Acta Mech 226:249–265 (2015)
DOI 10.1007/s00707-014-1171-8

A. P. S. Selvadurai · A. Katebi

An adhesive contact problem for an incompressible non-homogeneous elastic halfspace

Received: 21 November 2013 / Revised: 17 April 2014 / Published online: 15 June 2014

Abstract In this paper, we examine the axisymmetric adhesive contact problem for a rigid circular plate and an incompressible elastic halfspace where the linear elastic shear modulus varies exponentially with depth. The analytical solution of the mixed boundary value problem entails a set of coupled integral equations that cannot be solved easily by conventional integral transform techniques proposed in the literature. In this paper, we adopt a computational scheme where the contact normal and contact shear stress distributions are approximated by their discretized equivalents. The consideration of compatibility of deformations due to the indentation by a rigid indenter in adhesive contact gives a set of algebraic equations that yield the discretized equivalents of the contacts stresses and the axial stiffness of the medium.

1 Introduction

The topic of contact mechanics occupies an important position in the mechanics of solids where the results derived for the interaction of a rigid indenter and a halfspace form the basic problem of interest. The classical elasticity solutions to the mechanics of contact commenced with the well-known solutions by Hertz [1] and Boussinesq [2], and developments in this area are documented in several texts and review articles by Galin [3], Sneddon [4,5], Ufliand [6], Selvadurai [7,8], Gladwell [9] and Willner [10]. The majority of the classical studies in this area have focused on the contact problem where the halfspace region is homogeneous, and the indenter is both rigid and axisymmetric. Derivations from this model are documented by Gladwell [9], Willner [10], Selvadurai [11–14], Rajapakse and Selvadurai [15], Selvadurai and Dumont [16] and Oliveira et al. [17] who examine cases where the contacting element possesses flexural stiffness.

In this paper, we examine the axisymmetric problem of the adhesive contact between a rigid circular indenter with a flat base and an incompressible elastic inhomogeneous medium where the shear modulus varies exponentially with depth. The literature pertaining to contact problems involving non-homogeneous elastic media is presented in the references [7–9,18–22]. In this paper, we shall refer to the literature on contact problems that is germane to the title problem. Korenev [23] examined the problem of the axisymmetric indentation of a rigid smooth indenter and an isotropic elastic halfspace, whose modulus of elasticity varied exponentially with depth. This problem was also examined by Mossakovskii [24] who gives a corrected result for the contact stress distribution.

The model of an incompressible elastic halfspace where the shear modulus varies linearly with depth was developed by Gibson [25], where the results were applied to calculate the settlement of structures founded on London Clay. The majority of theoretical developments have focused on power law or exponential variations

A. P. S. Selvadurai (✉) · A. Katebi
Department of Civil Engineering and Applied Mechanics, McGill University, 817 Sherbrooke Street West, Montreal, QC H3A 0C3, Canada
E-mail: patrick.selvadurai@mcgill.ca
Tel.: +1 514 398 6672

in the linear elastic shear modulus, leading to significant simplifications in the solution of the associated elasticity problem. The linear variation in the shear modulus gives rise to convenient analytical results for the surface displacements of the loaded halfspace regions that gives rise to the definition of the *Gibson Soil*, a theoretical model for the *Winkler Medium*, which consists of a series of closely spaced independent springs. Both the exponential and linear variations in the shear modulus with depth give rise to unbounded values of the shear modulus when the theoretical developments are applied to semi-infinite regions. This limitation was first addressed by Selvadurai et al. [26], who examined the torsional indentation of an inhomogeneous elastic halfspace with an exponential but bounded variation in the shear modulus. Selvadurai [27] also examined the problem of the smooth indentation of an incompressible inhomogeneous elastic halfspace where the shear modulus exhibits a bounded exponential variation. Detailed discussions of the applicability of both linear and exponential variations in the shear modulus with depth, which is of special relevance to geomechanics problems encountered in soil mechanics and foundation engineering are presented by Selvadurai [7,8,27,28], Rajapakse [29], Rajapakse and Selvadurai [30], Selvadurai and Lan [31,32], Spencer and Selvadurai [33], Selvadurai and Katebi [34].

In this paper, we examine the problem of the adhesive contact problem between a rigid circular punch with a flat base and an incompressible elastic halfspace region where the shear modulus of the elastic material varies exponentially with depth. Also, unless otherwise stated, the rigid circular indenter is assumed to have a flat base, which makes the class of problems examined in the paper purely axisymmetric. In particular, attention is focused on estimating the influence of the elastic inhomogeneity on the adhesive elastic stiffness of the rigid circular indenter. The solution of the adhesive contact problem is achieved using a numerical scheme where the fundamental solutions for the axisymmetric loading of the non-homogeneous halfspace region due to normal and shear loads applied over discrete regions of the surface of the halfspace are combined to determine the axial and radial displacements over the region of the indenter. The distribution of the discrete values of the contact stresses is determined by considering the displacement constraints imposed on the contact region and the overall equilibrium of the indenter. This approach to the analysis of the adhesive contact problem related to the indentation of the inhomogeneous halfspace region is an extension to similar procedures that have been successfully used in both discretized approaches for solving soil-structure interaction problems (Selvadurai [7]).

2 Theoretical developments

Prior to the development of the numerical approach to the solution of the adhesive contact problem, it is instructive to record salient results applicable to the indentation of the *homogeneous* elastic halfspace region. To provide some generality, we consider the indentation of the homogeneous elastic halfspace problem with no restriction on Poisson's ratio. We consider an axisymmetric formulation of the contact problem where the displacement and stress fields referred to the cylindrical polar coordinate system are given by

$$\mathbf{u}(r,z) = \{u_r(r,z), u_z(r,z)\} \tag{1}$$

and

$$\boldsymbol{\sigma}(r,z) = \begin{pmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{rz} & 0 & \sigma_{zz} \end{pmatrix}. \tag{2}$$

The mixed boundary value problem governing the adhesive indentation of the homogeneous elastic halfspace is given by

$$\begin{aligned} u_z(r,0) &= \Delta; \quad 0 \le r \le a \\ u_r(r,0) &= 0; \quad 0 \le r \le a \\ \sigma_{zz}(r,0) &= 0; \quad a < r < \infty \\ \sigma_{rz}(r,0) &= 0; \quad a < r < \infty \end{aligned} \tag{3}$$

where Δ is the axial displacement of the rigid indenter. In addition, the displacement and stress fields should reduce to zero as $r, z \to \infty$. The solution to the indentation problem was presented in a general fashion by Mossakovskii [35] and Ufliand [36] where the axial displacement distribution was defined by $w(r)$, and the radial displacement was defined by $u(r)$. The *contact normal stress* $\sigma_{zz}(r)$ normalized with respect to the shear modulus and (Δ/a) is expressed as $\sigma_{zz}(r) = -\{G\Delta/a(1-\nu)\}\tilde{\sigma}(\tilde{\xi})$, and the *contact shear stress* $\sigma_{rz}(r)$ in the adhesive zone, normalized with respect to the shear modulus and (Δ/a) is expressed as $\sigma_{rz}(r) =$

