,j} + u_{j,i}) - \alpha \delta_{ij} \rho$$

(4)

$$p = -\alpha M \theta + M \zeta$$

(5)

where $u_i$ and $w_i$ ($i = x, y, z$) are the soil skeleton displacement and the pore-fluid average displacement relative to the soil skeleton, respectively; the subscripts $i$, $j$ and $i$, $j$ denote that the tensor operation and the summation convention is applied; the dots over $u_i$ and $w_i$ indicate the difference with respect to time $\tau$; $\lambda$ and $\mu$ are the Lamé constants of the soil skeleton; $M$ and $\alpha$ are Biot’s parameter accounting for the compressibility of the two phases; they are
expressed as \( \alpha = 1 - K/K_0 \) and \( 1/M = n/K_f + (\alpha - n)/K_0 \) with \( K, K_f \) and \( K_0 \) representing the bulk moduli of the soil skeleton, the pore fluid and the soil grain, respectively; \( \rho = n\rho_f + (1 - n)\rho_g \), where \( \rho_f \) and \( \rho_g \) are the mass densities of the soil skeleton and pore fluid, respectively; \( m = a_o\rho_f/\rho \), where \( a_o \) is the tortuosity factor and \( n \) is the porosity; \( \theta = u_{ij} \) and \( \zeta = -w_{ij} \) are the soil skeleton volumetric strain and the fluid dilatation, respectively; \( \sigma_{ij} \) is the total stress component and \( p \) is the pore pressure.

Here we pay special attention to the parameter \( b = \rho g K_0 \), where \( K_0 \) is the Darcy permeability of the soil medium and \( g \) is the acceleration due to gravity, as it determines the viscous drag force applied to the soil skeleton by the pore fluid. It is the viscous drag force that makes the three body waves (i.e., P1, P2 and S waves) dispersive in the saturated poroelastic medium (Shi et al., 2012). Since it is assumed that the poroelastic halfspace has infinite permeability, we have \( K_0 \to \infty \) and thus \( b = 0 \). Under this circumstance, the viscous drag force vanishes and the body waves are not dispersive.

### 2.2. Boundary conditions

The boundary conditions applicable to the problem depend on the contact conditions at the beam-poroelastic halfspace interface. For convenience of presentation, the region of the surface of the halfspace in contact with the beam is denoted by \( \Gamma_c \) (i.e. \( x \in (-\infty, \infty); y \in (-a, a) \) and \( z = 0 \)) and the combined region of the halfspace exterior to \( \Gamma_c \) is denoted by \( \Gamma_e \) (i.e. \( \Gamma_e = \Gamma_{e1} \cup \Gamma_{e2} \), where in \( \Gamma_{e1}, x \in (-\infty, \infty); y \in (a, \infty); z = 0 \) and in \( \Gamma_{e2}, x \in (-\infty, \infty); y \in (-a, -\infty); z = 0 \)). Also we denote \( \Gamma_c \cup \Gamma_e = \Gamma \) and \( \Gamma_c \cap \Gamma_e = \emptyset \). For bonded contact between an impermeable beam and a poroelastic halfspace where the external region is allowed to drain freely, the following boundary conditions are applicable:

\[
\begin{align*}
\sigma_{zz}(x, y, 0, t) = \sigma_{yy}(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma_c \\
\sigma_{xy}(x, y, 0, t) &= \sigma_{yy}(x, y, 0, t) = 0; \quad (x, y) \in \Gamma_c \\
\frac{\partial p}{\partial z} &\bigg|_{(x,y,0,t)} = 0; \quad (x, y) \in \Gamma_c \\
p(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma_c
\end{align*}
\]

(6a) (6b) (6c) (6d)

where \( \mathbf{u}(x, t), \sigma(x, t) \) and \( p(x, t) \) are the displacement vector, the Cauchy stress tensor and the pore fluid pressure referred to the rectangular coordinate system with a position vector \( \mathbf{x} \).

As is evident, the adhesive contact problem for the infinite beam on a poroelastic halfspace defined by (Eq. 6) has to consider not only the mixed boundary conditions applicable to \( \mathbf{u}(x, t) \) and \( \sigma(x, t) \) but also a set of mixed boundary conditions applicable to the pore fluid pressure \( p(x, t) \). The mathematical treatment of the resulting boundary value problem must take into consideration the oscillatory stress singularities in the stress field and the singularities in the pore fluid pressure at the boundary between the regions where Neumann and Dirichlet boundary conditions are prescribed; this makes the analysis extremely complicated both in terms of the numerical solution of the resulting coupled Fredholm-type integral equations and the inversion of the integral transforms. An alternative to this approach is the bounding technique proposed by Selvadurai (1984a, 2000, 2003a, b), Selvadurai and Au (1986) and Selvadurai and Shi (2015), where either the mechanical or the kinematic constraints are prescribed over the entire halfspace surface \( \Gamma \) for the soil-skeleton and pore-fluid phases, respectively, in order to make the problem analytically tractable. This bounding technique results in four sets of boundary conditions, i.e., pervious-frictionless (Case A), impervious-frictionless (Case B), pervious-inextensible (Case C) and impervious-inextensible (Case D) boundary conditions. Further we assume that the vertical contact stress between the beam and the halfspace surface is uniformly distributed beneath the beam section and the displacement compatibility between the two is required only along the central line of the beam.

**Case A:** Considering the shear traction free and pervious boundary conditions prescribed over \( \Gamma \), the resulting boundary value problem is given by

\[
\begin{align*}
\sigma_{zz}(x, y, 0, t) &= -\frac{q(x, t)}{2a} H(a - |y|); \quad (x, y) \in \Gamma \\
p(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma
\end{align*}
\]

(7a) (7b) (7c) (7d)

**Case B:** Considering the shear traction free and impervious boundary conditions prescribed over \( \Gamma \), the boundary value problem is given by

\[
\begin{align*}
\sigma_{zz}(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma \\
p(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma
\end{align*}
\]

(8a) (8b) (8c) (8d)

**Case C:** Considering the inextensibility and pervious boundary conditions prescribed over \( \Gamma \), the boundary value problem is given by

\[
\begin{align*}
\sigma_{zz}(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma \\
p(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma
\end{align*}
\]

(9a) (9b) (9c) (9d)

**Case D:** Considering the inextensibility and impervious boundary conditions prescribed over \( \Gamma \), the resulting boundary value problem is given by

\[
\begin{align*}
\sigma_{zz}(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma \\
p(x, y, 0, t) &= 0; \quad (x, y) \in \Gamma
\end{align*}
\]

(10a) (10b) (10c) (10d)

The solution of the boundary value problem described by Cases A–D will provide four sets of solutions that can “bound” the solution to the boundary value problem defined by Eq. (6), which are the most appropriate for describing the poroelastic halfspace with a free-draining surface in joined motion with the infinite beam with an impermeable interface.
3. Analysis of the boundary value problem

A Fourier transform is used to solve the boundary value problems described by Cases A–D. For convenience, the beam half width a, the shear modulus μ and the mass density ρ are introduced to render all physical quantities non-dimensional as

\[ x' = \frac{x}{a}, \quad t' = \frac{t}{a \sqrt{\frac{\mu}{\rho}}}, \quad u' = \frac{u}{a}, \quad w' = \frac{w}{a}, \quad \sigma' = \frac{\sigma}{\mu} \]

\[ p' = \frac{p}{\mu}, \quad \chi' = \frac{\chi}{\mu}, \quad m' = \frac{m}{\rho}, \]

\[ M' = \frac{M}{\mu a^2}, \quad \rho_0' = \frac{\rho_0}{\rho}, \quad b' = \frac{ab}{\sqrt{\mu \rho}}, \quad w'_b = \frac{w_b}{a}, \]

\[ M'_b = \frac{M_b}{\mu a^2}, \quad \delta(x') = a\delta(x), \]

\[ E' = \frac{E \mu a^2}{\rho}, \quad m'_b = \frac{m_b}{\rho_0 a^2}, \quad \delta_b' = \frac{\delta_b}{a \sqrt{\mu \rho}}, \quad q'_c = \frac{q_c}{\mu a}, \]

\[ f'_0 = \frac{f_0}{\mu a^2}, \quad c' = \frac{c}{\sqrt{\mu / \rho}} \]

(11)

In the ensuing, all physical quantities are expressed in non-dimensional forms, thus the superscript asterisks will be omitted for brevity. The double Fourier transforms with respect to the horizontal coordinates x and y and the Fourier transform with respect to the time t are introduced as follows:

\[ \tilde{\varphi}(k_1, k_2, z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(x, y, z, t) e^{-i(k_1 x + k_2 y)} dx dy \]

\[ \psi(x, y, z, t) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\varphi}(k_1, k_2, z, t) e^{i(k_1 x + k_2 y)} dk_1 dk_2 \]

(12)

\[ \tilde{\psi}(k_1, k_2, z, t) = \int_{-\infty}^{\infty} \tilde{\varphi}(x, y, z, t) e^{i(k_1 x + k_2 y)} dk_1 dk_2 \]

\[ \tilde{\varphi}(x, y, z, t) = \int_{-\infty}^{\infty} \tilde{\psi}(x, y, z, t) e^{-i(k_1 x + k_2 y)} dk_1 dk_2 \]

(13)

where k_1 and k_2 are the wavenumbers along the x- and y-axis, respectively; and ω is the frequency.

