INTERNATIONAL JOURNAL FOR NUMERICAL AND ANALYTICAL METHODS IN GEOMECHANICS
Int. J. Numer. Anal. Meth. Geomech. 2016; **40**:2003–2016
Published online 14 March 2016 in Wiley Online Library (wileyonlinelibrary.com). DOI: 10.1002/nag.2519

A note on the consolidation settlement of a rigid circular foundation on a poroelastic halfspace

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SUMMARY

This paper uses Biot's poroelasticity approach to examine the consolidation behaviour of a rigid foundation with a frictionless base in contact with a poroelastic halfspace. The mathematical development of the mixed boundary value problem involves a set of dual integral equations in the Laplace transform domain which cannot be conveniently solved by employing conventional procedures. In this paper, a numerical solution is developed using a scheme where the contact normal stress is approximated by a discretized equivalent. The influence of limiting drainage boundary conditions at the entire surface of the halfspace on the degree of consolidation of the rigid circular foundation is investigated. The results obtained in this study are compared with the corresponding results given in the literature. Copyright © 2016 John Wiley & Sons, Ltd.

Received 22 September 2015; Revised 12 January 2016; Accepted 6 February 2016

KEY WORDS: Biot poroelasticity; drainage boundary conditions; rigid circular foundation; discretization technique

1. INTRODUCTION

Consolidation is a typical problem of poroelasticity in geomechanics. The earliest recognition of soil consolidation is attributed to Terzaghi [1] who developed a one-dimensional theory. The complete theory, which includes the three-dimensional deformation of a porous medium, was developed by Biot [2–4] who studied the consolidation processes in porous media. Biot's studies were further developed by McNamee and Gibson [5, 6] who examined the circular and strip loading of the poroelastic halfspace. In their studies, the loading was uniformly distributed and normal to surface of the poroelastic halfspace. The consolidation because of a uniform tangential load applied over a circular region of the halfspace was examined by Schiffman and Fungaroli [7]. Booker [8] presented the solutions for the circular and strip loadings for a finite poroelastic layer. The solutions were examined for either smooth or bonded rigid base conditions. The sensitivity of the consolidation on the ratio of the loading size to the layer depth was also investigated. This study was then extended by Booker and Small [9, 10] for general surface loading (circular, strip, and square) and different surface (pervious/impervious) and base (smooth/rigid) conditions. The applications of Biot's theory are too numerous to be cited in their entirety; the reader is referred to review articles and volumes by Scheidegger [11], Paria [12], Zaretskii [13], Schiffman [14], Detournay and Cheng [15], Coussy [16], Selvadurai [17, 18], de Boer [19], and Ehlers and Bluhm [20] for further references.

Contact problems impose mixed initial boundary conditions at the contact surface in terms of stresses, displacements, and pore pressure. The mathematical solution of the mixed initial boundary

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value problem involves a set of dual integral equations in the Laplace transform domain which cannot be solved by using conventional procedures. Agbezuge and Deresiewicz [21, 22] developed approximate solutions to spherical and flat profiles. Chiarella and Booker [23] analytically solved the contact problem of rigid disc resting on a deep clay halfspace using an axisymmetric formulation. The circular disc was assumed to be frictionless, and the entire surface was considered permeable. Selvadurai and Yue [24] examined the indentation of the poroelastic layer by a rigid circular disc to include the compressibility of the pore fluid. Assuming that the base is impervious and rough, they considered two limiting cases for the surface boundary: (i) the entire surface is permeable and (ii) the entire surface is impervious. This study was extended for the poroelastic halfspace by Yue and Selvadurai [25]. They divided the surface into two regions, interior and exterior to the indenter and specified different boundary conditions to investigate the influence of drainage conditions on the consolidation process. Complete mathematical analyses for inclusion and asymmetric indentation problems are also given in Yue and Selvadurai [26, 27].

This paper examines the consolidation settlement of a rigid circular foundation resting in smooth contact with a poroelastic halfspace. An alternative approach, where the contact stress is approximated by its discretized equivalents, is employed to avoid the non-routine solution procedure for poroelastic contact problems. This discretization technique has been used quite extensively in elastic contact problems. Selvadurai [28] presented the analysis of consolidation settlement of a rigid circular plate resting on an elastic halfspace by using this discretization technique. This technique was also used by Rajapakse and Selvadurai [29] to examine the axisymmetric problem of flexible circular footings and anchor plates in a linearly non-homogeneous elastic halfspace. Selvadurai and Katebi [30] further extended this approach to include the adhesive contact stresses. In their study, the contact stress acts both normal and tangential to the surface of the elastic halfspace. In the current research, we extend the use of the discretization approach to contact problems in poroelasticity. Recently, Selvadurai and Shi [31] have extended Biot's classical problem of the flexure of a beam of finite width on an elastic halfspace to include poroelastic effects using the discretization technique. The solution of the problem is facilitated through the use of 'Selvadurai's bounding Method'. The problem solved in this research is an axisymmetric problem assuming that the foundation has a flat base. The influence of limiting drainage boundary conditions (i.e. pervious/impervious) at the entire surface of the halfspace on the degree of consolidation of the rigid circular foundation is investigated.

