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A frictionless contact problem for a flexible circular plate and an incompressible non-homogeneous elastic halfspace

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ABSTRACT

In this paper we apply an energy method to examine the axisymmetric contact problem for a flexible circular plate in smooth contact an incompressible elastic halfspace, where the linear elastic shear modulus varies exponentially with depth. The approach adopted approximates the deflected shape of the plate by a power series expansion which satisfies the kinematics of deformation of the plate and the Kirchhoff boundary condition at the edge of the plate. The coefficients in the series are evaluated by making use of the principle of minimum potential energy. Results are obtained for the maximum deflection, the relative deflection and the maximum flexural moment in the circular plate. The results derived from the proposed procedure are compared with equivalent results derived from a computational procedure.

1. Introduction

The flexural behavior of finite plates resting on the surface of deformable elastic media is of interest to several branches of engineering and in particular to the study of the interaction between foundations and geologic media. The flexural behavior of a loaded circular plate was investigated by Zemochkin [1], Habel [2], and Holmberg [3] who used discretization techniques to represent the contact pressures as a series of concentric annular regions of uniform stress. The classical study by Borowicka [4] examined the influence of the relative rigidity of circular plate, subjected to uniform external load and resting on an isotropic elastic halfspace using a power series expansion technique. Ishkova [5] and Brown [6] presented a modified solution to the problem in which they considered the effect of near edge singular terms in the approximation of the contact stress distribution. An extensive review of various investigations in this area is given by Selvadurai [7,8]. The study by Selvadurai [8] presented the first application of the energy method to examine the elastostatic contact problem where the deflected shape of the circular plate is presented in the form of a power series in terms of the radial coordinate r. Selvadurai [9] also applied the energy method to investigate the behavior of a circular flexible plate embedded in bonded contact with an elastic infinite halfspace. Zaman et al. [10] also used the energy approach advocated in [8,9] to examine the flexural behavior of a uniformly loaded flexible circular plate where the shape of the plate is approximated by an even-order power series expansion in terms of the radial coordinate. Selvadurai et al. [11] have also applied a variational technique to examine the mechanics of a flexible diaphragm in contact with an elastic medium. Pak et al. [12] investigated the tensionless contact of an annular flexible plate with a smooth halfspace under axisymmetric loads. Selvadurai and Dumont [13] utilized the energy method to investigate the contact problem for an isotropic elastic halfspace containing a Mindlin-type axial force and a flexible circular plate subjected simultaneously to an external load $p(r)$.

The variational approach presented by Selvadurai [8] assumes a deflected shape of the plate $w(r)$ in the form of an even-order power series in the radial coordinate. The total potential energy functional is then developed for the plate–elastic halfspace region, which is defined in terms of four undetermined constants characterizing the deflected shape of the plate. Invoking the Kirchhoff boundary conditions applicable to the free edge of the plate we can eliminate two constants in the series and the two remaining constants are evaluated by the minimization of the total potential energy functional of the system.

In this study, the incompressible elastic halfspace is assumed to be non-homogeneous with the shear modulus varying exponentially with depth. The majority of the classical studies have focused on problems where the halfspace region is homogeneous and the indenter is both rigid and axisymmetric e.g. [14–21]. Departures to this model are documented by Gorbunov-Posadov [22], Gladwell [23], Willner [24], Aleynikov [25], Selvadurai [8,9,26–29], Rajapakse

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and Selvadurai [30], Selvadurai and Dumont [13] and Oliveira et al. [31] who examine cases where the contacting body possesses flexural stiffness. A review of investigations related to contact problems relevant to a non-homogeneous elastic halfspace region is given by Selvadurai and Katebi [32].

The result of the investigation indicates the influence of the relative stiffness of the plate and the elastic non-homogeneity on the deflections and flexural moments in the plate. The results of this study have potential applications to the modeling of structural foundations resting on geologic media. Recent years, the indentation problem involving rigid indenters have been used to examine the mechanical behavior of functionally graded materials. The present study adds to the modeling by introducing both the flexibility of the indenter and the non-homogeneity of the elastic medium.

2. Proposed analytical procedure

Referring to Fig. 1, we examine the problem of the axisymmetric indentation of an incompressible ($\nu = 1/2$), non-homogeneous elastic halfspace by a flexible circular plate of thickness h and radius a. The plate is subjected to a uniform load of intensity p_0 over its entire surface. The non-homogeneity considered in the paper assumes that the shear modulus of the elastic medium varies exponentially according to

$$G(r,\ z) = G_0 e^{\tilde{\lambda} z}\ ;\ r \in (0,\ \infty)\ ;\ z \in (0,\ \infty) \tag{1}$$

where G_0 is constants. Non-dimensional parameter for non-homogeneity ($\tilde{\lambda}$) is then defined by $\lambda = \tilde{\lambda}/a$.