3.1. General solutions in the Fourier domain

Using the Fourier transforms defined in Eqs. (12) and (13), the general solutions to Eqs. (2)–(5) of the saturated poroelastic medium can be derived. The details of the solution are given in the paper by Shi et al. (2012); only the final expressions that will be used in this study are presented here. The expressions for the soil-skeleton displacements in the Fourier domain are

\[ \tilde{u}_s(k_1, k_2, z, \omega) = -\frac{i}{k_1^2} \left[ \chi_1 + a_1 (k_1^2 - L_1^2) \right] A e^{-i\omega z} + \chi_2 + a_2 (k_1^2 - L_1^2) B e^{-i\omega z} + (\gamma C + k_2 D) e^{-i\omega z} \]

\[ \tilde{u}_p(k_1, k_2, z, \omega) = -i k_2 (A a_1 e^{-i\omega z} + B a_2 e^{-i\omega z}) + i D e^{-i\omega z} \]

\[ \tilde{u}_s(k_1, k_2, z, \omega) = A a_1 \gamma_1 e^{-i\omega z} + B a_2 \gamma_2 e^{-i\omega z} + C e^{-i\omega z} \]

(14a)

(14b)

(14c)

The expressions for the pore fluid pressure and the stress components of the traction vector at the surface of the halfspace are

\[ \tilde{p}(k_1, k_2, z, \omega) = A e^{-i\omega z} + B e^{-i\omega z} \]

\[ \tilde{\sigma}_{xz}(k_1, k_2, z, \omega) = \frac{i}{k_1} [g \gamma_1 A e^{-i\omega z} + g_2 \gamma_2 B e^{-i\omega z} + (\gamma C + k_2 D) e^{-i\omega z}] \]

\[ \tilde{\sigma}_{yz}(k_1, k_2, z, \omega) = \frac{i}{k_1} [2 k_2 a_1 \gamma_1 A e^{-i\omega z} + 2 k_2 a_2 \gamma_2 B e^{-i\omega z} + (k_2 C - \gamma_2 D) e^{-i\omega z}] \]

\[ \tilde{\sigma}_{zx}(k_1, k_2, z, \omega) = g A e^{-i\omega z} + g_2 B e^{-i\omega z} - 2 \gamma C e^{-i\omega z} \]

(15a)

(15b)

(15c)

(15d)

where A, B, C and D are unknown constants in the Fourier domain; \( \chi_1, \chi_2, a_1 \text{ and } a_2 \) are functions of \( \omega \); \( g_1 \text{ and } g_2 \) are functions of \( k_1 \) and \( \omega \); \( g_3 \text{ and } g_4 \) are functions of \( k_2 \) and \( \omega \); the expressions of these are given in Appendix E; \( \gamma_1 (i = 1, 2, 3) \) is expressed as

\[ \gamma_1(k_1, k_2, \omega) = \sqrt{k_1^2 + k_2^2 - L_1^2} \]

\[ \gamma_2(k_1, k_2, \omega) = \sqrt{k_1^2 + k_2^2 - L_2^2} \]

\[ \gamma_3(k_1, k_2, \omega) = \sqrt{k_1^2 + k_2^2 - S^2} \]

in which \( L_1, L_2 \) and \( S \) are wavenumbers of the P1, P2 and S waves at frequency \( \omega \), respectively (see Appendix E). The proper behavior of the general solutions at large positive z can be accounted for by choosing the branches in the complex domain such that the radicals have a positive real part, i.e. Re(\( \gamma_1 \)) > 0.

Applying the Fourier transforms to the beam Eq. (1), the beam deflection in the transformed domain can be evaluated in the form

\[ \tilde{w}_b(k_1, \omega) = 2\pi f_0 \frac{\delta(\omega - k_1 c)}{Elk_1^4 - m_b \omega^2 - i\omega \delta_b + \tilde{K}(k_1, \omega)} \]

(17)

where \( \tilde{K}(k_1, \omega) \) is the equivalent stiffness of the saturated poroelastic halfspace defined in the Fourier domain that relates the transformed contact line force \( \tilde{q}_c(k_1, \omega) \) to the transformed beam deflection \( \tilde{w}_b(k_1, \omega) \) by

\[ \tilde{q}_c(k_1, \omega) = \tilde{K}(k_1, \omega) \tilde{w}_b(k_1, \omega) \]

(18)

In the next section, the equivalent stiffness will be determined by applying the boundary conditions described by Cases A–D.

3.2. Formulation of the equivalent stiffness

The equivalent stiffness of the four cases can be formulated in the same way. Here we illustrate the procedure by developing the solution for Case A. Transforming Eqs. (7b)–(7d) into the Fourier domain, the following set of boundary conditions can be obtained over the entire surface \( \Gamma \) of the poroelastic halfspace:

\[ \tilde{\sigma}_{xz}(k_1, k_2, 0, \omega) = -\tilde{q}_c(k_1, \omega) \frac{\sin k_2}{k_2} \]

\[ \tilde{\sigma}_{zz}(k_1, k_2, 0, \omega) = \tilde{\sigma}_{yz}(k_1, k_2, 0, \omega) = 0 \]

\[ \tilde{p}(k_1, k_2, 0, \omega) = 0 \]

(19a)

(19b)

(19c)

Substituting the general solution Eq. (15) into Eq. (19), the four unknowns A–D can be determined and the vertical surface displacement of the soil skeleton in Eq. (14c) can be obtained as

\[ \tilde{u}_s(k_1, k_2, 0, \omega) = \frac{(a_2 \gamma_2 - a_1 \gamma_1) S^2}{\Delta A} \frac{\sin k_2}{k_2} \tilde{q}_c(k_1, \omega) \]

(20)

where \( \Delta A \) is the Rayleigh wave equation of the poroelastic halfspace with boundary conditions applicable to Case A. Its expression is given in Appendix E. By transforming the displacement compatibility condition Eq. (7a) into the \( k_1 - \omega \) domain and substituting Eq. (20) into the resulting equation after performing the inverse Fourier transform with respect to \( k_2 \), the equivalent stiffness in Case A can be formulated in the \( k_1 - \omega \) domain as

\[ \tilde{K}(k_1, \omega) = \frac{\tilde{q}_c(k_1, \omega)}{\tilde{w}_b(k_1, \omega)} = \frac{2\pi}{\int_{-\infty}^{\infty} X_k \frac{\sin k_2}{k_2} dk_2} \]

(21)

where

\[ X_k = \frac{(a_2 \gamma_2 - a_1 \gamma_1) S^2}{\Delta A} \]

(22a)
For Cases B–D, Eq. (21) still applies after replacing $\chi_A$ by $\chi_B$, $\chi_C$ and $\chi_D$, respectively, where

$$
\chi_B = \frac{\gamma_1 \gamma_2 (\lambda_2 - \lambda_1) s^2}{\Delta_B} \tag{22b}
$$

$$
\chi_C = \frac{a_1 \gamma_1 (\lambda_3 - \lambda_1) - a_2 \gamma_2 (\lambda_3 - \lambda_2) + \chi_2 - \chi_1}{\Delta_C} \tag{22c}
$$

$$
\chi_D = \frac{\gamma_1 \gamma_2 [a_2 (\lambda_2 - \lambda_3) - a_1 (\lambda_1 - \lambda_2)] + \gamma_1 \chi_2 - \gamma_2 \chi_1}{\Delta_D} \tag{22d}
$$

and the relevant expressions for $\Delta_B$, $\Delta_C$ and $\Delta_D$ are given in Appendix E.

### 3.3. Evaluation of the equivalent stiffness

The equivalent stiffness in Eq. (21) contains an improper integral with respect to the wavenumber $k_y$, which needs to be evaluated with care. Since we have assumed that the poroelastic halfspace has infinite permeability, the conventional numerical inversion techniques (e.g., the trapezoidal rule, the rectangle method and Gaussian quadrature) cannot be used as they involve summations along the real axis of $k_y$, which may encounter poles and branch points of the integrand and thus contain singularities. Therefore, in this study, the improper integral is evaluated using the method of contour integration, which can explicitly take into consideration the contributions of the poles and branch cuts.

The velocities $C_{p1}$, $C_{p2}$ and $C_s$ of the first longitudinal (P1), second longitudinal (P2) and shear (S) waves will not be dispersive for the saturated medium of infinite permeability. These velocities are constants and dependent only on the material parameters. The relationship $C_{p1} > C_{p2} > C_s$ can hold for most water-saturated soil media. A poroelastic halfspace with a pervious and traction-free surface will have only one surface wave (Gerasik and Stastna, 2008), i.e. the Rayleigh wave governed by Eq. (E.8), due to the coupling of the P1 and SV waves, whose velocity $C_{p1}$ is slightly smaller than $C_s$. However, for a poroelastic halfspace with an impervious surface, a second surface wave exists in both Cases B and D, which is governed by Eqs. (E.9) and (E.11), respectively. In Case B, the second surface wave is of an evanescent type, where the imaginary part of the wavenumber is non-zero and its velocity $C_{p2}$ is almost equal to $C_{p1}$. In Case D, the second surface wave is generated by the coupling of the P1 and P2 waves and the velocity $C_{p2}$ is found to be slightly smaller than $C_{p1}$. For most water-saturated soil media we may assume that $C_{p2} > C_s$, thus an inequality is assumed for the analysis as

$$
C_{p1} > C_{p2} \geq C_{p1} > C_s > C_{p1} \tag{23}
$$

The analysis can be facilitated by introducing the following dimensionless quantities, as was proposed by Dieterman and Metrikine (1996):

$$
\bar{\xi} = k_y/k_1, V_{ph} = \omega/k_1, \beta_{p1} = V_{ph}/C_{p1}, \beta_{p2} = V_{ph}/C_{p2}, \beta_s = V_{ph}/C_s \tag{24}
$$

where $\xi$ denotes the ratio between the wavenumbers along the $y$- and $x$-axes; $V_{ph}$ is the phase velocity of the waves in the $x$ direction, which is equal to the velocity of the moving load according to the property of the Dirac delta function (see Eq. (17)); $\beta_{p1}$, $\beta_{p2}$ and $\beta_s$ denote the ratio between the phase velocity and the velocities of the P1, P2 and S waves, respectively.

#### 3.3.1. Equivalent stiffness in Cases A and B

Using Eq. (24), the improper integral in Eq. (21) for Case A can be mapped onto the complex $\xi$ plane as

$$
J = \frac{\beta_{p1}^2}{ik_1} \int_{-\infty}^{+\infty} a_2 a_2 (\xi, \omega) - a_1 \gamma_1 (\xi, \omega) \exp(ik_1 \xi) d\xi
$$

$$
J = \frac{\beta_{p1}^2}{ik_1} \int_{-\infty}^{+\infty} f(\xi) d\xi \tag{25}
$$

where

$$
\gamma_1 (\xi, \omega) = \sqrt{1 + \xi^2 - \beta_{p1}^2}, \gamma_2 (\xi, \omega) = \sqrt{1 + \xi^2 - \beta_{p2}^2},
$$

$$
\gamma_3 (\xi, \omega) = \sqrt{1 + \xi^2 - \beta_s^2} \tag{26}
$$

$$
\Delta_A (\xi, \omega) = 4 \gamma_1 (\xi, \omega) [a_2 \gamma_2 (\xi, \omega) - a_1 \gamma_1 (\xi, \omega) (1 + \xi^2)]
$$

$$
- (a_2 - a_1) [2 (1 + \xi^2) - \beta_s^2] \tag{27}
$$

The integrand in Eq. (25) is a multi-valued function due to the presence of the radicals in Eq. (26), and the integrand becomes infinite at $\xi = 0$ and at the Rayleigh pole defined by $\Delta_A (\xi, \omega) = 0$. Thus proper branch cuts need to be defined to make the radicals single-valued and the manner of integration around the poles must be determined before the integral can be evaluated by applying Cauchy’s residual theorem. By following the procedure given in Dieterman and Metrikine (1996), the branch cuts and the contour indentations around the poles can be defined in the upper half plane of the complex $\xi$ plane by assuming $k_1 > 0$. This is done for the five regimes of the phase velocity $V_{ph}$: $V_{ph} < C_{p1}, \beta_{p1} < \beta_{p2}, C_{p1} < V_{ph} < C_{p2}, C_{p2} < V_{ph} < C_{p1}$ and $V_{ph} > C_{p1}$, as shown in Figs. A1–A5 in Appendix A.