2. GOVERNING EQUATIONS

This study is restricted to the axisymmetric indentation problem of a fully-saturated poroelastic halfspace. The governing equations are given in relation to the state of deformation defined by the displacement components and pore pressure as $\mathbf{u}(\mathbf{x}) = \{u_r(r,z,t), 0, u_z(r,z,t)\}$; $p(\mathbf{x}) = p(r,z,t)$. The fully coupled equations governing the mechanics of a poroelastic medium are given in several references (see e.g. [2, 17, 18, 24]). The relevant final forms for the equations governing the displacement and pore pressure fields applicable to an axisymmetric state are given by

$$G\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) - (2\eta - 1)\frac{\partial \Theta}{\partial r} = \alpha \frac{\partial p}{\partial r} \tag{1}$$

$$G\nabla^2 u_z - (2\eta - 1)\frac{\partial \Theta}{\partial z} = \alpha \frac{\partial p}{\partial z} \tag{2}$$

$$\beta \frac{\partial p}{\partial t} - \gamma \frac{\partial \Theta}{\partial t} = c\nabla^2 p \tag{3}$$

where

DOI: 10.1002/nag

$$\alpha = \frac{3(\nu_u - \nu)}{B(1-2\nu)(1+\nu_u)} \quad ; \quad \beta = \frac{(1-2\nu_u)(1-\nu)}{(1-2\nu)(1-\nu_u)} \quad ; \quad \gamma = \frac{2GB(1-\nu)(1+\nu_u)}{3(1-2\nu)(1-\nu_u)}$$
$$c = \frac{2GB^2(1-\nu)(1+\nu_u)^2 k}{9(\nu_u - \nu)(1-\nu_u)\gamma_w} \quad ; \quad \eta = \frac{(1-\nu)}{(1-2\nu)} \quad ; \quad \Theta = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \tag{4}$$

In Eqns (1)–(4), G is the linear shear modulus and ν is Poisson's ratio of the porous skeleton (i.e. the drained elastic parameters); ν_u is the undrained Poisson's ratio of the fluid-saturated medium; k is the hydraulic conductivity; B is Skempton's pore pressure parameter [32]; and ∇^2 is the axisymmetric form of Laplace's operator given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{5}$$

Certain thermodynamic constraints need to be satisfied to ensure positive definiteness of the strain energy potential [33]; it can be shown that these constraints can be expressed in the forms: $G > 0$; $0 \le B \le 1$; $-1 < \nu < \nu_u \le 0.5$. Alternative but equivalent descriptions are also available and further references are given in Selvadurai [18].

2.1. Methods of solution of the governing equations

The method of solution of the governing equations largely depends on the type of initial boundary value problems being examined and usually the solution procedures are either direct or indirect. The direct approach [24, 27, 34] uses integral transform (Fourier, Hankel, and Laplace) representations of the dependent variables and establishes a system of ordinary differential equations for a set of unknown functions describing the dependent variables. The alternative approach is to introduce special functions that will reduce the governing equations to canonical forms such that the uniqueness of the solution is assured. This indirect approach has been used by many researchers including Mandel [35], de Josselin de Jong [36], Gibson and McNamee [37], and McNamee and Gibson [5, 6]. We adopt the approach proposed by McNamee and Gibson [5] where the partial differential Eqns (1)–(3) governing poroelastic behaviour for axisymmetric states of deformation can be represented in terms of two scalar functions $S(r,z,t)$ and $E(r,z,t)$, which satisfy

$$\nabla^2 S = 0 \tag{6}$$

$$c\nabla^4 E = \left(\beta + \frac{\alpha\gamma}{2G\eta}\right)\nabla^2 \frac{\partial E}{\partial t} - \frac{\beta}{\eta}\frac{\partial^2 S}{\partial z \partial t}. \tag{7}$$

The displacements, stresses, and pore fluid pressure can be uniquely represented in terms of $S(r,z,t)$ and $E(r,z,t)$ as follows:

$$u_r = -\frac{\partial E}{\partial r} + z\frac{\partial S}{\partial r}; \quad u_z = -\frac{\partial E}{\partial z} + z\frac{\partial S}{\partial z} - S \tag{8}$$

$$\Theta = \nabla^2 E; \quad p = \frac{2G}{\alpha}\left(\frac{\partial S}{\partial z} - \eta\Theta\right) \tag{9}$$

$$\left.\begin{array}{l} \dfrac{\sigma_{rr}}{2G} = \left(\dfrac{\partial^2}{\partial r^2} - \nabla^2\right)E - z\dfrac{\partial^2 S}{\partial r^2} + \dfrac{\partial S}{\partial z}; \quad \dfrac{\sigma_{zz}}{2G} = \left(\dfrac{\partial^2}{\partial z^2} - \nabla^2\right)E - z\dfrac{\partial^2 S}{\partial z^2} + \dfrac{\partial S}{\partial z} \\ \dfrac{\sigma_{rz}}{2G} = \dfrac{\partial^2 E}{\partial r \partial z} - z\dfrac{\partial^2 S}{\partial r \partial z} \end{array}\right\}. \tag{10}$$

DOI: 10.1002/nag

The accuracy of the representations in terms of $S(r,z,t)$ and $E(r,z,t)$ can be verified by back-substitution into Eqns (1)–(3). The analytical solutions for the time-dependent poroelastic problems are conveniently obtained by using integral transform techniques. The Laplace and the zeroth-order Hankel transforms are used to remove, respectively, time-dependency and dependency in the radial coordinate: i.e.