$-\{G\Delta/a(1-\nu)\}\tilde{\tau}(\tilde{\xi})$. The normalized stresses $\sigma(\tilde{\xi})$ and $\tau(\tilde{\xi})$ are governed by a pair of coupled integral equations of the form

$$\int_{\tilde{\xi}}^{1} \frac{t\tilde{\sigma}(t)\mathrm{d}t}{\sqrt{t^2-\tilde{\xi}^2}} - \gamma\left\{\int_0^1 \tilde{\tau}(t)\mathrm{d}t - \tilde{\xi}\int_0^{\tilde{\xi}} \frac{\tilde{\tau}(t)\mathrm{d}t}{\sqrt{\tilde{\xi}^2-t^2}}\right\} = \frac{\mathrm{d}}{\mathrm{d}\tilde{\xi}}\int_0^{\tilde{\xi}} \frac{t\tilde{w}(t)\mathrm{d}t}{\sqrt{\tilde{\xi}^2-t^2}} = \tilde{w}^*(\tilde{\xi}), \tag{4}$$

$$\tilde{\xi}\int_{\tilde{\xi}}^{1} \frac{\tilde{\tau}(t)\mathrm{d}t}{\sqrt{t^2-\tilde{\xi}^2}} - \gamma\int_0^{\tilde{\xi}} \frac{t\tilde{\sigma}(t)\mathrm{d}t}{\sqrt{\tilde{\xi}^2-t^2}} = \int_0^{\tilde{\xi}} \frac{\mathrm{d}}{\mathrm{d}t}\left[t\tilde{u}(t)\right]\frac{\mathrm{d}t}{\sqrt{\tilde{\xi}^2-t^2}} = \tilde{u}^*(\tilde{\xi}) \tag{5}$$

where $\tilde{\xi} = r/a, u_z(r) = \Delta\tilde{w}(\tilde{\xi}), u_r(r) = \Delta\tilde{u}(\tilde{\xi})$ and Δ is the axial indentation. Also, $\tilde{w}^*(\tilde{\xi})$ and $\tilde{u}^*(\tilde{\xi})$ are non-dimensional displacements that are prescribed in the contact zone and $\gamma = (1-2\nu)/(2-2\nu)$. For example, in the case of a rigid circular indenter in adhesive contact with an elastic halfspace region, $\tilde{w}^*(\tilde{\xi}) = 1$ and $\tilde{u}^*(\tilde{\xi}) = 0$. Using a Wiener-Hopf/Hilbert transform technique (Gladwell [9]) for the solution of the mixed boundary value problem, these authors [35,36] were able to develop explicit expressions for the distribution of contact stresses at the base of the indenter and use the results to obtain a relationship between the force P necessary to induce the induced displacement Δ; i.e.

$$\frac{P}{8\Delta a\mu} = \frac{\log_e(3-4\nu)}{2(1-2\nu)}. \tag{6}$$

An alternative to the development of an exact solution was first proposed and further developed by Selvadurai [37–42] and applied by Selvadurai and Au [43] to investigate the mechanics of anchors embedded at bimaterial interface regions. Selvadurai's bounding procedure involves the introduction of either inextensibility constraints or frictionless conditions at the interface between the two regions. The same approach can be adopted to "bound" the axial stiffness of the bonded rigid indenter. The mixed boundary value problem defined by Eq. (3) is replaced by reduced boundary value problems that impose either an inextensibility constraint over the entire surface of the halfspace or a frictionless constraint over the entire surface of the halfspace. The relevant sets of boundary conditions are, respectively,

$$\begin{aligned} u_z(r,0) &= \Delta; & 0 &\le r \le a \\ u_r(r,0) &= 0; & 0 &\le r < \infty \\ \sigma_{zz}(r,0) &= 0; & a &< r < \infty \end{aligned} \tag{7}$$

and

$$\begin{aligned} u_z(r,0) &= \Delta; & 0 &\le r \le a \\ \sigma_{rz}(r,0) &= 0; & 0 &\le r < \infty \\ \sigma_{zz}(r,0) &= 0; & a &< r < \infty. \end{aligned} \tag{8}$$

The solution of the mixed boundary value problems defined by Eqs. (7) and (8) is elementary (Sneddon [4]; Selvadurai [7]; Gladwell [9]) and, using these results, the bounds for the elastic stiffness of the bonded rigid disc can be expressed in the form

$$\frac{1}{2(1-\nu)} \le \frac{P}{8\Delta a\mu} \le \frac{2(1-\nu)}{(3-4\nu)}. \tag{9}$$

In the limit when $\nu = 0$, the exact solution Eq. (6) gives

$$\frac{P}{8\Delta a\mu} = \frac{\log_e 3}{2} \approx 0.55, \tag{10}$$

and the bounds (9) give

$$0.50 \le \frac{P}{8\Delta a\mu} \le 0.67. \tag{11}$$

In the limit when $\nu = 1/2$, the bounds Eq. (9) converge and yield the same result as the exact solution [Eq. (6)]. The influence of material incompressibility on the behaviour of the contact problem is clearly evident and, in the case of the isotropic homogeneous elasticity problem applicable to an incompressible elastic material, the absence of radial displacement on the surface of the halfspace due to Boussinesq's fundamental result for the

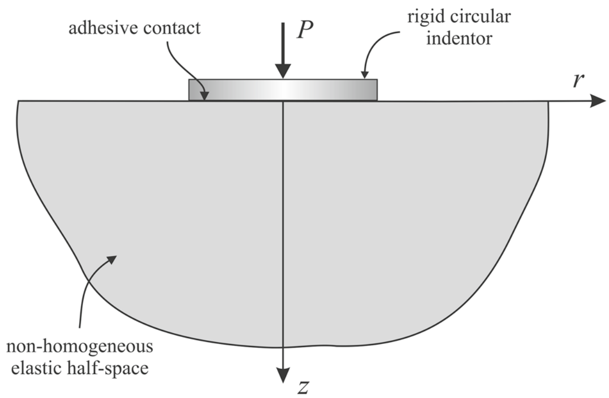

Fig. 1 Indentation of an incompressible non-homogeneous elastic halfspace

normal loading of the halfspace region (Boussinesq [2]; Westergaard [44]; Selvadurai [45,46]) pre-empts the need to consider the constraints imposed by bonding in the solution of the contact problem. This observation is strictly applicable to the case of a homogeneous elastic halfspace problem and does not generally apply in situations where the incompressible elastic halfspace region is non-homogeneous. Of related interest is a recent study by Labaki et al. [47] that examines the dynamics of an annular disc embedded at the interface of a bimaterial elastic domain.

3 The adhesive indentation of a non-homogeneous isotropic elastic halfspace

We now consider the problem of the indentation of an incompressible non-homogeneous isotropic elastic halfspace region by a rigid circular indenter of radius a with a flat base, which is bonded to the surface of the halfspace (Fig. 1). The specific non-homogeneity considered in the paper assumes that the shear modulus of the elastic medium varies exponentially according to

$$G(r,z) = G_0 e^{\lambda z};\ \ r \in (0,\infty);\ \ z \in (0,\infty) \tag{12}$$

and

$$\nu(r,z) = 1/2;\ \ r \in (0,\infty);\ \ z \in (0,\infty) \tag{13}$$

where G_0 is a constant. We now introduce the non-dimensional parameter $\tilde{\lambda}$ for characterizing the non-homogeneity, such that $\lambda = \tilde{\lambda}/a$.

The applicability of the exponential variation in the linear elastic shear modulus to geomaterials, and notably saturated clay deposits such as London Clay, was discussed by Selvadurai and Katebi [34]. These authors give references to further literature where the exponential variation in the linear elastic shear modulus was borne out by field investigations.

The objective of the study was to establish the influence of both the adhesive contact and the exponential variation in the shear modulus on the elastic stiffness of the rigid indenter. To the authors' knowledge, this problem has not been investigated in the literature on contact problems. In particular, the method of solution is based on the discretization of the contact normal stresses and contact shear stresses in the bonded region, which yields a set of algebraic equations that can be solved to develop results of engineering interest.