Here only the residual at the Rayleigh pole is presented in the text. The expressions of the integrations around the pole $\xi = 0$ and along the branch cuts defined in each velocity regime can be found in Appendix A. The Rayleigh pole in Case A that is a simple zero of $\Delta_A (\xi, \omega) = 0$ can be numerically determined after replacing $1 + \xi^2 - \beta_{p1}^2 = (V_{ph}/R_{11})^2$ in Eq. (27). When $V_{ph} < C_{p1}$ (i.e. $\beta_{p1} < 1$), the Rayleigh pole in the upper plane of the complex $\xi$ is located at $\xi = \sqrt{1 - R_{11}}$, at which the residual of the integrand $f(\xi)$ in Eq. (25) can be evaluated as

$$
S_1 = 2 \pi i \times \text{Res} \left[ \frac{f(\xi, \omega)}{\Delta_A (\xi, \omega)} \right] = 2 \pi i \left[ \frac{a_2 \gamma_2 - a_1 \gamma_1 \exp(ik_1 \xi)}{d\Delta_A (\xi, \omega)} \right]_{\xi = \sqrt{1 - R_{11}}} = S_{A1} \exp \left( -k_1 \sqrt{1 - R_{11}} \right) \tag{28}
$$

where

$$
S_{A1} = \frac{\pi i (a_2 R_{p2} - a_1 R_{p1}) / 2 (R_{11} - 1)}{2 (a_1 - a_2) (2 R_{11} - R_{p2}) - (a_1 R_{p1} - a_2 R_{p2}) (R_{11} - K_1) - K_1 R_{p1} (a_1 R_{p1} - a_2 R_{p2})} \tag{29}
$$

in which, $R_{p1} = \sqrt{R_{11} - \beta_{p1}^2}, R_{p2} = \sqrt{R_{11} - \beta_{p2}^2}$ and $K_1 = \sqrt{R_{11} - \beta_{s}^2}$. When $V_{ph} > C_{p1}$ (i.e. $\beta_{p1} > 1$), the Rayleigh pole in the upper plane of the complex $\xi$ moves to $\xi = \sqrt{R_{11} - 1}$, at which the residual of $f(\xi)$ becomes

$$
S_2 = 2 \pi i \times \text{Res} \left[ \frac{f(\xi, \omega)}{\Delta_A (\xi, \omega)} \right] = S_{A2} \exp \left( i k_1 \sqrt{R_{11} - 1} \right) \tag{30}
$$

As is evident from Eq. (29), the integral $J$ in Eq. (25) approaches infinity when $V_{ph} \rightarrow C_{p1}$ (i.e. $\beta_{p1} = 1$) because $S_{A1} \rightarrow \infty$. Under this circumstance, the equivalent stiffness in Eq. (21) defined by $R = 2 \pi J / f$ vanishes.

For Case B, two simple zeros can be determined numerically from $\Delta_A (\xi, \omega) = 0$. The first is at $1 + \xi^2 = \beta_{R1}^2$ with a real $\beta_{R1},$
which is similar to the Rayleigh pole in Case A, while the second is located at \(1 + \xi^2 = \beta_{\xi}^2\) with a complex \(\beta_{\xi}\), which implies an evanescent surface wave whose velocity can be determined as \(C_{\xi} = V_{ph}/Re(\beta_{\xi})\). The residual of the integrand in Case B at the second Rayleigh pole can be evaluated as

\[
S_3 = 2\pi i \times \text{Res} \left[ f(\xi), \xi = \sqrt{\beta_{\xi}^2} - 1 \right] \\
= 2\pi i \times \left[ \frac{\left(a_2 - a_1\right)\gamma_2 \exp(i\xi \beta_{\xi})}{d\Delta_n/d\xi} \right]_{\xi = \sqrt{\beta_{\xi}^2} - 1}
\]

where the expression of \(S_3\) is given in Appendix B.

With the addition of the second Rayleigh pole, the contour plots defined in Case A can be readily used in Case B for evaluating the equivalent stiffness. The expressions of the integrations around the pole \(\xi = 0\) and along the branch cuts as well as the residuals at the first Rayleigh pole are given in Appendix B for Case B. As in Case A, it is observed from the residual at the first Rayleigh pole in Eq. (B.3) that the equivalent stiffness in this case becomes zero when \(V_{ph} = C_0\), while for the second Rayleigh pole, its contribution is found to be negligible.

**3.3.2. Equivalent stiffness in Cases C and D**

Using Eq. (24) and Eq. (E.10), the improper integral in Eq. (21) for Case C can be mapped onto the complex \(\xi\) plane as

\[
J = \frac{k_1}{i(\lambda + 2)(\chi_2 - \chi_1)} \int_{-\infty}^{\infty} \gamma_1(\xi, \omega) \frac{\left(a_1 \gamma_1(\xi, \omega) - a_2 \gamma_2(\xi, \omega)\right)}{\gamma_2(\xi, \omega)} + (1 + \xi^2) \frac{\left(a_2 - a_1\right)}{\gamma_1(\xi, \omega)} \frac{\exp(i\xi \beta_{\xi})}{\xi} d\xi
\]

By comparing this result with Case A, it is seen from Eq. (32) that the integrand only has a simple pole at \(\xi = 0\) and no Rayleigh pole exists; i.e., for the saturated poroelastic half-space with an inextensible-pervious surface, no surface wave exists. Therefore, in this case, only four regimes of the phase velocity \(V_{ph}\), i.e., \(V_{ph} < C_0, C_0 < V_{ph} < C_{p_{2}}, C_{p_{2}} < V_{ph} < C_{p_{1}},\) and \(V_{ph} > C_{p_{1}}\), are needed to define the contour. The evolution of the branch cuts with respect to the four velocity regimes are the same as those defined in Case A (see Figs. A.1-A.5).

The contribution of the pole at \(\xi = 0\) is evaluated along a semi-circular contour \(C_0\) around it in the upper half plane of \(\xi\). After the polar coordinate parameterization for \(C_0\), the expressions of the integrations around the pole can be easily obtained when the radius of \(C_0\) approaches zero; taking the regime \(V_{ph} < C_0\) as an example, \(S_0 = -\int_{C_0} f(\xi) d\xi\) where

\[
S_0 = - \int_{C_0} \frac{\left(1 - \beta_{\xi}^2\right) \left(a_1 - a_2\right)}{\sqrt{1 - \beta_{\xi}^2} + \left(1 - \beta_{\xi}^2\right) a_1 - a_2} d\xi
\]

From Eq. (E.11), the simple zero of \(A_2 = 0\) can be obtained after replacing \(1 + \xi^2 = \beta_{\xi}^2\) as

\[
\beta_{\xi}^2 = \left(\frac{V_{ph}}{C_0}\right)^2 \frac{h_1 \beta_{p_{2}}^2 - h_2 \beta_{p_{1}}^2}{h_1 - h_2}
\]

where \(\beta_{\xi}\) is real-valued; \(h_1 = (\lambda + 2)\chi_1 - \alpha; h_2 = (\lambda + 2)\chi_2 - \alpha\).

The velocity \(C_{\xi}\) of the second Rayleigh wave is slightly smaller than that of the P2 wave in this case. Since we assume that \(C_{\xi} > C_0\), the second Rayleigh pole is located outside the domain enclosed by the contour defined in Figs. A.1-A.5. Thus, the contribution of the Rayleigh pole in Case D is not included. The expressions of the integrations along the branch cuts and around the pole at \(\xi = 0\) for this case are given in Appendix D for the four velocity regimes. As in Case C, the expressions of the integrations around the pole \(\xi = 0\) reveal that the equivalent stiffness in Case D becomes zero at \(V_{ph} = C_0\).

**3.3.3. Reduction from poroelasticity to elasticity**

The expressions in Cases A–D for poroelasticity can be reduced to elasticity by setting the pore-fluid related parameters to zero, i.e., \(\rho_1 = \alpha = n = 0\). By doing so, the frequency-dependent parameter \(a_1\) (see Eq. (E.4)) will be zero and the wavenumbers of the P1, P2 and S waves given in Eqs. (E.2) and (E.3) become \(k_1^2 = 0, k_2^2 = \alpha_2^2/(\lambda + 2)\) and \(S^2 = \alpha_2^2,\) respectively. Then all the expressions given in Case A and Case B can be reduced mathematically to those given by Dieterman and Metrikine (1996), who derived the equivalent stiffness for a traction-free elastic halfspace interacting with a beam using the method of contour integration. For example, the integrand in Eq. (21) for Case A and Case B can both be reduced to

\[
\chi_{A.B} = \frac{\gamma_2 S^2}{4\gamma_2 \gamma_3 \left(k_1^2 + 2k_2^2 - S^2\right)^2}
\]

which is the same as the integrand in Eq. (21) of Dieterman and Metrikine’s paper. However, we should keep in mind that the Lamé constants (\(\lambda\) and \(\mu\)) and the mass density \(\rho\) remain the same in the reduction process.