$$\overline{F}(\xi,z,t) = \int_0^\infty rJ_0(\xi r)F(r,z,t)dr \tag{11}$$

$$\tilde{\overline{F}}(\xi,z,s) = \frac{1}{2\pi i}\int_0^\infty e^{-st}\overline{F}(\xi,z,t)dt \tag{12}$$

where $(^{-})$ refers to the zeroth-order Hankel transform and $(^{\sim})$ refers to the Laplace transform of a particular function. After successive application of Laplace and zeroth-order Hankel transforms, the PDEs governing Eqns (6) and (7) give the following ODEs for the transformed variables of $\tilde{\overline{S}}(r,z,t)$ and $\tilde{\overline{E}}(r,z,t)$:

$$\left(\frac{d^2}{dz^2} - \xi^2\right)\tilde{\overline{S}} = 0 \tag{13}$$

$$\left(\frac{d^2}{dz^2} - \xi^2\right)\left\{\frac{d^2}{dz^2} - \left[\xi^2 + \frac{s}{c}\left(\beta + \frac{\alpha\gamma}{2G\eta}\right)\right]\right\}\tilde{\overline{E}} = -\frac{\beta s}{\eta c}\frac{d\tilde{\overline{S}}}{dz}. \tag{14}$$

3. CONTACT PROBLEM OF A RIGID CIRCULAR FOUNDATION ON A POROELASTIC HALFSPACE

We consider the axisymmetric problem of a rigid circular foundation resting in smooth contact with a poroelastic halfspace (Figure 1) where the foundation is subjected to an axial load $P_0\mathrm{H}(t)$, where $\mathrm{H}(t)$ is the Heaviside step function. A foundation of radius a rests on the surface $z=0$ of a poroelastic halfspace which occupies the region $r\in(0,\infty)$; $z\in(0,\infty)$. The rigid circular foundation has a flat base which is necessary to ensure axial symmetry of the indentation problem.

For a smooth contact problem, the boundary conditions ($t\geq 0$) are given by

$$u_z(r,0,t) = \Delta(t) \qquad 0\leq r\leq a \tag{15}$$

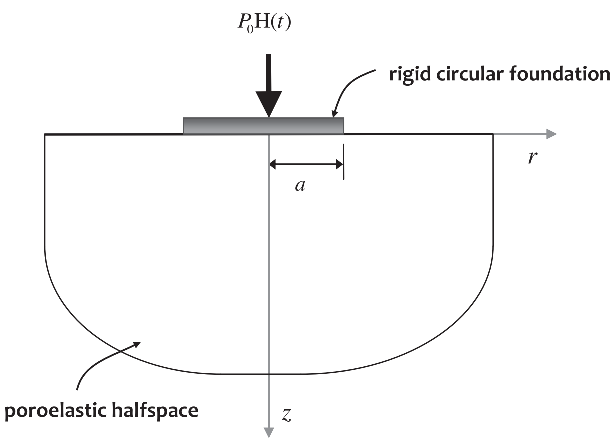

Figure 1. Indentation of a poroelastic halfspace.

DOI: 10.1002/nag

$$\sigma_{zz}(r,0,t) = 0 \qquad a < r < \infty \tag{16}$$

$$\sigma_{rz}(r,0,t) = 0 \qquad 0 < r < \infty \tag{17}$$

where Δ is the axial displacement of the rigid circular foundation. Two types of the drainage boundary conditions can be considered:

Case I: The entire surface is *completely permeable*

$$p(r,0,t) = 0 \qquad 0 < r < \infty \tag{18}$$

Case II: The entire surface is *completely impervious*

$$v_z(r,0,t) = 0 \qquad 0 < r < \infty \tag{19}$$

where v_z is the specific discharge in the z-direction.

In addition, the solutions in the above two cases should satisfy the regularity conditions that are applicable to a halfspace domain

$$\boldsymbol{\sigma}(\mathbf{x},t), \quad p(\mathbf{x},t), \quad \mathbf{u}(\mathbf{x},t) \to 0 \text{ as } |\mathbf{x}| \to \infty, \quad t \in (0,\infty). \tag{20}$$

4. ANALYTICAL SOLUTIONS FOR THE UNIFORM CIRCULAR LOADING OF A POROELASTIC HALFSPACE

Poroelastic contact problems for the halfspace impose mixed boundary conditions at the plane surface of the halfspace, and these will involve displacements, tractions, and pore fluid pressure (see Eqns (15)–(19)). The mathematical solution of the mixed boundary value problem can in general be reduced to a set of dual integral equation in the Laplace transform domain, which in turn can be reduced to a Fredholm integral equation of the second-kind in the Laplace transform domain. Examples of such a reduction are given by Chiarella and Booker [23] who examined the frictionless–pervious indentation problem and Yue and Selvadurai [25, 27] who examined contact and inclusion problems. An alternative approach can be developed by approximating the contact stress at the interface by its discretized equivalents; examples of this approach applicable to elastic contact problems are given by Selvadurai [28], Rajapakse and Selvadurai [29] and Selvadurai and Katebi [30]. Before we proceed further with the analysis of the poroelastic contact problem related to a rigid circular foundation, we first consider the problem where the halfspace region is subjected to a normal traction because of an external stress p^* over the circular area of radius a^* for two different drainage boundary conditions described in section 3.