It is further assumed that there is no loss of contact at the frictionless interface. Therefore, the interface displacement can be represented as the deflected shape of the plate, which is identical to the surface displacement of the halfspace in the z-direction over the contact region $0 \le r \le a$. The variational technique proposed by Selvadurai [8] assumes the plate deflection $w(r)$, can be specified to within a set of arbitrary constants, in which the deflected shape of the circular plate can be presented in the form of an even-order power series in terms of the radial coordinate r. The form of $w(r)$is also chosen to satisfy kinematic constraints of the axisymmetric plate deflection. The analysis of the interaction problem via the energy method requires the development of the total potential energy functional for the loading and the plate–elastic halfspace region, which consists of (i) the flexural energy of the plate, (ii) the strain energy of the halfspace region, and (iii) the potential energy of the applied loads. The total potential energy functional can be expressed in terms of the undetermined constants characterizing the deflected shape of the plate. Invoking the Kirchhoff boundary conditions applicable to the free edge of the plate we can eliminate two of the constants in the series and the two remaining constants are evaluated through the minimization of the total potential energy functional of the system.

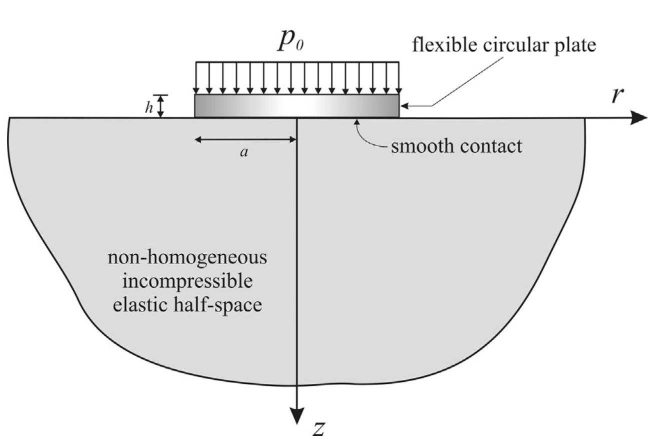

Fig. 1. Uniformly loaded flexible circular plate on the surface of an incompressible elastic halfspace.

3. Variational approach

The proposed formulation is discussed briefly in this section. It is assumed that the deflected shape of the plate can be approximated by the power series expansion:

$$w(r) = a \sum_{i=0}^{3} C_{2i} \left(\frac{r}{a}\right)^{2i} \tag{2}$$

where C_{2i} is the arbitrary constant. The assumed form of the plate deflection (2) has a kinematically admissible form which gives finite rotation and curvature in the plate region. Of the four arbitrary constants, two can be determined by invoking the Kirchhoff boundary conditions applicable to the free edge of the circular plate (Selvadurai [33]), i.e,

$$M_r(a) = -D\left[\frac{d^2 w(r)}{dr^2} + \frac{\nu_b}{r}\frac{dw(r)}{dr}\right]_{r=a} = 0 \tag{3}$$

$$Q_r(a) = -D\left[\frac{d}{dr}\{\nabla^2 w(r)\}\right]_{r=a} = 0 \tag{4}$$

where ν_b is Poisson's ratio of the plate material. The assumed expression for the plate deflection can be reduced to the form [8]

$$w(r) = a\left[C_0 + C_2\left\{\frac{r^2}{a^2} + l_1\frac{r^4}{a^4} + l_2\frac{r^6}{a^6}\right\}\right] \tag{5}$$

where

$$l_1 = \frac{-3(1+\nu_b)}{4(2+\nu_b)};\quad l_2 = \frac{(1+\nu_b)}{6(2+\nu_b)} \tag{6}$$

In the following, the total potential energy functional for the plate–elastic halfspace region is developed using the proposed plate deflection $w(r)$.