**3.4. Beam displacements and bending moments**

Using the expressions for the equivalent stiffness in the previous section, the steady-state vertical displacement of the beam can be obtained by applying double Fourier inversions to Eq. (17) with respect to \(\omega\) and \(k_1\), respectively.

\[
\frac{w_b(x, t)}{f_0} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\exp(ik_1 x')}{EIk_1^4 - m_b((ck_1)^2 - i\delta_b)ck_1 + \tilde{R}(k_1, c_1)} d\xi
\]

where \(x' = x - ct\) is an auxiliary spatial coordinate moving with the load. From the expressions of the equivalent stiffness it is found the real part of \(\tilde{R}\) is symmetrical and the imaginary part is asymmetrical with respect to \(k_1\), i.e.,

\[
\begin{align*}
\text{Re}[\tilde{R}(k_1, c_1)] &= \text{Re}[\tilde{R}(-k_1, -c_1)] \\
\text{Im}[\tilde{R}(k_1, c_1)] &= -\text{Im}[\tilde{R}(-k_1, -c_1)]
\end{align*}
\]

Using the symmetry relations of \(\tilde{R}\), Eq. (36) can be rewritten as

\[
\frac{w_b(x, t)}{f_0} = \frac{w_b^{\text{sym}}(x, t)}{f_0} + w_b^{\text{asym}}(x, t)
\]

where
Fig. 2. Comparisons with published results: (a) equivalent stiffness for $k_1 = 0.2$; (b) beam vertical displacement at $c = 0.93$ (Re = real part; Im = imaginary part).

\[
\frac{w_{b,\text{sym}}(x, t)}{f_0} = \frac{1}{\pi} \int_0^{+\infty} \frac{[EIk_1^4 - m_0c^2k_1^2 + \text{Re}(\tilde{K})] \cos(k_1x')}{[EIk_1^4 - m_0c^2k_1^2 + \text{Re}(\tilde{K})]^2 + [\delta_bck_1 - \text{Im}(\tilde{K})]^2} dk_1 \quad (39a)
\]

\[
\frac{w_{b,\text{sym}}(x, t)}{f_0} = \frac{1}{\pi} \int_0^{+\infty} \frac{[\delta_bck_1 - \text{Im}(\tilde{K})] \sin(k_1x')}{[EIk_1^4 - m_0c^2k_1^2 + \text{Re}(\tilde{K})]^2 + [\delta_bck_1 - \text{Im}(\tilde{K})]^2} dk_1 \quad (39b)
\]

From Eq. (38), the expressions for the bending moment of the beam can be easily obtained by taking the second order derivative with respect to the coordinate $x$. The steady-state solution in Eq. (38) can be used to study the critical velocities of the load, when the beam displacement under the load (i.e. $x' = 0$) is at the maximum.

4. Numerical results

In order to verify the accuracy of the derivations, the solutions for poroelasticity are reduced into those applicable for the associated elasticity problem and the results are compared with published data. The equivalent stiffness is then compared for Cases A–D to investigate the influence of the boundary conditions prescribed at the surface of the poroelastic halfspace. The beam displacement under the load is plotted versus the load velocity in each case to graphically locate the critical velocities at which the beam deflection reaches a maximum. Finally, the beam deflection and bending moment profiles are presented for selected load velocities.

4.1. Comparison with published results

Dieterman and Metrikine (1996, 1997) obtained the equivalent stiffness and the deflection responses of an infinite beam on a traction-free elastic halfspace under a moving point load. The solutions derived for Cases A and B are reduced from the poroelasticity to the elasticity cases through the procedure described in Section 3.3.3 and then compared with the solutions by Dieterman and Metrikine’s (1996, 1997), as shown in Fig. 2. The material parameters of the elastic halfspace and the beam are set to the values used by Dieterman and Metrikine (1997); i.e., $\lambda = 1.5$, $EI = 0.042$, $m_0 = 0.556$, $\delta_0 = 0.113$ and $f_0 = 1$, which characterize a “stiff” halfspace interacting with a “small” beam.

It can be seen from Fig. 2 that the comparisons with the results of Dieterman and Metrikine (1996, 1997) are generally good except for the discrepancy in the imaginary part of the equivalent stiffness when the phase velocity is high (see Fig. 2(a)). This may be attributed to the elementary functions that were used by Dieterman and Metrikine (1996) to approximate the integrations along the branch cuts. It should be noted that Case A and Case B give the same results after the reduction.

Steenbergen and Metrikine (2007) adopted the bonded contact condition for prescribing the boundary conditions at the interface of an infinite beam of finite width resting on the surface of an elastic halfspace, where the traction variation across the beam interface is approximated by discretizing the beam-halfspace interface into $N$ strips and the displacements of the halfspace under the beam are assumed constant. In order to assess the accuracy of the bonding techniques adopted in this study, Fig. 3 presents a comparison on the equivalent stiffness of the halfspace between the results given by Steenbergen and Metrikine (2007) and those obtained from Cases A and C after reducing from the poroelasticity to the elasticity cases. The modal parameters are taken as $\lambda = 1.5$, $k_1 = 0.16$ and $N = 1$, which are the same as those used in Fig. 4 of Steenbergen and Metrikine (2007).

From Fig. 3 it is evident that the results given by Steenbergen and Metrikine (2007) are consistent with the results derived from the bonding techniques described in Cases A and C. To be specific, the reference stiffness falls right between the stiffness bounds provided by Cases A and C when the phase velocity $V_{ph}$ is lower than 1, while it shifts closer to the stiffness obtained from Case A when $V_{ph}$ increases beyond 1. It should also be noted that Case C and Case D give the same results after the reduction.

4.2. Influence of the boundary conditions

The influence of the four sets of boundary conditions described in Cases A–D on the equivalent stiffness of the poroelastic halfspace in joined motion with an infinite beam is presented Fig. 4. Material parameters of the poroelastic medium of infinite permeability (i.e. $b = 0$) are chosen as those given by Jin (2004) and presented in Table 1. The velocities of the three body waves and the two surface waves of the poroelastic halfspace are calculated and presented in Table 2. The velocities of the body and surface waves of the elastic halfspace that are obtained through the reduction process described in Section 3.3.3 are given in Table 3. Material parameters of the beam and the moving load are the same as those in Fig. 2.
It is evident from Fig. 4(a) that the real part of the equivalent stiffness of the poroelastic halfspace with a traction-free surface (Cases A and B) reaches zero at the first Rayleigh wave velocity, while for the poroelastic halfspace with an inextensible surface (Cases C and D) the real part becomes zero at the shear wave velocity. This observation can be mathematically verified from Eqs. (29) and (33). The imaginary part of the equivalent stiffness is zero when $V_{ph} < C_{R1}$ for Cases A and B and $V_{ph} < C_s$ for Cases C and D, beyond which the imaginary part increases significantly as a result of the radiation of elastic waves. From trial computations, it is found that the contribution of the complex second Rayleigh pole to the equivalent stiffness in Case B is negligible.

By comparing Cases A–C and Cases B–D, it is seen from Fig. 4(a) that the poroelastic halfspace with an inextensible surface is stiffer (i.e. the larger real part) than that with a traction-free surface except in the velocity range from $C_s$ to $C_{p2}$; however the imaginary parts of them are very close for most of the phase velocity (see Fig. 4(b)). By comparing Case A to B and Case C to D, it is respectively

---

**Table 1**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
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<tr>
<td>Lamé constant</td>
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<td>Biot’s parameter</td>
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<td>Tortuosity factor</td>
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**Table 2**

<table>
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<th>Wave type</th>
<th>Symbol</th>
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<th>Case B</th>
<th>Case C</th>
<th>Case D</th>
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<tr>
<td>P2</td>
<td>$C_{p2}$</td>
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<tr>
<td>S</td>
<td>$C_s$</td>
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<td></td>
<td></td>
<td></td>
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<tr>
<td>R1</td>
<td>$C_{R1}$</td>
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<td>1.055</td>
<td>–</td>
<td>–</td>
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<tr>
<td>R2</td>
<td>$C_{R2}$</td>
<td>–</td>
<td>1.742</td>
<td>–</td>
<td>1.629</td>
</tr>
</tbody>
</table>

* R1 denotes the first Rayleigh wave that is generated by the coupling of the P1 and S waves.
* R2 denotes the second Rayleigh wave that is generated by the coupling of the P1 and P2 waves.

**Table 3**

<table>
<thead>
<tr>
<th>Wave type</th>
<th>Symbol</th>
<th>Value</th>
</tr>
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<tr>
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<tr>
<td>S</td>
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<td>R</td>
<td>$C_{R0}$</td>
<td>0.927</td>
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observed from Fig. 4(a) and (b) that the poroelastic halfspace with a pervious surface is stiffer and dissipates less energy than that with an impervious surface. The equivalent stiffness of the poroelastic halfspace with a free-draining surface in joined motion with the infinite beam with an impermeable interface would fall within the bounds defined by the four cases, which define a quite narrow interval for the equivalent stiffness as shown in Fig. 4.

When the poroelastic halfspace is reduced to the elastic one, the influence of the four sets of boundary conditions on the equivalent stiffness is shown in Fig. 5. It is observed from Fig. 5 that the velocities of the Rayleigh and the shear waves of the elastic halfspace, at which the equivalent stiffness changes substantially, are smaller than those of the poroelastic halfspace. The imaginary part of the equivalent stiffness for the elastic halfspace with a traction free surface (Cases A and B) almost overlaps that associated with an inextensible surface (Cases C and D).

![Fig. 5. Effects of boundary conditions described in Cases A–D on the equivalent stiffness of the elastic halfspace at \( k_i = 0.2 \).](image)

![Fig. 6. Beam displacement under the load as a function of the load velocity for the poroelastic and the elastic halfspace.](image)

4.3. Critical velocities of the moving constant load

Here we define critical velocities as velocities at which the beam displacement under the load reaches a maximum. Fig. 6 shows the beam displacement under the load (i.e. \( x' = 0 \)) versus the load velocity for the infinite beam on the poroelastic halfspace with the boundary conditions described in Cases A–D, respectively. Fig. 6 also includes the beam displacement associated with the elastic halfspace that is reduced from Cases A and C, respectively.

By referring to Figs. 4 and 5, it is seen from Fig. 6 that the beam displacement becomes infinite at velocities where the equivalent stiffness of the halfspace reduces to zero. With a null equivalent stiffness, the Fourier integral inversion in Eq. (39) for obtaining the beam displacement will diverge as a Cauchy principal value, which renders infinite displacement to the beam. These velocities have been mathematically predicted in Section 3.3 (see Eqs. (29) and (33)), which are referred to as the first critical velocity hereafter. It is also observed from Fig. 6 that a second peak in the beam displacement exists to the left of the infinite response in each case. This peak, which remains finite because viscosity is considered for the beam, is due to the mechanical interaction of the beam and the halfspace, and is thus dependent on the bending stiffness and mass of the beam in addition to the halfspace properties. The velocity at which the second peak occurs is termed the second critical velocity (\( C^\prime \)). The values of the two critical velocities are given in Table 4. It is observed that the second critical velocity is slightly smaller than the corresponding first critical velocity in each case.