Case I: The entire surface is *completely permeable*

$$p(r,0,t) = 0 \qquad 0 < r < \infty \tag{21}$$

$$\sigma_{rz}(r,0,t) = 0 \qquad 0 < r < \infty \tag{22}$$

$$\sigma_{zz}(r,0,t) = p^* \mathrm{H}(t) \qquad 0 \le r \le a^* \tag{23}$$

$$\sigma_{zz}(r,0,t) = 0 \qquad a^* < r < \infty \tag{24}$$

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Case II: The entire surface is *completely impervious*

$$v_z(r,0,t) = 0 \qquad 0 < r < \infty \tag{25}$$

$$\sigma_{rz}(r,0,t) = 0 \qquad 0 < r < \infty \tag{26}$$

$$\sigma_{zz}(r,0,t) = p^{*}\mathrm{H}(t) \qquad 0 \le r \le a^{*} \tag{27}$$

$$\sigma_{zz}(r,0,t) = 0 \qquad a^{*} < r < \infty \tag{28}$$

where P_0 is the total stress applied to the halfspace and p^* is defined as $p^* = \frac{P_0}{\pi a^{*2}}$.

The generalized solutions for ODEs (13) and (14) for $\tilde{\overline{S}}(\xi,z,s)$ and $\tilde{\overline{E}}(\xi,z,s)$ applicable to the poroelastic halfspace are

$$\begin{aligned}\tilde{\overline{S}}(\xi,z,s) &= Ae^{-\xi z}\\ \tilde{\overline{E}}(\xi,z,s) &= Be^{-\xi z} + Ce^{-\varphi z} + \Gamma Aze^{-\xi z}\end{aligned} \tag{29}$$

where

$$\varphi = \sqrt{\xi^2 + \frac{s}{c}\left(\beta + \frac{\alpha\gamma}{2G\eta}\right)}\ ;\ \Gamma = \frac{\beta G}{2\eta\beta G + \alpha\gamma}. \tag{30}$$

For each initial boundary value problem, three arbitrary functions of ξ and s are encountered in the formulation, and these can be determined by satisfying the boundary conditions given in Eqns (21)–(24) and Eqns (25)–(28), respectively. The coefficients *A, B, C* for Case I and II are given in Appendix A. The final expressions for the surface displacement in the z-direction are given by

Case I: The entire surface is *completely permeable*

$$u_z(r,0,t) = \frac{p^* a^*}{2G}\int_{\varsigma-i\infty}^{\varsigma+i\infty}\int_0^\infty \frac{\eta(\varphi+\xi)}{[\eta(\Gamma-1)(\varphi+\xi) - (2\Gamma\eta\xi - \xi)]\xi}\left(\frac{1}{s}\right)J_1(\xi a^*)J_0(\xi r)d\xi ds. \tag{31}$$

Case II: The entire surface is *completely impervious*

$$u_z(r,0,t) = \frac{p^* a^*}{2G}\int_{\varsigma-i\infty}^{\varsigma+i\infty}\int_0^\infty \frac{\eta\varphi(\varphi+\xi)}{[\eta\varphi(\Gamma-1)(\varphi+\xi) - \xi(2\Gamma\eta\xi - \xi)]\xi}\left(\frac{1}{s}\right)J_1(\xi a^*)J_0(\xi r)d\xi ds \tag{32}$$

where ς is a real number-associated Bromwich integral used in the Laplace transform inversion [38].

The solutions are not in exact closed form, and therefore it is necessary to evaluate the solutions numerically. In this study, the integral transforms are evaluated using the MATLAB® software. In particular, the inverse Laplace transform is evaluated by using the numerical inversion of the Laplace transform introduced in Crump [39]. The accuracy of the numerical results for both cases is confirmed through a comparison with the results given in McNamee and Gibson [6] for the axisymmetric surface loading, p^*, of a halfspace region. The numerical results and the analytical results given by [6] are presented in Table I. For Case I, the error substantially decreases when $t \geq 0.1$ and becomes less than 0.1% when $ct/a^{*2} = 10$. Case II shows the same pattern as Case I, but the errors in Case II are larger overall than for Case I. The procedures used are considered satisfactory for developing a full range of analytical results for the indentation problem of a rigid circular foundation on a poroelastic halfspace.

DOI: 10.1002/nag

Table I. Comparison of the numerical results with the analytical results in McNamee and Gibson [6].

	Permeable surface (Case I)			*Impermeable surface* (Case II)		
	McNamee and Gibson [6]			McNamee and Gibson [6]		
$\frac{ct}{a^2}$	$\frac{2G}{p*a*}\left(u_z - u_{z,t=0}\right)$	Present study	Error (%)	$\frac{2G}{p*a*}\left(u_z - u_{z,t=0}\right)$	Present study	Error (%)
0.01	0.1062	0.1128	6.2%	>0.01	0.0098	—
0.1	0.35	0.3528	0.8%	0.0920	0.0937	1.85%
1.0	0.7239	0.7288	0.68%	0.4832	0.4914	1.70%
10.0	0.9102	0.9105	0.03%	0.8174	0.8222	0.59%

5. THE DISCRETIZED APPROACH FOR THE SOLUTION OF THE POROELASTIC CONTACT PROBLEM

The solution method adopted in this study is based on the computational scheme for the solution of the analogous elasticity problem where the contact stress distribution between the rigid circular foundation and the elastic halfspace is discretized into equal annular areas (see Figure 2). More detailed descriptions can be found in Selvadurai [28] and Selvadurai and Katebi [30]. The radius of the equal annular area is obtained by

$$r_i = \left(\frac{i}{n}\right)^{\frac{1}{2}} a, \qquad i = 1, \quad 2, \quad 3, \ldots\ldots\ldots\ldots\ldots, \quad n. \tag{33}$$