3.1. Flexural energy of the plate

The first component of the total energy functional corresponds to the flexural energy of the plate. The flexural behavior of the elastic plate is described by the Poisson–Kirchhoff thin plate theory. This, the flexural energy of the plate with an axisymmetric deflection $w(r)$ is given as follows:

$$U_F = \frac{D}{2}\iint_S \left[\{\nabla^2 w(r)\}^2 - \frac{2(1-\nu_b)}{r}\frac{dw(r)}{dr}\frac{d^2 w(r)}{dr^2}\right] r dr d\theta \tag{7}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr};\quad D = \frac{G_b h^3}{6(1-\nu_b)} \tag{8}$$

and G_b and ν_b are the constant shear modulus and Poisson's ratio of the plate material, respectively, and s represents the plate region.

3.2. Strain energy of the halfspace region

The second component of the total potential energy functional corresponds to the strain energy of the incompressible non-homogeneous elastic halfspace, which is subjected to the displacement field $w(r)$ in the contact region $0 \le r \le a$. The elastic strain energy can be developed by evaluating the work component of the

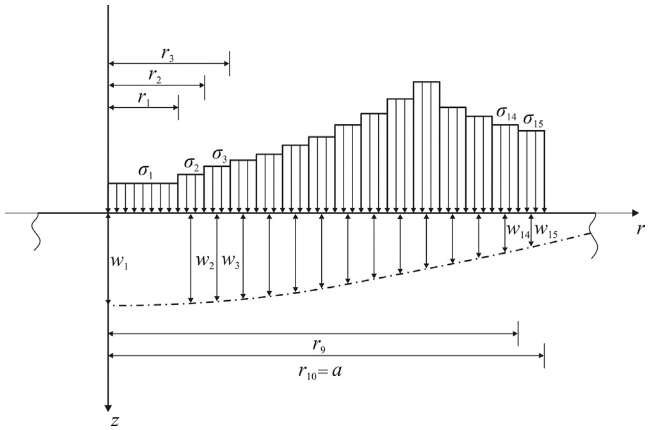

Fig. 2. Elastic incompressible non-homogeneous halfspace subjected to concentric distributions of annular loads.

surface tractions that compose the interface normal contact stresses at the frictionless interface. Since the interface is assumed to be smooth, only the normal tractions contribute to the strain energy. These normal tractions can be determined by making use of the discretization method described by Selvadurai and Katebi [32]. In order to obtain the contact stresses, we first consider the indentation of the non-homogeneous incompressible elastic half-space, which corresponds to axial displacement $w(r)$. The mixed boundary conditions are as follows:

$$u_z(r,0) = w(r); \quad 0 \le r \le a$$

$$\sigma_{rz}(r,0) = 0; \quad 0 \le r < \infty$$

$$\sigma_{zz}(r,0) = 0; \quad a < r < \infty \tag{9}$$

The method of solution assumes that the contact region of the indenter can be discretized into a finite number (15) of equal annular areas and the contact stress within each annular area is uniform. As can be seen from Fig. 2, annular normal loading of stress intensities $\sigma_1, \sigma_2, \ldots, \sigma_n$ $(n=15)$ acting within the annular areas of internal radii $0, r_1, r_2, \ldots, r_{n-1}$ and external radius $r_1, r_2, \ldots, r_n$ respectively. The surface displacement $w_1, w_2, w_3, \ldots, w_n$ at the *mid-section* of the annular region due to normal surface tractions can be obtained by superposition of the solutions for the uniform circular loading, which is a simplified version of the results developed by Selvadurai and Katebi [34]

$$u_z(r,0) = \frac{p_0 a}{G_0}\int_0^\infty \left(\frac{(k_1^2-k_2^2)}{(2k_1-q_1)(\xi^2+k_2^2)+(q_2-2k_2)(\xi^2+k_1^2)}\right) J_0(\xi r) J_1(\xi a)\, d\xi \tag{10}$$

where

$$k_1 = \frac{1}{2}\left[\tilde{\lambda}+\sqrt{\tilde{\lambda}^2+4i\tilde{\lambda}\xi+4\xi^2}\right] \quad ; \quad k_2 = \frac{1}{2}\left[\tilde{\lambda}+\sqrt{\tilde{\lambda}^2-4i\tilde{\lambda}\xi+4\xi^2}\right] \tag{11}$$

and

$$q_i = \frac{k_i^3}{\xi^2} - k_i - \frac{\lambda k_i^2}{\xi^2} - \tilde{\lambda} \quad ; \quad i = 1,\ 2 \tag{12}$$

in above equations $\tilde{\lambda} = \lambda/a$. The procedures for the numerical evaluation of these integrals are given in [34] and [35].

The compatibility between the settlement of the non-homogenous incompressible elastic halfspace and the settlement of the elastic plate, $w(r)$ is then established at the mid point location of each annular area.