From Table 4 it can be seen that the critical velocities associated with an inextensible halfspace surface (Cases C and D) are higher

![Table 4 First and second critical velocities of the moving constant load.](image)
Fig. 8. Beam displacement profile in Case A for six load velocities: (a) $c = 0.8$; (b) $c = 1.04$; (c) $c = 1.1$; (d) $c = 1.45$; (e) $c = 2.8$; and (f) $c = 4.5$.

than those associated with the traction-free surface (Cases A and B). Moreover, the critical velocities associated with a pervious half-space surface (Cases A and C) are higher than those associated with the impervious surface (Cases B and D). These observations can be explained by the fact that the kinematic constraint and the null pore pressure prescribed at the surface make the halfspace stiffer. Compared with poroelasticity, it is seen from Fig. 6 that the infinite beam in joined motion with the elastic halfspace has lower critical velocities but with larger amplitudes for the second peak.

It is interesting to look at the beam bending moment under the load as a function of the load velocity, as shown in Fig. 7. From this figure it is seen that the bending moment reaches its local maximum at the two critical velocities for each case, which agrees with the observations in Fig. 6. However, the bending moment, which is obtained by taking the second derivative of the beam displacement with respect to the coordinate $x$, remains finite at the first critical velocity in each case. This is different from the infinite displacement response observed at the first critical velocity in Fig. 6.

4.4. Displacement and bending moment profiles of the beam

For the boundary conditions described in Case A, Figs. 8 and 9 present the displacement and bending moment distributions along the beam versus the auxiliary spatial coordinate $x'$, respectively. From Table 4 we have the two critical velocities of the point load in Case A, i.e., $C_{R1} = 1.073$ and $C^{T} = 1.016$ for the first and second critical velocity, respectively. From Table 2 we have the velocities of the three body waves, i.e., $C_{S}$, $C_{P2}$ and $C_{P1}$ for the S, P2 and
P1 waves, respectively. Accordingly, six load velocities are considered for Figs. 8 and 9, which fall into the intervals $c < C_{R1}$, $C_{R1} < c < C_s$, $C_s < c < C_{p2}$, $C_{p2} < c < C_{p1}$ and $c > C_{p1}$, respectively.

It is seen from Fig. 8(a) that the beam displacement profile is almost symmetrical with respect to the loading point $x' = 0$ when the load velocity is lower than the second critical velocity (i.e. $c < C_{R1}$). When the load velocity increases to a value between the two critical velocities (i.e. $C_{R1} < c < C_s$), the displacement response shown in Fig. 8(b) becomes significantly larger and its variation along $x'$ is intensified when compared with Fig. 8(a). However, this does not mean that the wave radiates into the halfspace since the imaginary part of the equivalent stiffness of the poroelastic halfspace remains zero for $c < C_{R1}$ (see Fig. 4(b)). Until now the viscosity of the beam is the only way to dissipate the energy in the system; when the load velocity increases further to within the interval $C_{R1} < c < C_s$, the moving point load begins to radiate elastic waves into the poroelastic halfspace because the imaginary part of its equivalent stiffness substantially increases from zero to a non-zero value, as shown in Fig. 4(b). At this point the radiation results only in Rayleigh waves that travel along the halfspace surface, and the Doppler effect becomes quite obvious (see Fig. 8(c)) in the beam deflection that is compatible with the vertical displacement at the halfspace surface: the wavelength of the waves before the load is much smaller than that of the waves after the load. When the load velocity surpasses the Rayleigh wave velocity of the halfspace, the body waves (S, P2 and P1 waves) will be radiated in addition to the Rayleigh wave (see Fig. 4(b)), which causes
more energy dissipation to the system and the displacement response at the halfspace surface decreases significantly. Moreover, since the load is moving faster than the Rayleigh wave, it will lead the advancing Rayleigh wave front that propagates along the halfspace surface. Thus, little vibration in the beam displacement can be observed ahead of the load, while a large displacement response occurs behind the load. This is confirmed in Fig. 8(d), (e) and (f) where the load velocity surpasses the velocity of the S, P2 and P1 waves, respectively.

Similar observations can be made for the profiles of the beam bending moment versus the load velocity, as shown in Fig. 9. For boundary conditions described in Cases B–D, the variations of the profiles of the displacement and bending moment of the beam with respect to the load velocity are the same as those depicted in Figs. 8 and 9. Thus they have not been presented for brevity.

5. Conclusions

The dynamic response of an impermeable beam of finite width in joined motion with a saturated poroelastic medium with a free-draining surface involves mixed boundary conditions applicable not only for the soil-skeleton phase but also for the pore-fluid phase, which makes the resulting boundary value problem analytically intractable. Therefore we apply a “bounding technique” that incorporates either kinematic or mechanical constraints at the halfspace surface for the two phases of the saturated medium, which results in four sets of boundary conditions (four cases) that can “bound” the original situation. For each case the equivalent stiffness of the poroelastic halfspace of infinite permeability is analytically derived using the method of contour integration. Using the equivalent stiffness of the underlying halfspace, the displacement and bending moment responses of the infinite beam under a moving constant load can be evaluated without difficulty. The solutions developed for the poroelastic medium can be reduced analytically to recover the solution for an elastic medium under the assumption of the same Lamé constants and mass density for the two media. From the derivations and the numerical results the following conclusions can be drawn:

1. The phase velocity, at which the real part of the equivalent stiffness of the poroelastic halfspace vanishes and the imaginary part substantially increases, is equal to the first Rayleigh wave velocity (i.e., the Rayleigh wave due to the interaction of the P1 and S waves) for the halfspace with a traction-free surface, while it changes to the shear wave velocity for the halfspace with an inextensible surface.
2. The poroelastic halfspace with an inextensible or pervious surface is stiffer than that with a traction-free or impervious surface.
3. Two critical velocities exist for the moving constant load: the first one is equal to the phase velocity described in (1) for each case; the second velocity is slightly smaller than the first in each case (95% for the poroelasticity and 98% for the elasticity).
4. The critical velocities associated with an inextensible or pervious surface are higher than those with a traction-free or impervious surface. The critical velocities for the beam-elastic halfspace system are lower than those of the beam-poroelastic halfspace system.
5. A Doppler effect can be observed for the beam displacement and bending moment responses when the load speed surpasses the first and second critical velocities as well as the velocities of the three body waves in the poroelastic halfspace, respectively. The beam response increases significantly when the load speed falls into the interval between the first and second critical velocities.

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Appendix A

The contour of integration is defined in the upper half plane of the complex $\xi$ for five regimes of the phase velocity $V_{ph}$: (a) $V_{ph} < C_{R1}$; (b) $C_{R1} < V_{ph} < C_s$; (c) $C_s < V_{ph} < C_{p2}$; (d) $C_{p2} < V_{ph} < C_{p1}$ and (e) $V_{ph} > C_{p1}$. Since we assume that $k_1 > 0$, the complementary integration along the circular contours $C_1$ and $C_2$ vanish as $|\xi| \to \infty$. The integration along the circular contour around the branch points vanish as well when their radius decreases. Thus, the integral $J$ in Eq. (25) can be reduced by taking into consideration the integrations along the branch cuts and around the pole $\xi = 0$ as well as the residual at the Rayleigh pole. When $V_{ph} > C_s$ the pole at $\xi = 0$ becomes nonphysical, i.e., the imaginary part of its contribution becomes nonzero, which implies that it is radiating energy into the system. Thus, when $V_{ph} > C_s$ the

\[ J \]

Fig. A.1. Contour plot for regime (a): $V_{ph} < C_{R1}$.

Fig. A.2. Contour plot for regime (b): $C_{R1} < V_{ph} < C_s$.

Fig. A.3. Contour plot for regime (c): $C_s < V_{ph} < C_{p2}$.
contribution of pole at $\xi = 0$ is cancelled by subtracting “1” into the numerator of $f(\xi)$ and using the equality $\int_{-\infty}^{+\infty} \frac{Q_2(\xi, \omega) - a_1 Q_1(\xi, \omega)}{\Delta_1(\xi, \omega)} \frac{1}{\xi} d\xi = 0$. It is treated in the same way in Cases B–D.

(a) Regime $V_{ph} < C_R$\[\int_{-\infty}^{+\infty} f(\xi) d\xi = S_1 + S_0 - 2G_1 - 2G_2 - 2G_3\] \hspace{1cm} (A.1)\n
where $S_1$ is the contribution of the Rayleigh pole and its expression is given in Eq. (28); $S_0$ represents the integration along the semi-circular contour around $\xi = 0$; $G_1$, $G_2$ and $G_3$ are integrations along the branch cut lines 1 to 6. Their expressions are obtained as

\[S_0 = \frac{i\pi}{4\sqrt{1 - \beta_{p1}^2}} \left( a_2 \sqrt{1 - \beta_{p2}^2} - a_1 \sqrt{1 - \beta_{p1}^2} \right) \] \hspace{1cm} (A.2)\n
\[2G_1 = -2i \int_{1 - \beta_{p1}}^{1 - \beta_{p2}} \frac{a_1 Q_{p1}^+ - a_2 Q_{p2}^+}{(a_2 - a_1)E_2^2 + 4Q_2^+ (a_2 Q_{p2}^+ - a_1 Q_{p1}^+)(1 - \eta^2)} \exp(-\eta k_1) d\eta \] \hspace{1cm} (A.3)\n
\[2G_2 = 2i \int_{1 - \beta_{p1}}^{1 - \beta_{p2}} \frac{4Q_2^+ (1 - \eta^2) \left[ (a_2 Q_{p2}^+)^2 + (a_1 Q_{p1}^+)^2 \right] + a_2 Q_{p2}^+ (a_2 - a_1)E_2^2 + 4Q_2^+ (a_2 Q_{p2}^+ - a_1 Q_{p1}^+)^2}{(a_2 - a_1)E_2^2 + 4Q_2^+ (a_2 Q_{p2}^+ - a_1 Q_{p1}^+)^2} \exp(-\eta k_1) d\eta \] \hspace{1cm} (A.4)\n
\[2G_3 = 2i \int_{1 - \beta_{p1}}^{1 - \beta_{p2}} \frac{4Q_2^+ (1 - \eta^2) (a_2 Q_{p2} - a_1 Q_{p1})^2}{(a_2 - a_1)E_2^2 + 4Q_2^+ (a_2 Q_{p2} - a_1 Q_{p1})^2} \exp(-\eta k_1) d\eta \] \hspace{1cm} (A.5)\n
where

\[Q_{p1}^\pm = \sqrt{\pm \beta_{p1}^2 - \eta^2} \mp 1 \hspace{0.5cm} Q_{p2}^\pm = \sqrt{\pm \beta_{p2}^2 - \eta^2} \mp 1 \hspace{0.5cm} Q_2^\pm = \sqrt{\pm \beta_2^2 \pm \eta^2} \mp 1 \] \hspace{1cm} (A.6)\n
\[E_2 = 2(1 - \eta^2) - \beta_2^2 \] \hspace{1cm} (A.7)
(b) Regime $C_{R1} < V_{ph} < C_s$

\[ \int_{-\infty}^{+\infty} f(\xi)\,d\xi = S_2 + S_0 - 2G_1 - 2G_2 - 2G_3 \]  
(A.8)

where $S_2$ is the contribution of the Rayleigh pole and its expression is given in Eq. (30).