Prior to the application of the computational approach, it is instructive to illustrate the basic concepts of the adhesive contact problem related to the non-homogeneous elastic halfspace. Consider the problem where the surface of the non-homogeneous elastic halfspace region is subjected, separately, to (i) a normal ring load of intensity N (units of force/unit length) applied at the location $r = \rho$ (Fig. 2a) and (ii) a radially directed ring load of intensity T (units of force/unit length) applied at the location $r = \rho$ (Fig. 2b). The choice of these special forms of the fundamental results for N and T pre-supposes that the application of the discretization technique is restricted to axisymmetric contact problems related to an incompressible non-homogeneous elastic halfspace. The method of solution of the traction boundary value problems relevant to these basic loading configurations

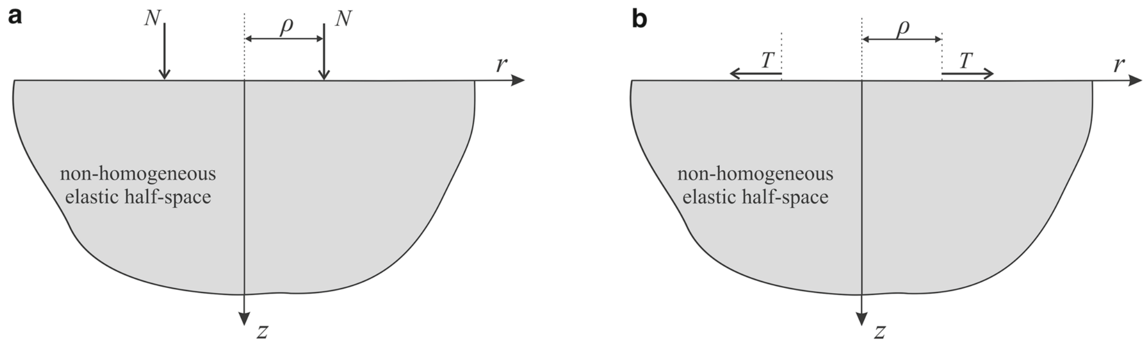

Fig. 2 **a** Normal ring load of intensity N (units of force/unit length) applied at the location $r = \rho$; **b** A radially directed ring load of intensity T (units of force/unit length) applied at the location $r = \rho$

is outlined in [34]. It can be shown that the axial and radial displacements at the surface of the halfspace region due to the normal ring loading N can be expressed in the general forms

$$u_z^N(r,0) = \frac{N}{G_0} I_{zz}^N(r,\rho); \quad 0 \le r < \infty,$$
$$u_r^N(r,0) = \frac{N}{G_0} I_{rz}^N(r,\rho); \quad 0 \le r < \infty \tag{14}$$

where the integrals $I_{zz}^N(r,\rho)$ and $I_{rz}^N(r,\rho)$ are given by

$$I_{zz}^N(r,\rho) = \int_0^\infty \left(\frac{(k_1^2 - k_2^2)}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \rho\xi J_0(\xi r) J_0(\xi\rho)\, \mathrm{d}\xi,$$
$$I_{rz}^N(r,\rho) = \int_0^\infty \left(\frac{k_2(\xi^2 + k_1^2) - k_1(\xi^2 + k_2^2)}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \rho J_1(\xi r) J_0(\xi\rho)\, \mathrm{d}\xi \tag{15}$$

with

$$k_1 = \frac{1}{2}\left[\lambda + \sqrt{\lambda^2 + 4i\lambda\xi + 4\xi^2}\right]; \quad k_2 = \frac{1}{2}\left[\lambda + \sqrt{\lambda^2 - 4i\lambda\xi + 4\xi^2}\right] \tag{16}$$

and

$$q_i = \frac{k_i^3}{\xi^2} - k_i - \frac{\lambda k_i^2}{\xi^2} - \lambda; \quad i = 1, 2. \tag{17}$$

Similarly, the axial and radial displacements on the surface of the halfspace region due to the radial ring loading T can be expressed in the general forms

$$u_z^T(r,0) = \frac{T}{G_0} I_{zr}^T(r,\rho); \quad 0 \le r < \infty,$$
$$u_r^T(r,0) = \frac{T}{G_0} I_{rr}^T(r,\rho); \quad 0 \le r < \infty \tag{18}$$

where the integrals $I_{rr}^T(r,\rho)$ and $I_{zr}^T(r,\rho)$ are given by

$$I_{zr}^T(r,\rho) = \int_0^\infty \left(\frac{(q_1 - 2k_1) + (2k_2 - q_2)}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \rho\xi^2 J_0(\xi r) J_1(\xi\rho)\mathrm{d}\xi,$$
$$I_{rr}^T(r,\rho) = \int_0^\infty \left(\frac{k_2 q_1 - k_1 q_2}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \rho\xi J_1(\xi r) J_1(\xi\rho)\mathrm{d}\xi. \tag{19}$$

The displacement fields given by Eqs. (14) and (18) will satisfy the regularity conditions required for the displacement field to vanish as $r \to \infty$ and the traction boundary conditions applicable to the ring loads. For the *normal ring load*

$$\begin{aligned} \sigma_{zz}(r,0) &= N\delta(r-\rho); \quad 0 \le r < \infty \\ \sigma_{rz}(r,0) &= 0; \qquad\qquad\quad 0 \le r < \infty \end{aligned} \tag{20}$$

where $\delta(r-\rho)$ is the Dirac delta function. Similarly, for the *radial ring load*

$$\begin{aligned} \sigma_{zz}(r,0) &= 0; \qquad\qquad\quad 0 \le r < \infty \\ \sigma_{rz}(r,0) &= T\delta(r-\rho); \; 0 \le r < \infty. \end{aligned} \tag{21}$$

The formal expressions for the axial and radial displacement fields generated by the contact normal stress $\sigma(r)$ and the contact shear stress $\tau(r)$ in the adhesive contact region of the rigid indenter can be written as

$$u_z(r,0) = \int_0^a \frac{\sigma(\rho)}{G_0} I_{zz}^N(r,\rho)\mathrm{d}\rho + \int_0^a \frac{\tau(\rho)}{G_0} I_{zr}^T(r,\rho)\mathrm{d}\rho; \quad 0 \le r < \infty, \tag{22}$$

$$u_r(r,0) = \int_0^a \frac{\tau(\rho)}{G_0} I_{rr}^T(r,\rho)\mathrm{d}\rho + \int_0^a \frac{\sigma(\rho)}{G_0} I_{rz}^N(r,\rho)\mathrm{d}\rho; \quad 0 \le r < \infty, \tag{23}$$

and ρ is an integration variable. The boundary conditions applicable to the adhesive contact problem are

$$\begin{aligned} u_z(r,0) &= \Delta; \; 0 \le r \le a \\ u_r(r,0) &= 0; \;\; 0 \le r \le a. \end{aligned} \tag{24}$$

An inspection of the integral expressions (15) and (19) for the influence functions $I_{zz}^N(r,\rho)$, $I_{rz}^N(r,\rho)$, etc., indicates that the integral Eqs. (22) and (23) are unlikely to be solved in a direct fashion to generate compact results of the type (6) for the elastic stiffness of the rigid circular indenter in adhesive contact with a non-homogeneous elastic halfspace. Therefore, recourse must be made to a computational approach that essentially discretizes the contact stress distributions.