The internal radii of the annular areas are 0, r_1, r_2, ….., r_{n-1}, and the external radii are r_1, r_2, r_3, ….., r_n. The contact stress of each annular area, $\tilde{\sigma}_i^{\,i}$, is assumed to be uniform over the annular region and evaluated at the mid-point of the internal and external radii. The mid-points are obtained by

$$r_{m_1} = 0, \quad r_{m_i} = \frac{(r_{i-1} + r_i)}{2}, \qquad i = 2,\ 3,\ 4, \ldots\ldots\ldots\ldots\ldots, \quad n. \tag{34}$$

The entire contact stress is the sum of the stresses evaluated at the mid-point of each annular area. Therefore, the entire contact stress applied to the poroelastic halfspace by the rigid circular indenter can be obtained by

$$\sigma = \sum_{i=1}^{n} \tilde{\sigma}_i p^*. \tag{35}$$

Using the expression (35), the vertical displacement of each annular area is expressed as

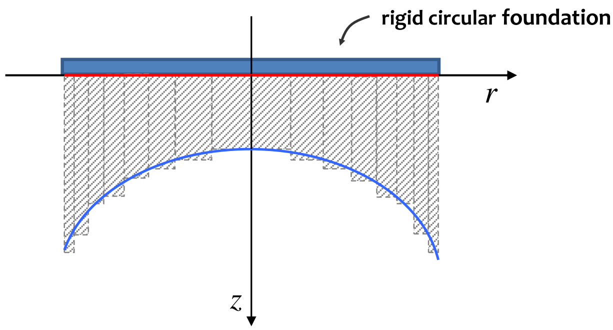

Figure 2. Discretization of contact loading by a rigid circular foundation.

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Case I: The entire surface is *completely permeable*

$$w_i = \frac{p^*a}{2G}\sum_{j=1}^{n}\int_{\varsigma-i\infty}^{\varsigma+i\infty}\int_0^\infty \frac{\eta(\varphi+\xi)}{[\eta(\Gamma-1)(\varphi+\xi)-(2\Gamma\eta\xi-\xi)]\xi}\left(\frac{1}{s}\right)\left(\frac{r_j}{a}J_1(\xi r_j)-\frac{r_{j-1}}{a}J_1(\xi r_{j-1})\right)J_0(\xi r_{m_i})d\xi ds. \tag{36}$$

Case II: The entire surface is *completely impervious*

$$w_i = \frac{p^*a}{2G}\sum_{j}^{n}\int_{\varsigma-i\infty}^{\varsigma+i\infty}\int_0^\infty \frac{\eta\varphi(\varphi+\xi)}{[\eta\varphi(\Gamma-1)(\varphi+\xi)-\xi(2\Gamma\eta\xi-\xi)]\xi}\left(\frac{1}{s}\right)\left(\frac{r_j}{a}J_1(\xi r_j)-\frac{r_{j-1}}{a}J_1(\xi r_{j-1})\right)J_0(\xi r_{m_i})d\xi ds \tag{37}$$

where $i, j = 1, 2, 3, \ldots, n$. The solution can also be written in the matrix form

$$\{\mathbf{w}\} = \frac{p^*a}{2G}[\mathbf{C}]\{\tilde{\boldsymbol{\sigma}}\} \tag{38}$$

where $\{\mathbf{w}\}$ and $\{\tilde{\boldsymbol{\sigma}}\}$ are, respectively, the vertical surface displacement and contact stress vectors and $[\mathbf{C}]$ is the coefficient matrix of the surface displacement, which is given by

$$[\mathbf{C}] = \begin{bmatrix} C_{11} & C_{12} & . & . & . & C_{1n-1} & C_{1n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ C_{n1} & C_{n2} & . & . & . & C_{nn-1} & C_{nn} \end{bmatrix} \tag{39}$$

where C_{ij} are the integral parts of Eqns (36) and (37).

Case I: The entire surface is *completely permeable* ($p = 0$ at $z = 0$)

$$c_{ij} = \int_{\varsigma-i\infty}^{\varsigma+i\infty}\int_0^\infty \frac{\eta(\varphi+\xi)}{[\eta(\Gamma-1)(\varphi+\xi)-(2\Gamma\eta\xi-\xi)]\xi}\left(\frac{1}{s}\right)\left(\frac{r_j}{a}J_1(\xi r_j)-\frac{r_{j-1}}{a}J_1(\xi r_{j-1})\right)J_0(\xi r_{m_i})d\xi ds \tag{40}$$

Case II: The entire surface is *completely impervious* ($v_z = 0$ at $z = 0$)

$$c_{ij} = \int_{\varsigma-i\infty}^{\varsigma+i\infty}\int_0^\infty \frac{\eta\varphi(\varphi+\xi)}{[\eta\varphi(\Gamma-1)(\varphi+\xi)-\xi(2\Gamma\eta\xi-\xi)]\xi}\left(\frac{1}{s}\right)\left(\frac{r_j}{a}J_1(\xi r_j)-\frac{r_{j-1}}{a}J_1(\xi r_{j-1})\right)J_0(\xi r_{m_i})d\xi ds \tag{41}$$

In Eqn (38), the coefficient matrix $[\mathbf{C}]$ is known; however, $\{\mathbf{w}\}$ and $\{\tilde{\boldsymbol{\sigma}}\}$ are still unknown. In order to solve the problem, two more constraints are required.