(c) Regime $C_s < V_{ph} < C_{p2}$

\[ \int_{-\infty}^{+\infty} f(\xi)\,d\xi = S_2^* - 2G_1^* - 2G_2^* - 2G_4 - 2G_6 \]  
(A.9)

\[ S_2^* = S_A \times \left[ \exp \left( \frac{i}{\sqrt{2}k_1}\right) - 1 \right] \]  
(A.10)

\[ 2G_1 = -2i \int_{\sqrt{1 - \beta_{p1}^2}}^{\infty} \frac{a_1Q_{p1}^+ - a_2Q_{p2}^+}{(a_2 - a_1)E_2^2 + 4Q_{p2}^+ (a_2Q_{p2}^+ - a_1Q_{p1}^+)} \frac{\left[ \exp (-\eta k_1) - 1 \right]}{\eta} \,d\eta \]  
(A.11)

\[ 2G_2 = 2i \int_{\sqrt{1 - \beta_{p1}^2}}^{\infty} \frac{\left[ (a_2Q_{p2}^+ - a_1Q_{p1}^+) \right]^2 + 4Q_{p2}^+(a_2 - a_1)E_2^2}{4a_2Q_{p2}^+Q_{p1}^+ (1 - \eta^2) + (a_2 - a_1)E_2^2 \left[ 4(1 - \eta^2)a_1Q_{p2}^+Q_{p1}^+ \right]^2} \frac{\left[ \exp (-\eta k_1) - 1 \right]}{\eta} \,d\eta \]  
(A.12)

\[ 2G_4 = 2i \int_{\sqrt{1 - \beta_{p1}^2}}^{\infty} \frac{\left[ a_2Q_{p2}^+ - a_1Q_{p1}^+ \right]^2}{\left[ (a_2 - a_1)E_2^2 \right]^2 + \left[ 4(1 - \eta^2)a_1Q_{p2}^+Q_{p1}^+ \right]^2} \frac{\left[ \exp (-\eta k_1) - 1 \right]}{\eta} \,d\eta \]  
(A.13)

\[ 2G_5 = -2i \int_{0}^{\sqrt{\beta_{p1}^2 - 1}} \left[ (1 - \xi^2) \left( \frac{a_2Q_{p2}^+ - a_1Q_{p1}^+}{a_2 - a_1}E_2^2 \right)^2 + 4(1 + \xi^2) P_\xi \left( a_2Q_{p2}^+ - a_1Q_{p1}^+ \right) \right]^2 \frac{\left[ \exp (i\xi k_1) - 1 \right]}{\xi} \,d\xi \]  
(A.14)

where

\[ Q_{p1}^+ = \sqrt{\beta_{p1}^2 + \xi^2} \pm 1, \quad P_{p1}^+ = \sqrt{\beta_{p2}^2 + \xi^2} \pm 1, \quad P_s = \sqrt{\beta_s^2 + \xi^2} \pm 1 \]  
(A.15)

\[ e_k = 2(1 + \xi^2) - \beta_s^2 \]  
(A.16)

(d) Regime $C_{p2} < V_{ph} < C_{p1}$

\[ \int_{-\infty}^{+\infty} f(\xi)\,d\xi = S_2^* - 2G_5 - 2G_6 - 2G_7 - 2G_8 \]  
(A.17)

\[ 2G_6 = 2i \int_{0}^{\sqrt{\beta_{p1}^2 - 1}} \left[ (1 - \xi^2) \left( \frac{a_2Q_{p2}^+ - a_1Q_{p1}^+}{a_2 - a_1}E_2^2 \right)^2 + 4(1 - \eta^2)a_1Q_{p2}^+Q_{p1}^+ \right] \frac{\left[ \exp (-\eta k_1) - 1 \right]}{\eta} \,d\eta \]  
(A.18)

\[ 2G_7 = -2i \int_{0}^{\sqrt{\beta_{p1}^2 - 1}} \left[ (1 + \xi^2) \left( \frac{a_2Q_{p2}^+ - a_1Q_{p1}^+}{a_2 - a_1}E_2^2 \right)^2 + 4(1 + \xi^2) a_2P_{p2}^+P_\xi \left( a_2Q_{p2}^+ - a_1Q_{p1}^+ \right) \right] \frac{\left[ \exp (i\xi k_1) - 1 \right]}{\xi} \,d\xi \]  
(A.19)

\[ 2G_8 = -2i \int_{0}^{\sqrt{\beta_{p1}^2 - 1}} \left[ (1 + \xi^2) \left( \frac{a_2Q_{p2}^+ - a_1Q_{p1}^+}{a_2 - a_1}E_2^2 \right)^2 + 4(1 + \xi^2) a_2P_{p2}^+P_\xi \left( a_2Q_{p2}^+ - a_1Q_{p1}^+ \right) \right] \frac{\left[ \exp (i\xi k_1) - 1 \right]}{\xi} \,d\xi \]  
(A.20)

(e) Regime $V_{ph} > C_{p1}$

\[ \int_{-\infty}^{+\infty} f(\xi)\,d\xi = S_2^* - 2G_9 - 2G_{10} - 2G_{11} - 2G_8 \]  
(A.21)

\[ 2G_9 = -2i \int_{0}^{\sqrt{\beta_{p1}^2 - 1}} \frac{a_1Q_{p1}^+ - a_2Q_{p2}^+}{(a_2 - a_1)E_2^2 + 4Q_{p2}^+ (a_2Q_{p2}^+ - a_1Q_{p1}^+)} \frac{\left[ \exp (-\eta k_1) - 1 \right]}{\eta} \,d\eta \]  
(A.22)

\[ 2G_{10} = -2i \int_{0}^{\sqrt{\beta_{p1}^2 - 1}} \frac{a_2P_{p2}^+ - a_1P_{p1}^-}{(a_2 - a_1)E_2^2 + 4P_\xi \left( a_2P_{p2}^+ - a_1P_{p1}^- \right)} \frac{\left[ \exp (i\xi k_1) - 1 \right]}{\xi} \,d\xi \]  
(A.23)

\[ 2G_{11} = -2i \int_{0}^{\sqrt{\beta_{p1}^2 - 1}} \frac{4P_s^+ \left( a_2P_{p2}^+ - a_1P_{p1}^- \right)}{4a_2P_{p2}^+P_\xi \left( 1 + \xi^2 \right) + (a_2 - a_1)E_2^2 \left[ 4(1 + \xi^2)a_1P_{p1}^+P_\xi \right]^2} \frac{\left[ \exp (i\xi k_1) - 1 \right]}{\xi} \,d\xi \]  
(A.24)
Appendix B

(a) Regime $V_{ph} < C_{R_1}$

\[
\int_{-\infty}^{\infty} f(\xi) d\xi = S_1 + S_3 + S_0 - 2G_1 - 2G_2 - 2G_3
\]  
(B.1)

where $S_1$ and $S_3$ are the contributions of the first and second Rayleigh pole, respectively; the expression of $S_3$ is given in Eq. (31); $S_0$ represents the integration along the semi-circular contour around $\xi = 0$; $G_1$, $G_2$ and $G_3$ are integrations along the branch cut lines 1 to 6. Their expressions are obtained as

\[
S_1 = S_B \times \exp \left( -ik_1 \sqrt{1 - \beta_R^2} \right)
\]  
(B.2)

\[
S_B = \frac{2\pi i(a_2 - a_1)R_1 R_2 / \left( \beta_R^2 - 1 \right)}{4(a_2 - a_1)\left( \beta_R^2 + 2\Omega_1 - \left( 2\beta_R^2 - \beta_L^2 \right) \left[ 4(a_2 R_1 - a_1 R_2) + \frac{\xi_1^2}{\xi_{1p}^2} - \frac{\xi_2^2}{\xi_{2p}^2} \right] + 4(\xi_{1p}^2 R_1 - \xi_{2p}^2 R_2) \right)}
\]  
(B.3)

where

\[
R_1 = \sqrt{\beta_R^2 - \beta_{R_1}^2}, \quad R_2 = \sqrt{\beta_R^2 - \beta_{R_2}^2}, \quad R_s = \sqrt{\beta_R^2 - \beta_s^2}
\]  
(B.4)

\[
\gamma_3^0 = (\lambda + 2)a_1 \beta_{R_1}^2 - 2a_1 - \alpha / k_1^2, \quad \gamma_3^4 = (\lambda + 2)a_2 \beta_{R_2}^2 - 2a_2 - \alpha / k_1^2
\]  
(B.5)

\[
\Omega_1 = R_1 R_2 / R_s + R_1 R_2 / R_s + R_1 R_2 / R_s, \quad \Omega_2 = R_1 R_2 R_s, \quad \Omega_2 = R_1 R_2 R_s
\]  
(B.6)

For the first and second Rayleigh pole, $\beta_R$ in Eqs. (B.2)-(B.6) should be replaced by $\beta_{R_1}$ and $\beta_{R_2}$, respectively.