4 The computational approach for the solution of the adhesive contact problem

We now focus on the development of a computational approach for the solution of the axisymmetric problem of a rigid circular indenter of radius a, which is in adhesive contact with an incompressible non-homogeneous isotropic elastic halfspace where the linear elastic shear modulus varies exponentially with depth and is subjected to an axisymmetric load equivalent to P. For the analysis of the adhesive contact problem, we assume that both the contact stress distributions $\sigma(r)$ and $\tau(r)$ can be represented as discrete regions of uniform stress acting over annular regions. This approach for the solution of contact problems has been developed by a number of investigators, and a comprehensive review of the approach is presented in Selvadurai [7]. The representation of the contact stress distributions as discrete values of finite magnitude allows no provision for accommodating the singularities that will be present at the boundary of the rigid indenter. In the case of bonded contact with a homogeneous elastic medium, the normal and shear stresses can exhibit oscillatory forms of the stress singularity that arises in the solution of Wiener-Hopf/Hilbert type problems. The oscillatory stress singularities are, however, integrable and contribute to finite strain energy in the halfspace region and yield exact closed form results for the axial stiffness of the bonded indenter. While consideration of the exact stress state is important to the assessment of fracture generation at the boundary of the bonded region, its influence on the estimation of the elastic stiffness is relatively small. Selvadurai [48–52] has shown that, in the case of a homogeneous elastic halfspace problem related to a bonded circular indenter, the incorporation of the oscillatory form of the stress singularity does not significantly influence the axial stiffness of the bonded rigid indenter. When the stress singularity at the boundary of the bonded rigid indenter of radius a is represented by a regular singularity of the form $1/(a^2 - r^2)^{1/2}$, the problem can be reduced to the solution of a Fredholm integral equation of the second kind, and a comparison of the results obtained from the exact solution and the above approximation indicates that the difference is less than 0.5 % when $\nu = 0$ and the solutions coincide when $\nu = 1/2$. We further note that in the case of an incompressible non-homogeneous elastic halfspace the local stress field at the boundary of the contact region will not be oscillatory in order to provide a continuous transition to the analogous problem for the adhesive contact problem associated with a homogeneous incompressible elastic halfspace. In keeping with these observations, we proceed to represent the contact stress distributions by their discrete equivalent distributions, and the geometries of the discrete annular regions are altered to account for sharp stress gradients that could be present in regions close to the boundary of the contact zone.

4.1 Smooth contact problem for a rigid circular indenter

First, we consider the indentation of a non-homogeneous incompressible elastic halfspace by a rigid circular indenter of radius a with smooth contact subjected to an axial load P. The method of solution assumes that the plan area of the indenter is discretized into equal annular areas and the contact stress within each annular area is uniform. Considering the annular normal load of stress intensity $\sigma_1, \sigma_2, \ldots, \sigma_n$ acting within the annular region of internal radius $0, r_1, r_2, \ldots, r_{n-1}$ and external radius $r_1, r_2, \ldots, r_n$, respectively, the surface displacement $w_1, w_2, w_3, \ldots, w_n$ at the mid-section of the annular region due to normal surface tractions can be obtained by superposition of Eqs. (25) and (26), which are a simplified version of the results developed by Selvadurai and Katebi [34],

$$u_z(r,0) = \frac{p_0 a}{G_0}\int_0^\infty \left(\frac{(k_1^2-k_2^2)}{(2k_1-q_1)(\xi^2+k_2^2)+(q_2-2k_2)(\xi^2+k_1^2)}\right) J_0(\xi r) J_1(\xi a)\,\mathrm{d}\xi, \tag{25}$$

$$u_r(r,0) = \frac{p_0 a}{G_0}\int_0^\infty \left(\frac{k_2(\xi^2+k_1^2)-k_1(\xi^2+k_2^2)}{\xi(2k_1-q_1)(\xi^2+k_2^2)+\xi(q_2-2k_2)(\xi^2+k_1^2)}\right) J_1(\xi r) J_1(\xi a)\,\mathrm{d}\xi \tag{26}$$

where k_1, k_2, q_1 and q_2 are defined by Eqs. (16) and (17). The procedures for the numerical evaluation of these integrals are given in [34] and [53].

The compatibility of displacement (Δ) between the non-homogenous incompressible elastic halfspace and the indenter is then established at the mid-point location of each annular region. The physical domain of interest is taken to be a non-homogeneous incompressible ($v = 1/2$) elastic halfspace in which the shear modulus has an exponential variation over the entire depth of the halfspace (Eq. (12)).

In order to assign equal annular areas, the dimensions of r_i take the following form:

$$r_i = \left(\frac{i}{n}\right)^{\frac{1}{2}} a; \quad i = 1, 2, 3, \ldots.., n. \tag{27}$$

Similarly for the mid-points,

$$r_{m1} = 0, \quad r_{mi} = \left(\frac{r_{i-1}+r_i}{2}\right); \quad i = 2, 3, \ldots, n. \tag{28}$$

We further assume that the uniform normal stress elements σ_i can be represented as multiples of the average pressure p_0 that is applied externally to the rigid indenter,

$$\sigma_i = \tilde{\sigma}_i p_0 \quad \text{where} \quad i = 1, 2, 3, \ldots, n \tag{29}$$

where

$$p_0 = \frac{P}{\pi a^2}. \tag{30}$$

Using the above reductions, it is possible to express the surface displacements w_i due to normal contact stresses $\tilde{\sigma}_i$ in the form of the matrix relation

$$\{\mathbf{w}\} = \frac{p_0 a}{G_0}[\mathbf{C}]\{\tilde{\boldsymbol{\sigma}}\} \tag{31}$$

Table 1 Comparison of analytical and numerical solutions

n	5	10	15	20	Boussinesq [2]
Δ^*	0.400676	0.395437	0.394525	0.378847	0.392699

where $\{\mathbf{w}\}$ and $\{\tilde{\boldsymbol{\sigma}}\}$ are column vectors, and the coefficients of the square matrix $[\mathbf{C}]$ are as follows:

$$[\mathbf{C}] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \dots & w_{1j} \\ w_{21} & w_{22} & w_{23} & \dots & w_{2j} \\ w_{31} & w_{32} & w_{33} & \dots & w_{3j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & w_{i2} & \dots & w_{ij} \end{bmatrix} \tag{32}$$

where w_{ij} are the surface displacements at the annuli mid-point location i due to the normal contact stress $\sigma_j (j = i)$ (see the "Appendix A").

The results in the form presented here are applicable to the calculation of surface displacements involving any arbitrary distributed loading with an axisymmetric profile, which is a useful result for geotechnical engineering studies in general.

We shall now focus attention on the rigid indenter problem. For compatibility at the soil-indenter interface, w_i will have the same value as the rigid displacement of the indenter Δ; this constitutes an additional unknown of the problem. The remaining equation required for the solution of Eq. (31) is furnished by the equilibrium equation for the entire indenter, i.e.

$$\sum_{i=1}^{n} \tilde{\sigma}_i = n. \tag{33}$$

The matrix formed by combining Eqs. (31) and (33) can be inverted to determine the non-dimensional contact stresses $\tilde{\sigma}_i$ and the non-dimensional indenter displacement,

$$\Delta^* = \frac{G_0 \Delta}{p_0 a}. \tag{34}$$

The accuracy of this discretized solution can be verified by comparing the computational estimates with the exact closed form results given by the classical result of Boussinesq [2].