In order to assign equal areas to the annular regions, the dimension of r_i takes the following form:

$$r_i = \left(\frac{i}{n}\right)^{\frac{1}{2}} a \ ; \quad i = 1,\ 2,\ 3,\ \ldots..,\ 15 \tag{13}$$

Similarly for the mid-points,

$$r_{m1} = 0, \quad r_{mi} = \left(\frac{r_{i-1}+r_i}{2}\right); \quad i = 2,\ 3,\ \ldots.,\ 15 \tag{14}$$

We further assume that the uniform normal stress elements σ_i can be represented as multiples of the average pressure p_0 that is applied externally to the rigid plate i.e.:

$$\sigma_i = \tilde{\sigma}_i p_0 \quad \text{where } i = 1,\ 2,\ 3,\ \ldots.,\ 15 \tag{15}$$

Using the above relations, it is possible to express the surface displacements w_i at the mid-points of the annular areas due to normal contact stresses $\tilde{\sigma}_i$, in the form of the matrix relation

$$\{\mathbf{w}\} = \frac{p_0 a}{G_0}[\mathbf{C}]\{\tilde{\boldsymbol{\sigma}}_i\} \tag{16}$$

where $\{\mathbf{w}\}$ and $\{\tilde{\boldsymbol{\sigma}}_i\}$ are column vectors and the square matrix of coefficients, $[\mathbf{C}]$ is as follows:

$$[\mathbf{C}] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \ldots & w_{1j} \\ w_{21} & w_{22} & w_{23} & \ldots & w_{2j} \\ w_{31} & w_{32} & w_{33} & \ldots & w_{3j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & w_{i2} & \ldots & w_{ij} \end{bmatrix} \tag{17}$$

where w_{ij} is the surface displacement at the annuli mid-point location i due to the normal contact stress σ_j $(i=j)$.

Since there is no loss contact, for compatibility at the elastic halfspace-indenter interface, w_i will have the same value as displacement field $w(r)$ at annuli mid point locations r_{mi}; then the Eq. (16) can be inverted to determine the non-dimensional contact stresses $\tilde{\sigma}_i$ and therefore σ_i in terms of the arbitrary parameter of $w(r)$ (i.e. expressed in terms of C_0 and C_2). By inverting Eq. (16) and using Eq. (15) we have:

$$\{\boldsymbol{\sigma}_i\} = \frac{G_0}{a}[\mathbf{B}]\{\mathbf{w}\} \tag{18}$$

In which

$$[\mathbf{B}] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \ldots & m_{1j} \\ m_{21} & m_{22} & m_{23} & \ldots & m_{2j} \\ m_{31} & m_{32} & m_{33} & \ldots & m_{3j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{i1} & m_{i2} & m_{i2} & \ldots & m_{ij} \end{bmatrix} \tag{19}$$

The strain energy in the halfspace region U_E can be obtained from

$$U_E = \frac{1}{2} f_i w_i; \quad i = 1,\ 2,\ 3,\ \ldots..,\ 15 \tag{20}$$

where f_i is the force in each equal annular area, calculated by multiplying the contact stress σ_i with the area of the annular region (which is equal to $\pi/15$).

3.3. Potential energy of the applied loads

The total potential energy of the uniform external load applied to the plate is given by

$$U_p = -p_0 \iint_S w(r)\, r\, dr\, d\theta \tag{21}$$

4. Total potential energy functional

The total potential energy functional for the loading-plate-non-homogeneous incompressible elastic medium system is given by

$$U = U_F + U_E + U_p \tag{22}$$

By making use of $w(r)$ as defined by (5), and using Eqs. (7), (20) and (21), the total potential energy functional (22) reduces to the form

$$U = G_0\left[C_0^2\chi_1 + C_0C_2\chi_2 + C_2^2\chi_3\right] + \pi D C_2^2\chi_4 - \pi p_0 a^3\left[C_0 + C_2\chi_5\right] \tag{23}$$

where the constants $\chi_1, \chi_2, \chi_3, \chi_4$ and χ_5 are given by

$$\chi_1 = \frac{\pi}{30}m_{ij} \quad , i \text{ and } j = 1\ldots15 \tag{24}$$

$$\chi_2 = \frac{\pi}{15}m_{ij}\left(\frac{r_{mj}^2}{a^2} + L_1\frac{r_{mj}^4}{a^4} + L_2\frac{r_{mj}^6}{a^6}\right) \quad , i \text{ and } j = 1\ldots15 \tag{25}$$