\[
S_0 = \frac{i\pi (a_2 - a_1) \sqrt{1 - \beta_{R_1}^2} \sqrt{1 - \beta_{R_2}^2}}{(g_3^0 \sqrt{1 - \beta_{R_1}^2} - g_3^4 \sqrt{1 - \beta_{R_2}^2}) \left( 2 - \beta_{R_1}^2 \right) + 4(a_2 - a_1) \sqrt{1 - \beta_{R_1}^2} \sqrt{1 - \beta_{R_2}^2} \left( 1 - \beta_{R_1}^2 \right) \left( 1 - \beta_{R_2}^2 \right)}
\]  
(B.7)

where

\[
\gamma_3^0 = (\lambda + 2)a_1 \beta_{R_1}^2 - 2a_1 - \alpha / k_1^2, \quad \gamma_3^4 = (\lambda + 2)a_2 \beta_{R_2}^2 - 2a_2 - \alpha / k_1^2
\]  
(B.8)

\[
2G_1 = -2i \int_{-\infty}^{\infty} \frac{(a_2 - a_1) Q_{p1} Q_{p2}^*}{\left( g_3^0 \sqrt{1 - \beta_{R_1}^2} - g_3^4 \sqrt{1 - \beta_{R_2}^2} \right) \left( 2 - \beta_{R_1}^2 \right) + 4(a_2 - a_1) \sqrt{1 - \beta_{R_1}^2} \sqrt{1 - \beta_{R_2}^2} \left( 1 - \beta_{R_1}^2 \right)} \exp \left( -\eta k_1 \right) \frac{1}{\eta} \, d\eta
\]  
(B.9)

where

\[
\gamma_3^0 = (\lambda + 2)a_1 \beta_{R_1}^2 - 2a_1 - \alpha / k_1^2, \quad \gamma_3^4 = (\lambda + 2)a_2 \beta_{R_2}^2 - 2a_2 - \alpha / k_1^2
\]  
(B.10)

\[
2G_2 = -2i \int_{-\infty}^{\infty} \frac{(a_2 - a_1) Q_{p1} Q_{p2}^*}{\left( g_3^0 \sqrt{1 - \beta_{R_1}^2} - g_3^4 \sqrt{1 - \beta_{R_2}^2} \right) \left( 2 - \beta_{R_1}^2 \right) + 4(a_2 - a_1) \sqrt{1 - \beta_{R_1}^2} \sqrt{1 - \beta_{R_2}^2} \left( 1 - \beta_{R_1}^2 \right)} \exp \left( -\eta k_1 \right) \frac{1}{\eta} \, d\eta
\]  
(B.11)

\[
2G_3 = 2i \int_{-\infty}^{\infty} \frac{4(1 - \eta^2)(a_2 - a_1) Q_{p1} Q_{p2}^*}{\left( g_3^0 \sqrt{1 - \beta_{R_1}^2} - g_3^4 \sqrt{1 - \beta_{R_2}^2} \right) \left( 2 - \beta_{R_1}^2 \right) + 4(a_2 - a_1) \sqrt{1 - \beta_{R_1}^2} \sqrt{1 - \beta_{R_2}^2} \left( 1 - \beta_{R_1}^2 \right)} \exp \left( -\eta k_1 \right) \frac{1}{\eta} \, d\eta
\]  
(B.12)

(b) Regime $C_{R_1} < V_{ph} < C_s$

\[
\int_{-\infty}^{\infty} f(\xi) d\xi = S_2 + S_3 + S_0 - 2G_1 - 2G_2 - 2G_3
\]  
(B.13)

where

\[
S_2 = S_B \times \exp \left( ik_1 \sqrt{\beta_{R_1}^2} \right)
\]  
(B.14)

(c) Regime $C_s < V_{ph} < C_{R_2}$

\[
\int_{-\infty}^{\infty} f(\xi) d\xi = S_2 + S_3 - 2G_1 - 2G_2 - 2G_4 - 2G_5
\]  
(B.15)

\[
S_2 = S_B \times \exp \left( ik_1 \sqrt{\beta_{R_1}^2} - 1 \right)
\]  
(B.16)

\[
S_3 = S_B \times \exp \left( ik_1 \sqrt{\beta_{R_2}^2} - 1 \right)
\]  
(B.17)
where

\[ g_3^4 = (\lambda + 2)a_1^3\beta_1^2 - \alpha/k_1^2 - 2a_1(1 + \xi^2)^2 \]

(B.22)

(d) Regime \( C_{p_1} < V_{ph} < C_{p_2} \)

\[ \int_{-\infty}^{\infty} f(\xi) \, d\xi = S_2 + S_3 - 2C_7 - 2G_6 - 2G_7 - 2G_8 \]  

(B.23)

\[ \int_{-\infty}^{\infty} (a_2 - a_1)Q_{p_1}Q_{p_2} + \frac{\exp(-\eta k_1) - 1}{\eta} \, d\eta \]

(B.18)

\[ \int_{-\infty}^{\infty} (a_2 - a_1)Q_{p_1}Q_{p_2} + \frac{\exp(-\eta k_1) - 1}{\eta} \, d\eta \]

(B.20)

\[ \int_{0}^{\infty} (a_2 - a_1)^2 \left( \frac{Q_{p_1}Q_{p_2}}{g_2^4E_{p_1}^2} \right)^2 \frac{\exp(-\eta k_1) - 1}{\eta} \, d\eta \]

(B.21)

(e) Regime \( V_{ph} > C_{p_1} \)

\[ \int_{-\infty}^{\infty} f(\xi) \, d\xi = S_2 + S_3 - 2G_9 - 2G_{10} - 2G_{11} - 2G_8 \]  

(B.27)

\[ \int_{0}^{\infty} (a_2 - a_1)Q_{p_1}Q_{p_2} + \frac{\exp(-\eta k_1) - 1}{\eta} \, d\eta \]

(B.24)

\[ \int_{0}^{\infty} (a_2 - a_1)^2 \left( \frac{P_{p_1}P_{p_2}}{g_2^4E_{p_1}^2} \right)^2 \frac{\exp(-\eta k_1) - 1}{\eta} \, d\eta \]

(B.25)

\[ \int_{0}^{\infty} (a_2 - a_1)^2 \left( \frac{P_{p_1}P_{p_2}}{g_2^4E_{p_1}^2} \right)^2 \frac{\exp(-\eta k_1) - 1}{\eta} \, d\eta \]

(B.26)

Appendix C

(a) Regime \( V_{ph} < C_3 \)

\[ \int_{-\infty}^{\infty} f(\xi) \, d\xi = S_0 - 2G_1 - 2G_2 - 2G_3 \]  

(C.1)

where \( S_0 \) represents the integration along the semi-circular contour around \( \xi = 0 \) and its expression is given in Eq. (33). The expressions of integrations along the branch cuts are given as

\[ 2G_1 = 2i \int_{-\infty}^{\infty} \frac{(a_2 - a_1)Q_{p_1}Q_{p_2} + (1 - \eta^2)(a_1Q_{p_1}^*-a_2Q_{p_2}^*)}{Q_{p_2}} \frac{\exp(-\eta k_1) - 1}{\eta} \, d\eta \]

(C.2)

\[ 2G_2 = 2i \int_{-\infty}^{\infty} \frac{(a_2 - a_1)Q_{p_1}Q_{p_2} + (1 - \eta^2)(a_2 - a_1)Q_{p_1}Q_{p_2}}{Q_{p_2}} \frac{\exp(-\eta k_1) - 1}{\eta} \, d\eta \]

(C.3)
\[ 2G_3 = 2i \int_{1 / \sqrt{1 - \beta_{11}^2}}^{1 / \sqrt{1 - \beta_{12}^2}} \frac{(1 - \eta^2)(a_2 - a_1) \exp(-\eta k_1) \, d\eta}{Q^e} \]  
(C.4)

(b) Regime \( C_s < V_{ph} < C_{p2} \)

\[ \int_{-\infty}^{+\infty} f(\xi) \, d\xi = 2G_1 - 2G_2 - 2G_4 - 2G_5 \]  
(C.5)

\[ 2G_1 = 2i \int_{1 / \sqrt{1 - \beta_{11}^2}}^{1 / \sqrt{1 - \beta_{12}^2}} \frac{(1 - \eta^2)(a_2 - a_1) - Q^e \left( a_1 Q_{p1}^+ - a_2 Q_{p2}^+ \right) \exp(-\eta k_1) - 1 \, d\eta}{\eta} \]  
(C.6)

\[ 2G_2 = 2i \int_{1 / \sqrt{1 - \beta_{11}^2}}^{1 / \sqrt{1 - \beta_{12}^2}} \frac{a_2 Q_{p1} Q^+ + (1 - \eta^2)(a_2 - a_1) \exp(-\eta k_1) - 1 \, d\eta}{Q^e} \]  
(C.7)

\[ 2G_4 = 2i \int_{1 / \sqrt{1 - \beta_{11}^2}}^{1 / \sqrt{1 - \beta_{12}^2}} \frac{(1 - \eta^2)(a_2 - a_1) \exp(-\eta k_1) - 1 \, d\eta}{\eta} \]  
(C.8)

\[ 2G_5 = -2i \int_0^{\sqrt{\beta_{11}^2 - 1}} \frac{(1 + \xi^2)(a_2 - a_1) \exp(ik_1 \xi) - 1 \, d\xi}{P^-} \]  
(C.9)

(c) Regime \( C_{p2} < V_{ph} < C_{p1} \)

\[ \int_{-\infty}^{+\infty} f(\xi) \, d\xi = -2G_1 - 2G_6 - 2G_7 - 2G_8 \]  
(C.10)

\[ 2G_6 = 2i \int_0^{\sqrt{\beta_{11}^2 - 1}} \frac{a_2 Q_{p1} Q^e + (1 - \eta^2)(a_2 - a_1) \exp(-\eta k_1) - 1 \, d\eta}{Q^e} \]  
(C.11)

\[ 2G_7 = -2i \int_0^{\sqrt{\beta_{11}^2 - 1}} \frac{a_2 P_{p1} P^- + (1 + \xi^2)(a_2 - a_1) \exp(ik_1 \xi) - 1 \, d\xi}{P^-} \]  
(C.12)

\[ 2G_8 = -2i \int_0^{\sqrt{\beta_{11}^2 - 1}} \frac{(1 + \xi^2)(a_2 - a_1) \exp(ik_1 \xi) - 1 \, d\xi}{P^-} \]  
(C.13)

(d) Regime \( V_{ph} > C_{p1} \)

\[ \int_{-\infty}^{+\infty} f(\xi) \, d\xi = -2G_9 - 2G_{10} - 2G_{11} - 2G_8 \]  
(C.14)

\[ 2G_9 = 2i \int_0^{\sqrt{\beta_{11}^2 - 1}} \frac{(1 - \eta^2)(a_2 - a_1) - Q^e \left( a_1 Q_{p1}^+ - a_2 Q_{p2}^+ \right) \exp(-\eta k_1) - 1 \, d\eta}{P^-} \]  
(C.15)

\[ 2G_{10} = -2i \int_0^{\sqrt{\beta_{11}^2 - 1}} \frac{(1 + \xi^2)(a_2 - a_1) - P^- \left( a_1 P_{p1} - a_2 P_{p2} \right) \exp(ik_1 \xi) - 1 \, d\xi}{P^-} \]  
(C.16)

\[ 2G_{11} = -2i \int_0^{\sqrt{\beta_{11}^2 - 1}} \frac{a_2 P_{p2} P^- + (1 + \xi^2)(a_2 - a_1) \exp(ik_1 \xi) - 1 \, d\xi}{P^-} \]  
(C.17)