In this paper, the rigid indenter is discretized into 5, 10, 15 and 20 equal annular areas (i.e.$n = 5, 10, 15, 20$). Table 1 presents the comparison between the Boussinesq [2] and the discretized solution for $n = 5, 10, 15, 20$, respectively. The accuracy of the solution depends on the number of annular regions. However, the number of annular regions cannot be increased infinitely; such a procedure usually results in an ill-conditioned set of equations. These difficulties stem from the singular behaviour of the contact stress at the indenter boundary. It can be seen from Table 1 that the accuracy of the results increases by increasing the discretized areas from 5 to 15 but the accuracy is reduced by increasing the number of discretized areas from 15 to 20 due to the singularity effect at the indenter boundary.

Ideally both the contact stresses and the indentation should be predicted with equal accuracy in a discretization approach. Due the simplification of representing a singular stress field as a finite value over an annular region, the discretized contact stresses will exhibit a departure from the theoretical Boussinesq [2] profile as n is increased. Furthermore, the direct comparison of the stress values obtained at the mid-points of the annular regions from the discretization scheme with the stress values computed at the corresponding points applicable to the analytical solution for the compressive contact stress will not capture the overall effect of the stress gradients associated with Boussinesq's result for the contact stress at the base of a smooth rigid indenter given by

$$\sigma_{zz}(r, 0) = \frac{P}{2\pi a \sqrt{a^2 - r^2}} \tag{35}$$

where P is the total load applied to the indenter and a is the radius of the indenter. For this reason, it is necessary to calculate the average stress within either the central element or an intermediate element is calculated using Boussinesq's distribution. Consider the case where the contact region is discretized into n equi-areal regions with a circular central region and $(n - 1)$ annular exterior regions. Consider an annular region bounded by

Table 2 Comparison of contact stress between Boussinesq results and current study

n	Center		Outermost annular boundary region	
	10	15	10	15
$\tilde{\sigma}_i$ (Boussinesq [2])	0.5132	0.5086	3.1622	3.8729
$\tilde{\sigma}_i$ (current study $\tilde{\lambda} = 0$)	0.5157	0.5098	2.9832	3.8463

$r \in (\tilde{\rho}_i a, \tilde{\rho}_e a)$, where $\tilde{\rho}_i a$ is the interior radius of the annulus and $\tilde{\rho}_e a$ is the external radius. The relationship between the radii $\tilde{\rho}_i$ and $\tilde{\rho}_e$ is given by

$$\tilde{\rho}_e = \left(\tilde{\rho}_i^2 + \frac{1}{n}\right)^{1/2}. \tag{36}$$

The total force acting over the interval $r \in (\tilde{\rho}_i a, \tilde{\rho}_e a)$ due to the Boussinesq distribution is given by

$$P^* = \int_{\tilde{\rho}_i a}^{\tilde{\rho}_e a} \int_0^{2\pi} \frac{P\, r \mathrm{d}r\, \mathrm{d}\theta}{2\pi a\sqrt{a^2 - r^2}} = \frac{P}{a} \int_{\tilde{\rho}_i a}^{\tilde{\rho}_e a} \frac{r \mathrm{d}r}{\sqrt{a^2 - r^2}}. \tag{37}$$

The average stress in the annular region $r \in (\tilde{\rho}_i a, \tilde{\rho}_e a)$ due to the Boussinesq distribution is

$$\sigma_i = \frac{nP}{\pi a^2}\left(\sqrt{1 - \tilde{\rho}_i^2} - \sqrt{1 - \tilde{\rho}_e^2}\right). \tag{38}$$

The average stress within the annulus normalized with respect to the average stress over the entire contact region, $P/\pi a^2$, is given by

$$\tilde{\sigma}_i = n\left[\sqrt{1 - \tilde{\rho}_i^2} - \sqrt{1 - \tilde{\rho}_e^2}\right]. \tag{39}$$

Table 2 shows the comparison for contact stress between the techniques proposed in this paper and those determined using Eq. (39) using Boussinesq's results.

In these computations, for example, (i) when $n = 10$, the discrepancy in the displacement is $-0.7\,\%$ and the discrepancy in the contact stresses at the centre of the indenter is $-0.48\,\%$ and the discrepancy in the contact stress at the annular boundary region is $-5.66\,\%$, and (ii) when $n = 15$, the discrepancy in the displacement is $-0.465\,\%$, and the discrepancy in the contact stresses at the centre of the indenter is $-0.24\,\%$, and the discrepancy in the contact stress at the annular boundary region is $-0.68\,\%$. In the discretized analysis of the adhesive contact problem for the inhomogeneous incompressible elastic halfspace, the discretizations are selected as $n = 10$ and $n = 15$. It should also be noted that any computational estimates for the contact problem, based on either finite element or boundary element approaches will also display similar manifestations, unless special provisions are made to include singularity elements at the boundary point of the indenting region, and infinite elements are used to capture the semi-infinite domain.

Figure 3 shows the displacement of the rigid indenter on a non-homogeneous medium obtained using the discretization approach to the homogeneous medium normalized with respect to the analytical result of Boussinesq [2] and presented as a function of the depth variation in the shear modulus characterized by $\tilde{\lambda}$ and for different values of the discretization intervals n. Figure 3 clearly shows that the displacement of the indenter is influenced by the degree of non-homogeneity in the linear elastic shear modulus. As would be expected, the axial displacement of the indenter decreases as the shear modulus increases with depth.

Figure 4 shows the contact stress distribution beneath a rigid indenter resting on a non-homogeneous elastic halfspace, for different shear moduli, with $n = 10$. A comparison has also been made between the current results for the contact stress distribution $\tilde{\sigma}_i$ at mid-point locations r_{mi} and those determined from the exact result given by Boussinesq [2]. Similar results are presented in Fig. 5 for the contact pressure distributions obtained for a numerical procedure where $n = 15$. As evident from Figs. 4 and 5, there is good correlation between the results obtained from the discretization method and the analytical results given by Boussinesq [2], when $\tilde{\lambda} = 0$. It should also be noted that for the incompressible homogeneous elastic halfspace the normal contact stresses within the indented area are independent of the elasticity parameters of the halfspace and invariant of the adhesion constraint, i.e. when $\nu = 1/2$, $\gamma \to 0$, and (4) gives rise to the conventional Abel integral equation for the normal contact stress $\sigma(r)$, which yields Boussinesq's result for the indentation problem [4,5]. It can be inferred that discrepancies between the results for the indentation of the incompressible non-homogeneous elastic halfspace and Boussinesq's result are likely to be due to the elastic non-homogeneity.

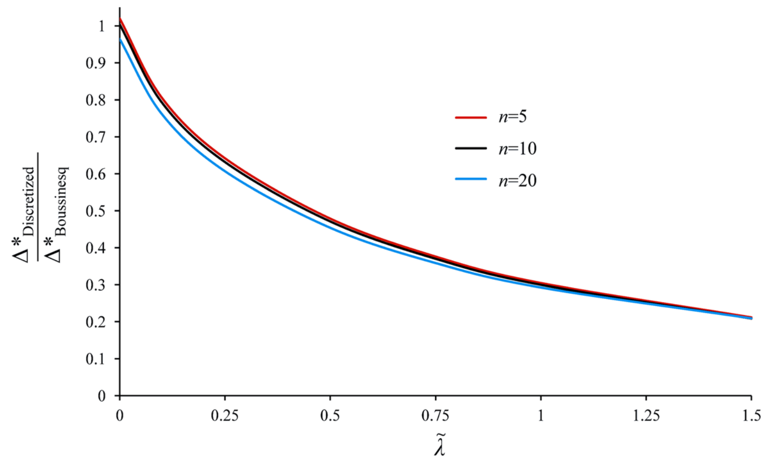

Fig. 3 Ratio of displacement of the rigid indenter on a non-homogeneous medium (numerical discretized solution) to homogeneous medium (Boussinesq [2]) for different $\tilde{\lambda}$, for $n = 5, 10, 20$