For the problem of a rigid circular indentation, the vertical surface displacements of the halfspace w_i have the same values at the interface between the indenter and the poroelastic halfspace and are equal to the displacement of the foundation.

$$w_1 = w_2 = \cdots\cdots = w_n = \Delta \tag{42}$$

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Because the contact stresses are related to the vertical surface displacements through the coefficient matrix, the following equation is applied to satisfy global equilibrium:

$$\sum_{i=1}^{n} \widetilde{\sigma}_i = n. \tag{43}$$

The vertical displacements $\{\mathbf{w}\}$ and the contact stresses $\{\widetilde{\boldsymbol{\sigma}}\}$ can now be determined by inverting the coefficient matrix $[\mathbf{C}]$.

6. NUMERICAL RESULTS AND COMPARISON TO THE RESULTS GIVEN IN THE LITERATURE

The accuracy of the solution method used in this study is validated by comparing the results with those given in Chiarella and Booker [23] and Yue and Selvadurai [25]. The consolidation rate is defined as

$$U_z = \frac{w(0,0,t^*) - w(0,0,0)}{w(0,0,\infty) - w(0,0,0)} \tag{44}$$

where t^* (=ct/a^2) is the non-dimensional time step. The initial and final responses, which are basically the elastic solutions, are obtained by taking the limits of the Laplace transforms (Eqns (31) and (32)) as $s \to \infty$ and $s \to 0$, respectively. In this study, the initial and final responses for both cases are obtained as

$$u_z(0,0,0) = \lim_{s\to\infty} u_z(0,0,s) = \frac{p^* a^*}{2G} \int_0^\infty -\frac{1}{\xi(\Gamma - 1)} J_1(\xi a^*) J_0(\xi r) d\xi \tag{45}$$

$$u_z(0,0,\infty) = \lim_{s\to 0} u_z(0,0,s) = \frac{p^* a^*}{2G} \int_0^\infty -\frac{2\eta}{\xi(1 - 2\eta)} J_1(\xi a^*) J_0(\xi r) d\xi. \tag{46}$$

The coefficient matrix of the surface displacement is modified accordingly. Table II shows the comparison of the analytical solutions obtained in this study with the results given by Chiarella and Booker [23] and Yue and Selvadurai [25] when the surface boundary is completely pervious. The agreement between the results obtained in this study and the values given in the literature is generally good. The maximum discrepancy to the results given by Chiarella and Booker [23] is approximately 2% for the cases of $\nu = 0$, $\nu_u = 0.5$ and $\nu = 0.2$, $\nu_u = 0.5$ but increases up to 5.6% for the case of $\nu = 0.4$, $\nu_u = 0.5$. The comparison with the results given by Yue and Selvadurai [25]

Table II. Comparison of the analytical results to the results given in Chiarella and Booker [23] and Yue and Selvadurai [25].

	$\nu = 0.0$			$\nu = 0.2$			$\nu = 0.4$		
$\sqrt{\frac{ct}{a^2}}$	Current study	Chiarella and Booker [23]	Yue and Selvadurai [25]	Current study	Chiarella and Booker [23]	Yue and Selvadurai [25]	Current study	Chiarella and Booker [23]	Yue and Selvadurai [25]
0.2	0.231	0.230	0.250	0.288	0.285	0.287	0.339	0.359	0.339
0.4	0.392	0.384	0.412	0.457	0.448	0.456	0.515	0.517	0.513
0.6	0.509	0.509	0.529	0.573	0.569	0.572	0.627	0.647	0.625
0.8	0.600	0.603	0.614	0.654	0.655	0.653	0.701	0.716	0.700
1.0	0.655	0.655	0.676	0.712	0.709	0.710	0.752	0.770	0.751
1.2	0.701	0.707	0.722	0.754	0.750	0.752	0.790	0.816	0.788
1.4	0.736	0.741	0.757	0.786	0.779	0.784	0.818	0.840	0.816

DOI: 10.1002/nag

shows a maximum discrepancy of 7.6% when $\nu = 0$, $\nu_u = 0.5$ but decreases to 0.4% when $\nu = 0.2$, $\nu_u = 0.5$ and $\nu = 0.4$, $\nu_u = 0.5$.

Figure 3 compares the contact stress ratio, $\sigma_{zz}(0,0,t^*)/\sigma_{zz}(0,0,0)$, with the results given in Chiarella and Booker [23]. The trend of the present results is consistent with that given by Chiarella and Booker [23]. The discrepancy is larger around the maximum contact stress at approximately 3.5%. It decreases as t^* increases. The discrepancy is 1.6% at $t^* = 1$.

Figure 4 compares the results obtained for Case I (*completely pervious surface*) and Case II (*completely impervious surface*). The consolidation rate increases as ν increases for both cases. It is

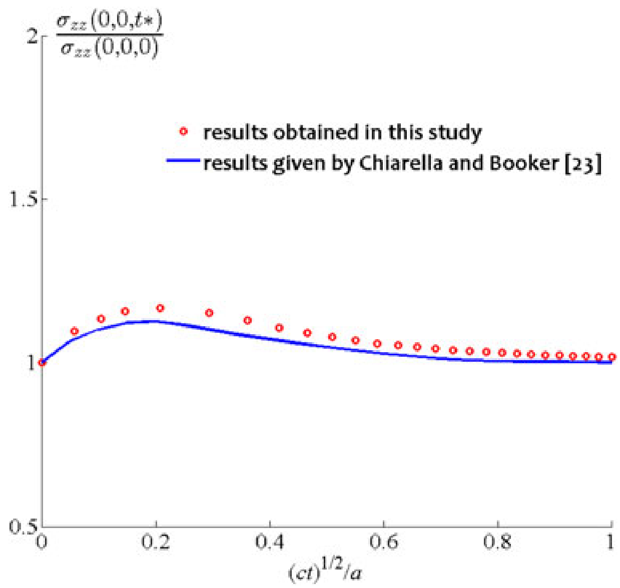

Figure 3. Comparison of the contact stress ratio at the centre of a rigid circular foundation with results given by Chiarella and Booker [23].