$$\chi_3 = \frac{\pi}{30}m_{ij}\left(\frac{r_{mj}^2}{a^2} + L_1\frac{r_{mj}^4}{a^4} + L_2\frac{r_{mj}^6}{a^6}\right)\left(\frac{r_{mj}^2}{a^2} + L_1\frac{r_{mj}^4}{a^4} + L_2\frac{r_{mj}^6}{a^6}\right) \quad , i \text{ and } j = 1\ldots15 \tag{26}$$

$$\chi_4 = \left\{8 + 32l_1 + \frac{144}{3}l_2 + \frac{128}{3}l_1^2 + \frac{648}{5}l_2^2 + 144l_1l_2 - (1-\nu_b)(4 + 16l_1 + 24l_2 + 16l_1^2 + 36l_2^2 + 48l_1l_2)\right\} \tag{27}$$

$$\chi_5 = \frac{1}{2} + \frac{l_1}{3} + \frac{l_2}{4} \tag{28}$$

The constants C_0 and C_2 can be evaluated by minimization conditions for the total potential energy functional i.e.:

$$\frac{\partial U}{\partial C_0} = 0; \quad \frac{\partial U}{\partial C_2} = 0 \tag{29}$$

The deflected shape of the uniformly loaded circular foundation corresponding to (2) can be expressed in the form

$$w(r) = \frac{\pi a p_0}{4G_0\Omega}\left[\chi_2\chi_5 - 2\chi_3 - R\chi_4 + (\chi_2 - 2\chi_1\chi_5)\left\{\frac{r^2}{a^2} + l_1\frac{r^4}{a^4} + l_2\frac{r^6}{a^6}\right\}\right] \tag{30}$$

where

$$\Omega = \chi_2^2 - 4\chi_1\chi_3 - 2R\chi_1\chi_4 \tag{31}$$

and R is the relative rigidity parameter defined by

$$R = \frac{\pi a p_0}{12(1-\nu_b)}\left(\frac{G_b}{G_0}\right)\left(\frac{h}{a}\right)^3 = \frac{\pi K_r}{6(1-\nu_b^2)} \tag{32}$$

and K_r is a similar parameter defined by Brown [6]. The accuracy of the solution for the deflection of the plate determined by variational approach can be investigated by assigning various limits to the relative rigidity parameter (K_r or R).

5. Limiting cases

In order to verify the accuracy of the developments with known analytical results, two limiting case are considered in this section. In both cases, it is assumed that the non-homogeneous parameter $\lambda \to 0$, which corresponds to a homogeneous incompressible elastic halfspace.

(i) *Rigid circular plate*; $R \to \infty$: the indentation problem reduces to Boussinesq's classical result [36–39] for the smooth indentation of a rigid circular plate on an isotropic homogeneous halfspace. The corresponding value for the rigid displacement of the circular indenter with an incompressible halfspace is given by

$$w_0 = 0.394525\frac{p_0 a}{G_0}, \quad 0 \le r < a \tag{33}$$

This result is in agreement with result obtained by Selvadurai and Katebi [32] using the discretized method. Boussinesq's classical result [36–39] for smooth indentation of a rigid circular plate on an incompressible isotropic homogeneous halfspace is given by

$$w(r) = 0.392699\frac{p_0 a}{G_0}, \quad 0 \le r < a \tag{34}$$

By comparing the results (33) and (34), we can see that the current method gives results accurate within 0.465%.

(ii) *Flexible circular loading*; as $R \to 0$: the problem reduces to the loading of the incompressible halfspace by a uniform circular load of radius a and stress intensity p_0. The two important results of specific interest for geotechnical engineering are the maximum surface deflection and the differential deflection within the loaded area. Thus limiting case 2 Eq. (30) reduces to

$$\lim_{R\to 0} w(0) = \frac{p_0 a}{4G_0}\{2.087\} \tag{35}$$

The exact result corresponding to the central surface displacement of the uniformly loaded area ($\nu_s = 1/2$) is given by (see, e.g., Timoshenko and Goodier [39])

$$w(0) = \frac{p_0 a}{4G_0}\{2.00\} \tag{36}$$

By comparing the two results (35) and (36), it can be seen that the energy method over-estimates the central deflection by 4.35%.