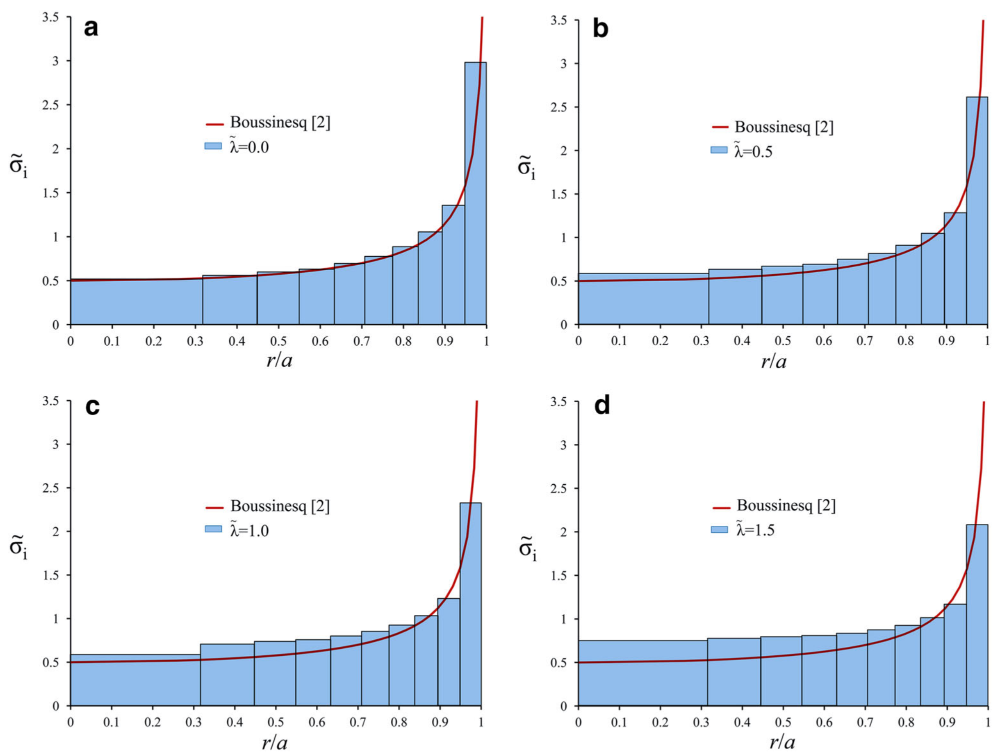

Fig. 4 Comparison of contact stress distribution ($\tilde{\sigma}_i$) of the rigid indenter between the current results and results given by Boussinesq [2] ($n = 10$)

4.2 Adhesive contact problem for a rigid circular indenter

We consider the adhesive contact problem for a rigid circular indenter with a flat base, which is in contact with a non-homogeneous incompressible elastic halfspace in which the shear modulus varies exponentially with depth (Eq. (12) and is) subjected to an axial load P. The boundary conditions applicable to the adhesive contact are given in Eq. (3).

In order to solve this problem using a discretization approach similar to that described in Sect. 4.1, we first assume that a non-homogenous incompressible elastic halfspace is subjected to uniform normal surface tractions σ_i^b and uniform shear surface tractions τ_i^b at equi-areal annular regions with locations defined by r_i. Then, the compatibility between the displacement of the non-homogenous incompressible halfspace within

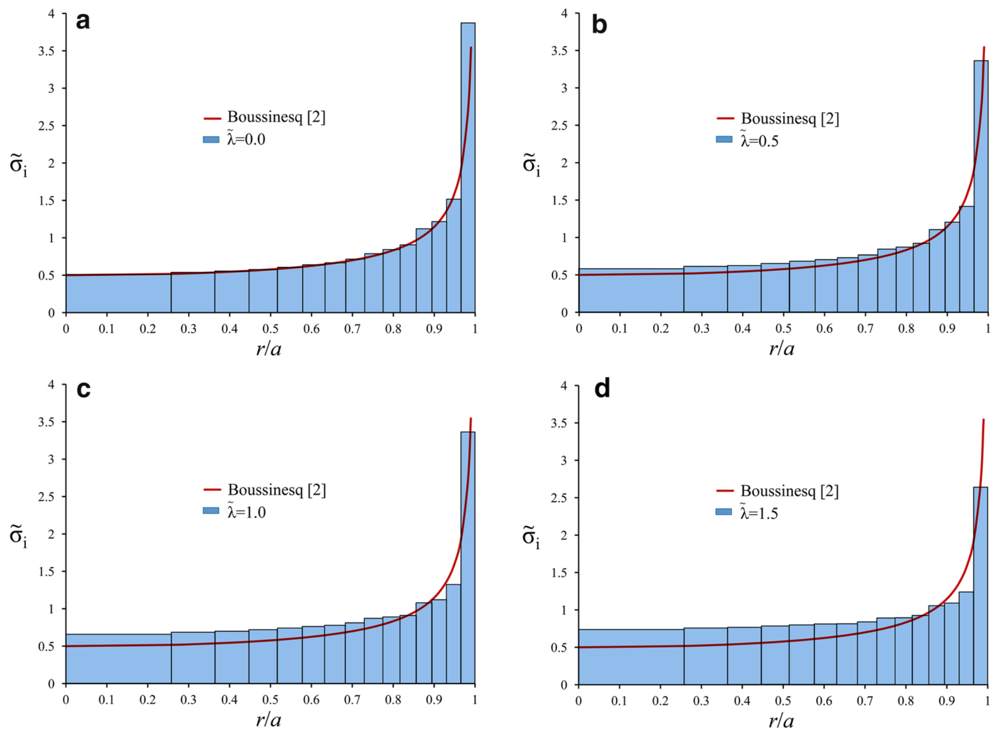

Fig. 5 Comparison of contact stress distribution ($\tilde{\sigma}_i$) of the rigid indenter between the current results and results given and Boussinesq [2] ($n = 15$)

the contact region due to the combined actions of normal and shear surface tractions and the axial one of the indenter, Δ, and null radial displacement is established at the mid-point location of each annular region.

We further assume that the uniform normal stress elements σ_i^b and uniform shear stress elements τ_i^b can be presented as multiples of the externally applied average pressure p_0 [see Eq. (30)] and an average shear traction τ_0 (to be determined), respectively, in the forms

$$\sigma_i^b = \tilde{\sigma}_i^b p_0 \quad \text{where} \quad i = 1, 2, 3, \ldots, n, \tag{40}$$

$$\tau_i^b = \tilde{\tau}_i^b \tau_0 \quad \text{where} \quad i = 1, 2, 3, \ldots, n. \tag{41}$$