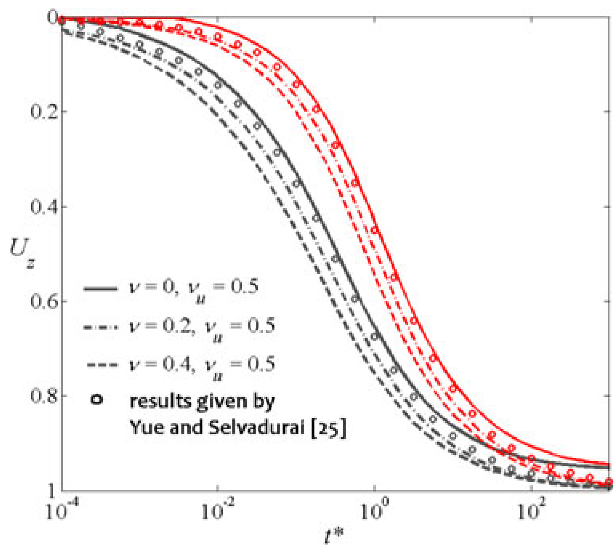

Figure 4. Effect of the drainage condition on the consolidation rate: black colour indicates the results obtained for Case I (*completely pervious boundary*); red colour shows the results obtained for Case II (*completely impervious boundary*).

DOI: 10.1002/nag

observed that the consolidation rate is slower when the surface is completely impervious. The results given by Yue and Selvadurai [25] are represented by circles; the trend shown in the present results matches well with that of Yue and Selvadurai [25].

Figure 5 shows the effect of Poisson's ratio on the consolidation rate for Case I. When v is set to 0, the consolidation rate decreases as v_u increases. On the other hand, when v_u is fixed at 0.5, the consolidation rate increases as v increases (see also Figure 4). The corresponding results given by Yue and Selvadurai [25] are denoted by circles in Figure 5. The agreement is very good (less than 4% when $t^* > 10^{-2}$) except for the case, $v = 0, \quad v_u = 0.5$. The discrepancy becomes very large near the initial and final responses (see Table II).

The effect of Poisson's ratio on the consolidation rate for Case II is presented in Figure 6. As observed in Figure 5, the consolidation rate increases as v_u decreases when v is fixed as 0, while the consolidation rate increases as v increases when v_u is 0.5 (see also Figure 4). The corresponding results given by Yue and Selvadurai [25] are denoted by circles in Figure 6. The agreement between

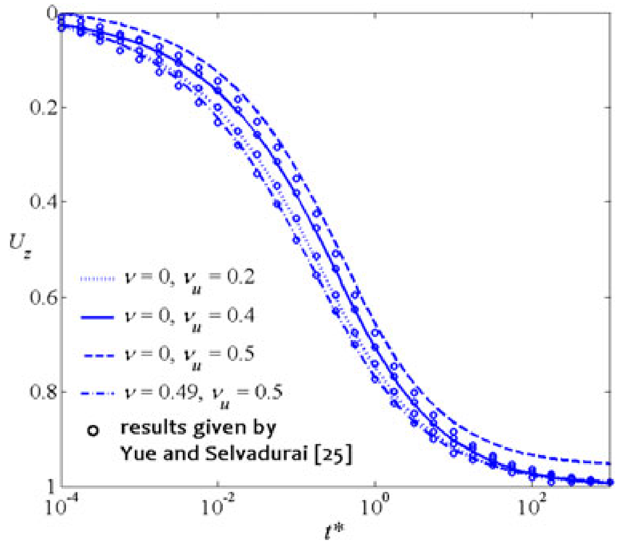

Figure 5. Consolidation rates for different Poisson's ratios when *the surface is completely pervious*.

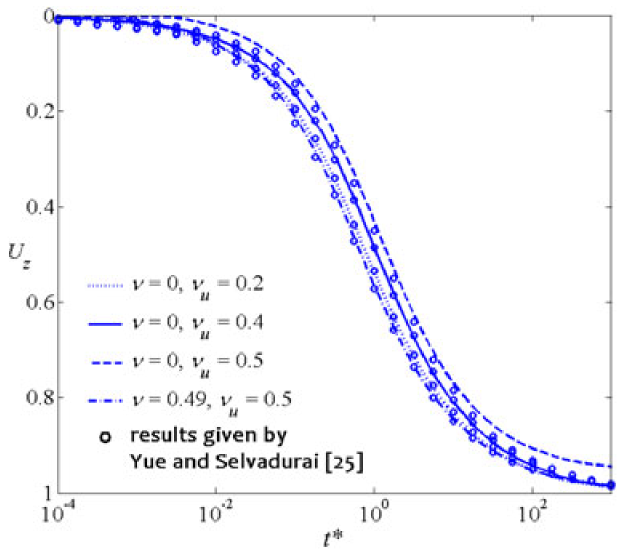

Figure 6. Consolidation rates for different Poisson's ratios when *the surface is completely impervious.*

DOI: 10.1002/nag

the present results and the results given by Yue and Selvadurai [25] is again good with maximum error of 6% except for the case $\nu=0, \quad \nu_u=0.5$.