The result for the differential deflection obtained from the energy method is as follows:

$$\lim_{R\to 0}[w(0) - w(a)] = \frac{p_0 a}{4G_0}\{0.7327\} \tag{37}$$

and corresponding exact solution is as follows:

$$[w(0) - w(a)] = \frac{p_0 a}{4G_0}\left\{\frac{2\pi - 4}{\pi}\right\} \tag{38}$$

This shows that the estimate given by the energy method over-predicts the exact solution by only 0.8%.

6. Maximum flexural moments

The flexural moment in the plate can, in principle, be computed from the expressions

$$M_r = -D\left\{\frac{d^2w(r)}{dr^2} + \frac{\nu_b}{r}\frac{dw(r)}{dr}\right\} \tag{39}$$

$$M_\theta = -D\left\{\frac{1}{r}\frac{dw(r)}{dr} + \nu_b\frac{d^2w(r)}{dr^2}\right\} \tag{40}$$

As observed by Selvadurai [8], due to the presence of derivatives up to the second order, the flexural moment calculated by Eqs. (39) and (40) is considerably less accurate compared to the estimation of the plate deflection. A more accurate estimate of the flexural moment in the plate can be obtained by considering the combined action of the external load $p_0$and the contact stress distribution (σ_i). A solution for the maximum flexural moment can be computed by superposing the two solutions: (i) a circular plate simply supported along its boundary subjected to a uniform external load p_0, and (ii) a circular plate simply supported along its boundary subjected to the contact stress distribution (σ_i). Implicit in the procedure is the equilibrium of the plate under the action of the external load and the contact stresses; i.e. since the two sets of loading refer to same total load the edge reaction cancel out when the two solutions are superposed.

(i) From the results given by Timoshenko and Woinowsky-Krieger [40], the maximum flexural moment at the center of an edge-supported plate due to the external load p_0 is given by

$$M_{\max}^{p} = \frac{p_0 a^2 (3+\nu_b)}{16} \tag{41}$$

(ii) The maximum flexural moment due to the contact stress (σ_i) acting on an edge-supported plate is given by

$$M_{\max}^{c} = \sum_{i=1}^{15} \int_{r_{i-1}}^{r_i} \frac{\xi \sigma_i(\xi)}{2} \left[\frac{(1-\nu_b)}{2} \left(\frac{(r_i - r_{i-1})^2 - \xi^2}{(r_i - r_{i-1})^2} \right) - (1-\nu_b) \ln\left(\frac{\xi}{r_i - r_{i-1}} \right) \right] d\xi, \quad i = 1, 2, \ldots, 15 \tag{42}$$

where $\sigma(\xi)$ is defined in Eq. (15) and $r_0 = 0$.

7. Numerical results and discussion

The objective of this paper is to investigate the influence of the elastic non-homogeneity of an incompressible halfspace and the relative stiffness of the plate–elastic halfspace system on the deflections and flexural moments in the plate. The results are presented for the various relative rigidities from $\log_{10} K_r = 0$ to 3 in which $\log_{10} K_r = 0$ represents a relatively flexible plate and $\log_{10} K_r = 3$ represents a nearly rigid plate. Poisson's ratio for the plate is chosen to be $\nu_b = 0.3$. To demonstrate the effect of non-homogeneity on the results, the parameter $\tilde{\lambda}$, which is indicative of the depth dependent variation in the shear modulus (see Eq. 1), has been used for the range $\tilde{\lambda} = 0$ to1.5 where $\tilde{\lambda} = 0$ represents an incompressible homogeneous elastic halfspace. In the case when $\tilde{\lambda} = 0$, the results can be compared with the existing numerical results presented by Brown [6] and Selvadurai [8]. It should be pointed out that all numerical results are presented in a non-dimensional form.

Fig. 3 shows the variation of the central displacement of a uniformly loaded circular plate for various relative rigidities for the incompressible homogeneous elastic halfspace ($\tilde{\lambda} = 0$). The current result is compared with existing solution by Brown [6] and Selvadurai [8]. The results are also compared with a finite-element analysis of the problem of the smooth indentation of a rigid circular plate on an isotropic homogeneous halfspace using COMSOL™ Multiphysics software. It can be seen from Fig. 3, that there is reasonable correlation between the analytical results and the results derived from the computational scheme COMSOL-TM.

The deflected shape and the number of terms in the expression chosen to truncate the power series can influence the accuracy of the solution. The other parameter influencing the solution is the accuracy of the numerical technique used to evaluate the infinite integrals involved. These factors can of course be improved by increasing the terms involved in the power series and the accuracy of quadrature scheme used to evaluate the integrals is firmly established. As for computational results (COMSOL™), the accuracy of the results can influence from various parameters such as modeling a semi-infinite elastic medium with finite domain, the mesh size and element type chosen. Further discussions are given by Selvadurai et al. [41].