Using the above reductions, it is possible to express the surface axial and radial displacements w_i and u_i due to the non-dimensional contact stresses $\tilde{\sigma}_i^b$ and $\tilde{\tau}_i^b$ in the form of the matrix relations

$$\{\mathbf{w}\} = \frac{p_0 a}{G_0}[\mathbf{C}_1]\{\tilde{\boldsymbol{\sigma}}^b\} + \frac{\tau_0 a}{G_0}[\mathbf{C}_2]\{\tilde{\boldsymbol{\tau}}^b\}, \tag{42}$$

$$\{\mathbf{u}\} = \frac{p_0 a}{G_0}[\mathbf{C}_3]\{\tilde{\boldsymbol{\sigma}}^b\} + \frac{\tau_0 a}{G_0}[\mathbf{C}_4]\{\tilde{\boldsymbol{\tau}}^b\} \tag{43}$$

where $\{\mathbf{w}\}$, $\{\mathbf{u}\}$, $\{\tilde{\boldsymbol{\sigma}}^b\}$ and $\{\tilde{\boldsymbol{\tau}}^b\}$ are column vectors, and the coefficients of the square matrices $[\mathbf{C}_i]$ ($i = 1, 2, 3, 4$) are as follows:

$$[\mathbf{C}_1] = \begin{bmatrix} w_{11} & w_{12} & w_{13} & \ldots & w_{1j} \\ w_{21} & w_{22} & w_{23} & \ldots & w_{2j} \\ w_{31} & w_{32} & w_{33} & \ldots & w_{3j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{i1} & w_{i2} & w_{i2} & \ldots & w_{ij} \end{bmatrix}, [\mathbf{C}_2] = \begin{bmatrix} w'_{11} & w'_{12} & w'_{13} & \ldots & w'_{1j} \\ w'_{21} & w'_{22} & w'_{23} & \ldots & w'_{2j} \\ w'_{31} & w'_{32} & w'_{33} & \ldots & w'_{3j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w'_{i1} & w'_{i2} & w'_{i2} & \ldots & w'_{ij} \end{bmatrix}, \tag{44}$$

$$[\mathbf{C}_3] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1j} \\ u_{21} & u_{22} & u_{23} & \dots & u_{2j} \\ u_{31} & u_{32} & u_{33} & \dots & u_{3j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{i1} & u_{i2} & u_{i2} & \dots & u_{ij} \end{bmatrix}, [\mathbf{C}_4] = \begin{bmatrix} u'_{11} & u'_{12} & u'_{13} & \dots & u'_{1j} \\ u'_{21} & u'_{22} & u'_{23} & \dots & u'_{2j} \\ u'_{31} & u'_{32} & u'_{33} & \dots & u'_{3j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u'_{i1} & u'_{i2} & u'_{i2} & \dots & u'_{ij} \end{bmatrix} \tag{45}$$

where u_{ij} and u'_{ij} are radial surface displacements, and w_{ij} and w'_{ij} are the axial surface displacement at the annuli mid-point locations due to the normal and radial stress traction σ_j^b and τ_j^b, respectively (see "Appendix A").

The displacement of a non-homogeneous incompressible elastic halfspace subjected to a surface distributed radial load of radius a with stress intensity τ_0 can be found by

$$u_z(r,0) = \frac{\tau_0}{G_0}\int_0^\infty \left(\frac{(q_1 - 2k_1) + (2k_2 - q_2)}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \xi^2 J_0(\xi r) \left[\int_0^a r J_1(\xi r) \mathrm{d}r \right] \mathrm{d}\xi, \tag{46}$$

$$u_r(r,0) = \frac{\tau_0}{G_0}\int_0^\infty \left(\frac{k_2 q_1 - k_1 q_2}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \xi J_1(\xi r) \left[\int_0^a r J_1(\xi r) \mathrm{d}r \right] \mathrm{d}\xi \tag{47}$$

where k_1, k_2, q_1 and q_2 are defined by Eqs. (16) and (17).

Equations (46) and (47) can be numerically evaluated using procedures presented in [34] and [53].

The axial and radial surface displacement due to the normal tractions σ_j^b at the central location can be obtained by a superposition of the results for the axial and radial displacements given in Eqs. (25), (26), (46) and (47).

The axial and radial surface displacement due to the shear stress traction τ_j^b can be generated from Eqs. (46) and (47) by superposition. The normal and shear surface tractions are related to each other by the constraint given in Eq. (7). Using (7) along with Eq. (43) results in:

$$\frac{p_0 a}{G_0}[\mathbf{C}_3]\{\tilde{\boldsymbol{\sigma}}^b\} + \frac{\tau_0 a}{G_0}[\mathbf{C}_4]\{\tilde{\boldsymbol{\tau}}^b\} = \{\mathbf{0}\}, \tag{48}$$

and this result along with Eqs. (40) and (41) can be used to determine the shear tractions based on the normal tractions

$$\{\boldsymbol{\tau}^b\} = [B]\{\boldsymbol{\sigma}^b\}. \tag{49}$$

By using Eq. (49), we can rewrite Eq. (42) as follows:

$$\{\mathbf{w}\} = \frac{p_0 a}{G_0}[\mathbf{H}]\{\tilde{\boldsymbol{\sigma}}^b\} \tag{50}$$

in which $[\mathbf{H}]$ was obtained by adding the two matrices $[\mathbf{C}_1]$ and $[\mathbf{C}_2]$, which also incorporates the result Eq. (49). The results in the form presented here are applicable to the calculations of displacements involving any distributed loading with an axisymmetric profile.

For compatibility of the displacements at the incompressible non-homogeneous elastic halfspace-indenter interface, w_i will have the same value as the rigid displacement of the indenter Δ; this constitutes an additional unknown of the problem. The remaining equation required for the solution of Eq. (50) is the equilibrium equation for the entire indenter given in Eq. (33). The matrix formed by combining Eqs. (33) and (50) can be inverted to determine the non-dimensional contact stresses $\tilde{\sigma}_i^b$ and $\tilde{\tau}_i^b$ and the non-dimensional axial displacement of the indenter in bonded contact with the incompressible non-homogeneous elastic halfspace. The contact normal stresses for the indenter, which is adhesively bonded to the incompressible elastic halfspace are shown in Figs. 4 and 5, developed, respectively, by considering discretizations $n = 10$ and $n = 15$.

Figure 6 presents a comparison of the surface displacement of the rigid indenter for smooth contact, adhesive contact, and for the case where a surface with inextensibility (in which $u_r = 0$ throughout the surface of the halfspace) is enforced. Results are provided for different values of the non-homogeneity parameter $\tilde{\lambda}$. These displacements are normalized with the surface displacement given by Boussinesq [2].

The results show that the stiffness of the indenter in adhesive contact is bounded by the corresponding values for indentation with frictionless contact and the result for the indentational stiffness that incorporated the surface

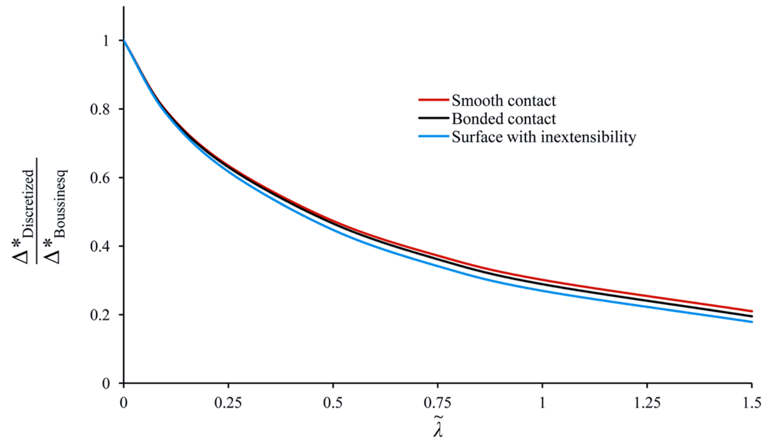

Fig. 6 Ratio of displacement of the rigid disc on a non-homogeneous medium (numerical discretized solution) to homogeneous medium (Boussinesq [2]) for different $\tilde{\lambda}$, comparison between smooth and bonded contact