7. CONCLUDING REMARKS

Contact problems for the poroelastic halfspace involve mixed boundary conditions at the plane surface of the halfspace in terms of the displacements, stresses and pore fluid pressure. The solution procedures for the mixed boundary conditions involve a set of dual integral equations in the Laplace transform domain that cannot be solved using a conventional integral transform (Hankel and Laplace transforms) approach. An alternative approach, where the contact stress is discretized into equivalents, has been developed to examine indentation problems for an elastic halfspace. This paper extends the discretization approach to examine the behaviour of a rigid circular foundation with a frictionless base in contact with a poroelastic halfspace. The results are used to estimate the influence of the Poisson's ratio on the vertical displacement of the poroelastic halfspace. For $\nu_u=0.5$, the consolidation rate increases as ν increases, while the consolidation rate decreases as ν_u increases when $\nu=0$. The results are used to compare the two limiting cases where the surface is either completely pervious or completely impervious. Observations show that the consolidation rate is slower when the surface is completely impervious than when the surface is completely pervious. The results compare favourably with the results given in the literature with a maximum discrepancy of 8%, which shows that the discretization approach is a convenient mathematical technique for examining poroelastic contact problems.

APPENDIX A

The coefficients obtained for Case I and II are.

Case I: The entire surface is *completely pervious*.

$$A=\frac{aJ_1(\xi a)}{\xi s}\frac{\eta(\xi+y)}{[\eta(\Gamma-1)(\xi+y)-(2\eta\Gamma\xi-\xi)]\xi}$$

$$B=\frac{\Gamma\eta\left(\varphi^2-\xi^2\right)-\varphi(2\eta\Gamma\xi-\xi)}{\eta\xi\left(\varphi^2-\xi^2\right)}A$$

$$C=\frac{(2\eta\Gamma\xi-\xi)}{\eta\left(\varphi^2-\xi^2\right)}A$$

Case II: The entire surface is *completely impervious*.

$$A=\frac{aJ_1(\xi a)}{\xi s}\frac{\eta\xi\varphi(\xi+y)}{\left[\eta\xi\varphi(\Gamma-1)(\xi+y)-\xi^2(2\eta\Gamma\xi-\xi)\right]\xi}$$

$$B=\frac{\Gamma\eta\left(\varphi^2-\xi^2\right)-\xi(2\eta\Gamma\xi-\xi)}{\eta\xi\left(\varphi^2-\xi^2\right)}A$$

$$C=\frac{\xi(2\eta\Gamma\xi-\xi)}{\eta\varphi\left(\varphi^2-\xi^2\right)}A$$

DOI: 10.1002/nag

APPENDIX B

The final expressions for the displacement in the z-direction are given by

Case I: The entire surface is *completely permeable*

$$u_z(r,z,t) = \frac{p^* a^*}{2G} \int_{\varsigma - i\infty}^{\varsigma + i\infty} \int_0^\infty \left\{ \frac{\left[\eta\left(\varphi^2 - \xi^2\right)(\Gamma + \xi z(\Gamma - 1) - (\Gamma + 1)) - \varphi(2\Gamma\eta\xi - \xi)\right] e^{-\xi z}}{(\varphi - \xi)[\eta(\Gamma - 1)(\varphi + \xi) - (2\Gamma\eta\xi - \xi)]\xi} \right. \tag{31}$$

$$\left. + \frac{\varphi(2\Gamma\eta\xi - \xi) e^{-\varphi z}}{(\varphi - \xi)[\eta(\Gamma - 1)(\varphi + \xi) - (2\Gamma\eta\xi - \xi)]\xi} \right\} \left(\frac{1}{s}\right) J_1(\xi a^*) J_0(\xi r) d\xi ds$$

Case II: The entire surface is *completely impervious*

$$u_z(r,z,t) = \frac{p^* a^*}{2G} \int_{\varsigma - i\infty}^{\varsigma + i\infty} \int_0^\infty \left\{ \frac{\left[\eta\left(\varphi^2 - \xi^2\right)(\Gamma + \xi z(\Gamma - 1) - (\Gamma + 1)) - \xi(2\Gamma\eta\xi - \xi)\right] \varphi e^{-\xi z}}{(\varphi - \xi)[\eta\varphi(\Gamma - 1)(\varphi + \xi) - \xi(2\Gamma\eta\xi - \xi)]\xi} \right.$$

$$\left. + \frac{\xi\varphi(2\Gamma\eta\xi - \xi) e^{-\varphi z}}{(\varphi - \xi)[\eta\varphi(\Gamma - 1)(\varphi + \xi) - \xi(2\Gamma\eta\xi - \xi)]\xi} \right\} \left(\frac{1}{s}\right) J_1(\xi a^*) J_0(\xi r) d\xi ds$$

ACKNOWLEDGEMENTS

The work described in this paper was supported by a NSERC Discovery Grant awarded to A.P.S. Selvadurai. The first author would like to acknowledge the partial research support provided by the Faculty of Engineering in the form of MEDA.

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