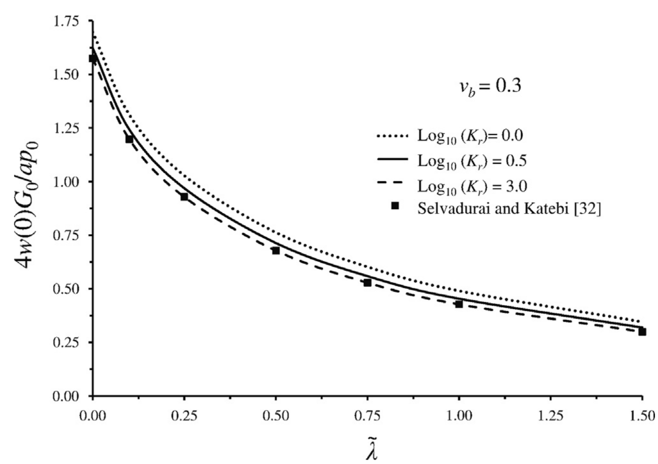

Fig. 4. Variation in the central displacement of the circular plate for different values of $\tilde{\lambda}$ and K_r.

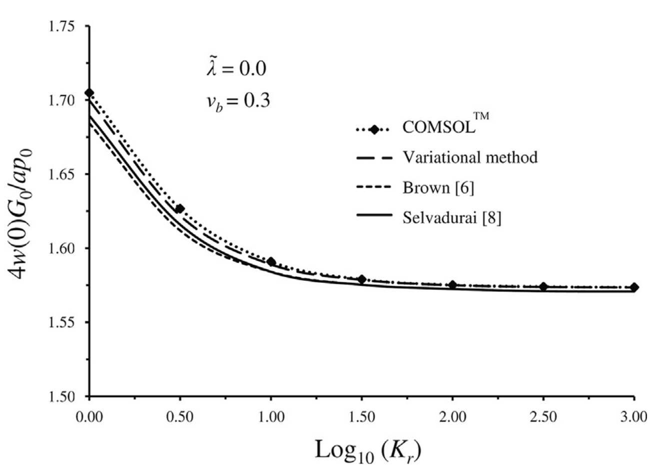

Fig. 3. Variation in the central displacement of the circular plate with $K_r(\tilde{\lambda} = 0$, homogeneous).

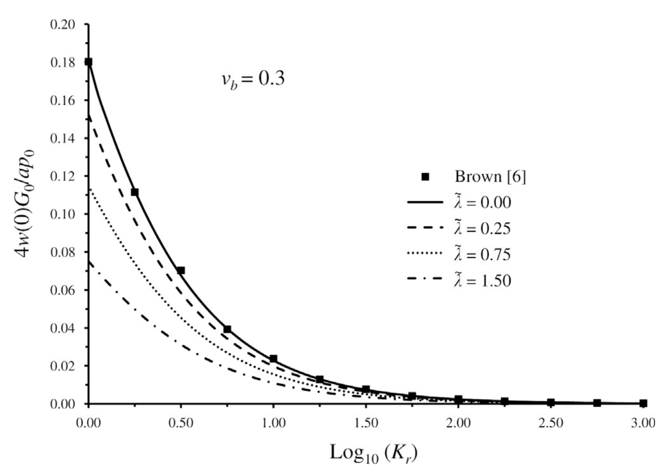

Fig. 5. Variation in the differential deflection of the circular plate for different values $\tilde{\lambda}$ and K_r.

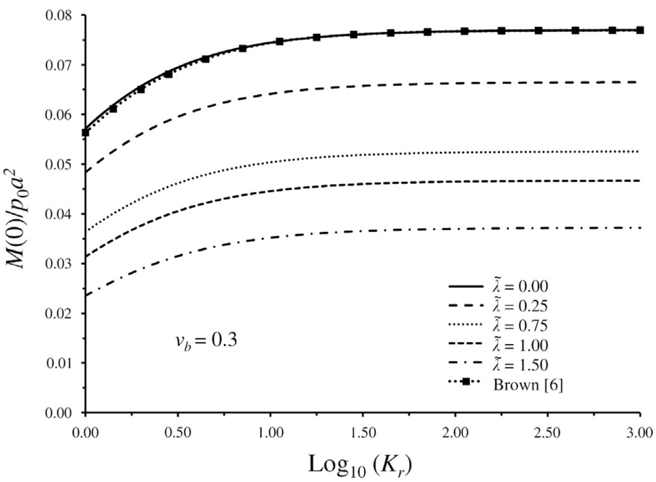

Fig. 6. Variation in the central flexural moment for different values of $\tilde{\lambda}$ and K_r.