**Appendix D**

(a) Regime \( V_{ph} < C_s \)

\[ \int_{-\infty}^{+\infty} f(\xi) \, d\xi = S_0 - 2G_1 - 2G_2 - 2G_3 \]  
(D.1)

where \( S_0 \) represents the integration along the semi-circular contour around \( \xi = 0 \); \( G_1, G_2 \) and \( G_3 \) are integrations along the branch cut lines 1–6. Their expressions are obtained as

\[ S_0 = \pi a_1 \sqrt{1 - \beta_{11}^2 - a_1 \sqrt{1 - \beta_{12}^2}} + (a_1 - a_2) \sqrt{1 - \beta_{11}^2} \sqrt{1 - \beta_{12}^2} \sqrt{h_2 \sqrt{1 - \beta_{12}^2} - h_1 \sqrt{1 - \beta_{12}^2}} \]  
(D.2)

where

\[ h_1 = (\lambda + 2) \chi_1 - \alpha, \ h_2 = (\lambda + 2) \chi_2 - \alpha \]  
(D.3)
\[ 2G_1 = -2i \int_{\sqrt{1-\beta_p^2}}^{\infty} \frac{(a_1 - a_2) Q_{p1}^+ Q_{p2}^+ Q_{p3}^+ + (1 - \eta^2)(a_1 Q_{p2}^+ - a_2 Q_{p3}^+)}{Q_{p1}^+ (h_z Q_{p1}^+ - h_i Q_{p2}^+)} \exp(-\eta k_1) \, d\eta \]  
(D.4)

\[ 2G_2 = 2i \int_{\sqrt{1-\beta_p^2}}^{\infty} \frac{(a_1 - a_2) h_1 a_1 (Q_{p2}^+)^2 + h_2 a_2 (Q_{p3}^+)^2 - (a_1 - a_2) h_2 Q_{p2}^+ Q_{p3}^+ (Q_{p1}^+)^2}{Q_{p1}^+ (h_z Q_{p1}^+ - h_i Q_{p2}^+)} \exp(-\eta k_1) \, d\eta \]  
(D.5)

\[ 2G_3 = 2i \int_{\sqrt{1-\beta_p^2}}^{\infty} \frac{(a_1 - a_2) h_1 a_1 (Q_{p2}^+)^2 + h_2 a_2 (Q_{p3}^+)^2 - (a_1 - a_2) h_2 Q_{p2}^+ Q_{p3}^+ (Q_{p1}^+)^2}{Q_{p1}^+ (h_z Q_{p1}^+ - h_i Q_{p2}^+)} \exp(-\eta k_1) \, d\eta \]  
(D.6)

(b) Regime \( C_s < V_{ph} < C_{p2} \)

\[ \int_{-\infty}^{\infty} f(\xi) \, d\xi = -2G_1 - 2G_2 - 2G_4 - 2G_5 \]  
(D.7)

\[ 2G_1 = -2i \int_{\sqrt{1-\beta_p^2}}^{\infty} \frac{(a_1 - a_2) Q_{p1}^+ Q_{p2}^+ Q_{p3}^+ + (1 - \eta^2)(a_1 Q_{p2}^+ - a_2 Q_{p3}^+)}{Q_{p1}^+ (h_z Q_{p1}^+ - h_i Q_{p2}^+)} \exp(-\eta k_1) \, d\eta \]  
(D.8)

\[ 2G_2 = 2i \int_{\sqrt{1-\beta_p^2}}^{\infty} \frac{(1 - \eta^2) (a_2 Q_{p1}^- - a_1 Q_{p2}^-)}{Q_{p1}^+ (h_z Q_{p1}^+ - h_i Q_{p2}^+)} \exp(-\eta k_1) - 1 \, d\eta \]  
(D.9)

\[ 2G_4 = 2i \int_{0}^{\sqrt{1-2\beta_p^2 - 2G_1}} \frac{(1 + \xi^2) (a_2 P_{p1}^+ - a_1 P_{p2}^+)}{P_{p1}^+ (h_z P_{p1}^+ - h_i P_{p2}^+)} \exp(i\xi k_1) - 1 \, d\xi \]  
(D.10)

\[ 2G_5 = -2i \int_{0}^{\sqrt{1-2\beta_p^2 - 2G_1}} \frac{(1 + \xi^2) (a_1 P_{p1}^- - a_2 P_{p2}^-) \exp(i\xi k_1) - 1}{\xi} \, d\xi \]  
(D.11)

(c) Regime \( C_{p2} < V_{ph} < C_{p1} \)

\[ \int_{-\infty}^{\infty} f(\xi) \, d\xi = -2G_1 - 2G_4 - 2G_7 - 2G_8 \]  
(D.12)

\[ 2G_6 = 2i \int_{0}^{\sqrt{1-2\beta_p^2 - 2G_1}} \frac{(1 - \eta^2) (a_1 P_{p1}^+ + a_2 P_{p2}^+)}{P_{p1}^+ (h_z P_{p1}^+ - h_i P_{p2}^+)} \exp(-\eta k_1) - 1 \, d\eta \]  
(D.13)

\[ 2G_7 = -2i \int_{0}^{\sqrt{1-2\beta_p^2 - 2G_1}} \frac{(1 + \xi^2) (a_1 P_{p1}^+ + a_2 P_{p2}^+)}{P_{p1}^+ (h_z P_{p1}^+ - h_i P_{p2}^+)} \exp(i\xi k_1) - 1 \, d\xi \]  
(D.14)

\[ 2G_8 = -2i \int_{0}^{\sqrt{1-2\beta_p^2 - 2G_1}} \frac{(1 + \xi^2) (a_2 P_{p1}^- - a_1 P_{p2}^-) \exp(i\xi k_1) - 1}{\xi} \, d\xi \]  
(D.15)

(d) Regime \( V_{ph} > C_{p1} \)

\[ \int_{-\infty}^{\infty} f(\xi) \, d\xi = -2G_3 - 2G_{10} - 2G_{11} - 2G_8 \]  
(D.16)

\[ 2G_9 = -2i \int_{0}^{\infty} \frac{(a_1 - a_2) Q_{p1}^+ Q_{p2}^+ Q_{p3}^+ + (1 - \eta^2)(a_1 Q_{p2}^+ - a_2 Q_{p3}^+)}{Q_{p1}^+ (h_z Q_{p1}^+ - h_i Q_{p2}^+)} \exp(-\eta k_1) - 1 \, d\eta \]  
(D.17)

\[ 2G_{10} = 2i \int_{0}^{\sqrt{1-2\beta_p^2 - 2G_1}} \frac{(a_1 - a_2) P_{p1}^- P_{p2}^- P_{p3}^- + (1 + \xi^2) (a_1 P_{p2}^- - a_2 P_{p3}^-) \exp(i\xi k_1) - 1}{\xi} \, d\xi \]  
(D.18)

\[ 2G_{11} = -2i \int_{0}^{\sqrt{1-2\beta_p^2 - 2G_1}} \frac{(1 + \xi^2) (h_1 a_1 (P_{p2}^+)^2 + h_2 a_2 (P_{p3}^+)^2) - (a_1 - a_2) h_2 P_{p2}^- P_{p3}^- (P_{p1}^+)^2}{\xi} \, d\xi \]  
(D.19)
Appendix E

$$\beta_1(\omega) = \frac{m\omega^2(\lambda + \alpha^2 M + 2) + (1 - 2\alpha \rho_1) M\omega^2}{(\lambda + 2) M} : \beta_2(\omega) = \frac{m - \rho_1^2}{(\lambda + 2) M} \omega^4$$  \hspace{1cm} (E.1)

$$L_1^2(\omega) = \frac{\beta_1 - \sqrt{\beta_1^2 - 4\beta_2}}{2} : L_2^2(\omega) = \frac{\beta_1 + \sqrt{\beta_1^2 - 4\beta_2}}{2}$$  \hspace{1cm} (E.2)

$$S^2(\omega) = \omega^2(1 - \vartheta \rho_1) : \vartheta = \rho_1/m$$  \hspace{1cm} (E.3)

$$a_1(\omega) = \frac{\chi_1(\lambda + 1) - \alpha + \vartheta}{S^2 - L_1^2} : a_2(\omega) = \frac{\chi_2(\lambda + 1) - \alpha + \vartheta}{S^2 - L_2^2}$$  \hspace{1cm} (E.4)

$$\chi_1(\omega) = a_1 L_1^2; \chi_2(\omega) = a_2 L_2^2$$  \hspace{1cm} (E.5)

$$g_1(k_1, \omega) = 2a_1 k_1^2; g_2(k_1, \omega) = 2a_2 k_1^2$$  \hspace{1cm} (E.6)

$$g_3(k_1, k_2, \omega) = \lambda \chi_1 - 2a_1 k_1^2 - \alpha; g_4(k_1, k_2, \omega) = \lambda \chi_2 - 2a_2 k_2^2 - \alpha$$  \hspace{1cm} (E.7)

$$\Delta_A(k_1, k_2, \omega) = 4\gamma_3(a_2 k_2 - a_1 \gamma_1)(k_1^2 + k_2^2) - (a_2 - a_1)[2(k_1^2 + k_2^2) - S^2]^2$$  \hspace{1cm} (E.8)

$$\Delta_B(k_1, k_2, \omega) = (g_4 \gamma_1 - g_3 \gamma_2)[2(k_1^2 + k_2^2) - S^2] + 4\gamma_1 \gamma_2 \gamma_3 (a_2 - a_1)(k_1^2 + k_2^2)$$  \hspace{1cm} (E.9)

$$\Delta_C(k_1, k_2, \omega) = \gamma_3(\lambda + 2)(\chi_2 - \chi_1)$$  \hspace{1cm} (E.10)

$$\Delta_D(k_1, k_2, \omega) = \gamma_3(\lambda + 2)(\chi_2(\gamma_1 - \chi_1) + \chi_1(\gamma_1 - \gamma_2))$$  \hspace{1cm} (E.11)

References