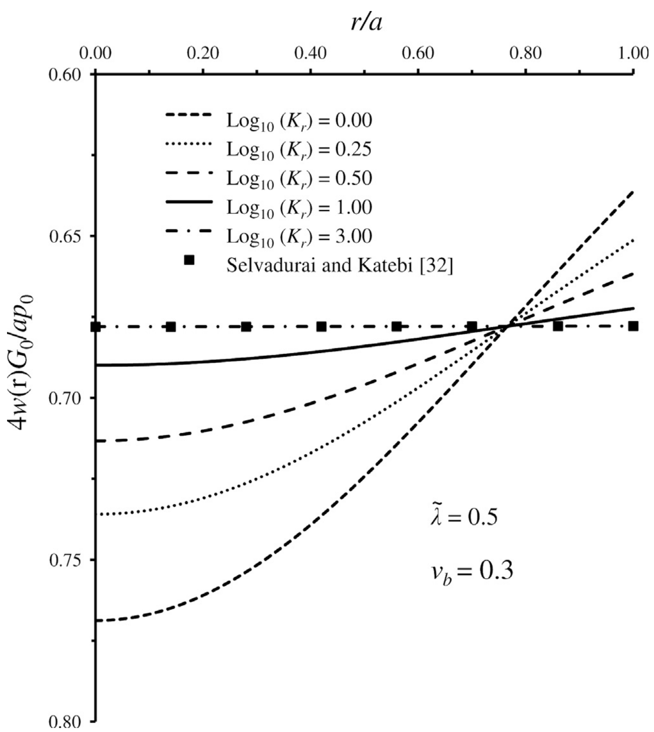

Fig. 7. Variation in the vertical surface displacement along the r-axis for different values of K_r, $\tilde{\lambda} = 0.5$.

Fig. 4 shows the variation in the central displacement of a uniformly loaded circular plate for different $\tilde{\lambda}$ and for different values of the relative rigidity K_r. It is evident that the presence of elastic non-homogeneity condition has a significant effect on the central displacement of the plate. The results are compared with smooth indentation problem of a rigid circular plate and an incompressible non-homogeneous elastic halfspace presented by Selvadurai and Katebi [32]. In this comparison the flexible plate with $\log_{10} K_r = 3$ considered as being a nearly rigid plate. As is evident, there is a good correlation between the two set of results. The variation in the differential deflection of a uniformly loaded circular plate for different $\tilde{\lambda}$ and different relative rigidities is shown in Fig. 5. The current results for homogenous case ($\tilde{\lambda} = 0$) are compared with numerical results presented in [6].

The central flexural moment for different values of $\tilde{\lambda}$ and different relative rigidities of the plate K_r is shown in Fig. 6 and it is compared with results in [6] for homogeneous case. There is good correlation between the two results.

Fig. 7 shows the variation in the vertical surface displacement along the radial direction for different relative rigidities of the plate. The effect of the relative rigidity on the shape of the plate deflection is investigated; when $R \to \infty$ (or $K_r \to \infty$), the problem reduces to the indentation of the circular rigid plate on an incompressible elastic halfspace and the results are compared with the known solution presented by Boussinesq [37] and Selvadurai and Katebi [34] for $\tilde{\lambda} = 0.5$ respectively.

8. Conclusion

The energy method has been used to analyze the mechanics of axisymmetric smooth contact between a flexible plate and incompressible isotropic non-homogeneous elastic halfspace where the shear modulus varies exponentially with depth. The effect of relative rigidity of the plate as well as the non-homogeneity of the incompressible elastic halfspace on the response is clearly elaborated in the numerical results. The results have been compared with existing solutions as well as the COMSOL software to validate the accuracy of the solution. The accuracy of the results for deflection of the plate depends on the relative rigidity and varies from 0.465% to 4.35% for $R \to \infty$ (or $K_r \to \infty$) and $R \to 0$ (or $K_r \to 0$) respectively.

Disclaimer

The use of the computational code COMSOLTM is for demonstration purposes only. The authors do not advocate or recommend the use of this code without conducting suitable validation procedures to test the accuracy of the computational scheme in a rigorous fashion.

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