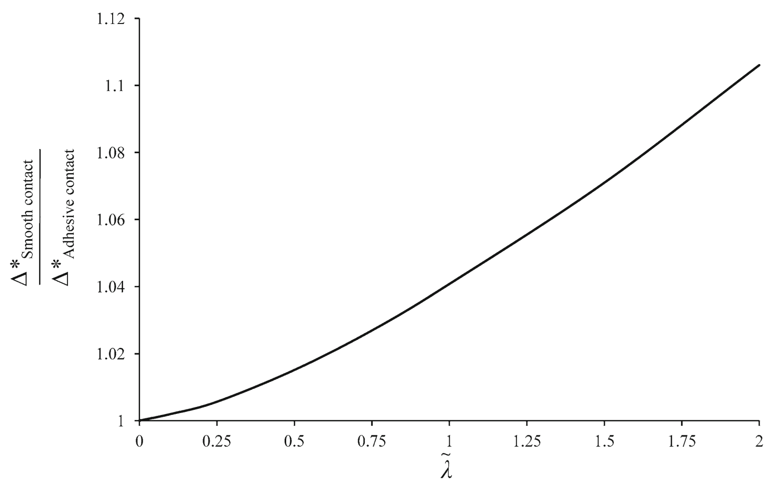

Fig. 7 Ratio of displacement of the bonded contact rigid indenter to the smooth contact rigid indenter for different values of $\tilde{\lambda}$

inextensibility constraint [see also [42]). Figure 7 shows the influence of adhesion and the non-homogeneity parameter on the indentational stiffness the rigid circular indenter. The effect of adhesive contact increases from zero for an incompressible homogenous halfspace with $\tilde{\lambda} = 0$ to an almost 10 % difference between the settlement of smooth contact and adhesive contact, for a non-homogeneous incompressible halfspace with $\tilde{\lambda} = 2$. However, the change in displacement for $\tilde{\lambda} < 0.25$, which is applicable to a range of variations in the shear modulus with depth observed in saturated clays found in regions of the British Isles, is less than 2 % (see e.g. [34]).

5 Concluding remarks

The mechanics of indentation is visualized as a technique for the estimation of deformability characteristics of materials as well as for the estimation of settlements of geotechnical structures. The interface conditions between the deformable region and the rigid indenter can influence the indentational stiffness. In the classical problem for the adhesive indentation of an isotropic homogeneous elastic halfspace, the indentational stiffness is controlled by Poisson's ratio for the deformable medium and, in the case of an *incompressible homogeneous elastic halfspace region*, the interface conditions (either frictionless or fully bonded) have no influence on the elastic stiffness. This is due to the zero radial displacement at the surface of the halfspace due to Boussinesq's problem for the concentrated normal force. This paper examines the problem of the axisymmetric adhesive indentation of an incompressible *non-homogeneous elastic halfspace*, with an exponential variation of the shear modulus with depth, by an indenter with a flat base. The formulation of the integral equations for the normal and shear stresses at the adhesive zone indicates that the solution cannot be readily accomplished using integral equation techniques commonly employed in the study of adhesive contact problems. The paper develops a numerical scheme where the contact normal stresses and shear stresses are represented by discretized equivalents, and the unknown values are evaluated by considering the kinematic and mechanical constraints on the adhesive zone. These results are used to estimate the axial stiffness of the adhesive indentation of a rigid circular indenter with an inhomogeneous elastic halfspace that has an exponential variation in the shear modulus. The results are also compared with estimates for the indentation stiffness when the contact is frictionless and when the entire surface of the non-homogeneous halfspace is considered to be radially inextensible. It is observed that for the exponential variation in shear modulus in an incompressible elastic halfspace, the contact constraints, either adhesive contact or frictionless contact, have very small influence on the indentational stiffness of the rigid circular indenter and for the exponent in the exponential variation $\tilde{\lambda} \in (0, 0.25)$. The discrepancy is of the order of 10 % for $\tilde{\lambda} \approx 2$. The representation of the contact stress distribution in terms of a discretized distribution is a convenient mathematical approximation for the solution of a contact problem where the analytical solution would otherwise be intractable. The discretization technique offers a convenient solution scheme for the adhesive contact problem for the non-homogeneous incompressible elastic halfspace problem. Furthermore, the general concepts can be extended to include flexible contact zones and partial adhesion in the contact zone. The discretization techniques can also be extended to examine indenters with arbitrary-shaped plan forms that will not be amenable to exact solution. In situations where processes such as indentational fracture needs to be examined (e.g. [54]), the singularity at the boundary of the indenter needs to be incorporated in the discretization scheme, so that the stress state can be more precisely defined to generate stress intensity factors and energy release rates important to crack extension analysis can be accurately determined. Also, when semi-infinite domains are encountered, the accurate modelling is accomplished through the use of special infinite elements [55].

Acknowledgments The authors are indebted to the reviewers whose comments led to improvements in the presentation of the material. The work described in the paper was supported by a NSERC Discovery Grant awarded to A. P. S. Selvadurai. The authors acknowledge the comments made by a reviewer.

Appendix A

The w_{ij}, w'_{ij}, u_{ij} and u'_{ij} in Eqs. (32), (44) and (45) can be calculated by using superposition technique, we have

w_{ij}: by using Eq. (25), we have

for $j = 1$ and $i = 1, 2, \ldots, n$:

$$w_{ij}(r_{mi}, 0) = \int_0^\infty \left(\frac{(k_1^2 - k_2^2)}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) J_0(r_{mi}\xi) \frac{r_1}{a} J_1(r_1\xi) \mathrm{d}\xi, \tag{A1}$$

and for $j = 2, 3, \ldots, n$ and $i = 1, 2, \ldots, n$:

$$w_{ij}(r_{mi}, 0) = \int_0^\infty \left(\frac{(k_1^2 - k_2^2)}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) J_0(r_{mi}\xi) \left[\frac{r_j}{a} J_1(r_j\xi) - \frac{r_{j-1}}{a} J_1(r_{j-1}\xi) \right] \mathrm{d}\xi; \tag{A2}$$

w'_{ij}: by using Eq. (46), we have

for $j = 1$ and $i = 1, 2, \ldots, n$:

$$w'_{i1}(r_{mi}, 0) = \int_0^\infty \left(\frac{(q_1 - 2k_1) + (2k_2 - q_2)}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \xi^2 J_0(r_{mi}\xi) \left[\int_0^{r_1} \frac{r}{a} J_1(\xi r) \mathrm{d}r \right] \mathrm{d}\xi, \tag{A3}$$

and for $j = 2, 3, \ldots, n$ and $i = 1, 2, \ldots, n$:

$$w'_{ij}(r_{mi}, 0) = \int_0^\infty \left(\frac{(q_1 - 2k_1)+(2k_2 - q_2)}{(2k_1 - q_1)(\xi^2 + k_2^2)+(q_2 - 2k_2)(\xi^2+k_1^2)} \right) \xi^2 J_0(r_{mi}\xi) \left[\int_{r_{j-1}}^{r_j} \frac{r}{a} J_1(\xi r)\mathrm{d}r \right] \mathrm{d}\xi; \quad \text{(A4)}$$

u_{ij}: by using Eq. (26), we have
for $j = 1$ and $i = 1, 2, \ldots, n$:

$$u_{i1}(r_{mi}, 0) = \int_0^\infty \left(\frac{k_2(\xi^2 + k_1^2) - k_1(\xi^2 + k_2^2)}{\xi(2k_1 - q_1)(\xi^2 + k_2^2) + \xi(q_2 - 2k_2)(\xi^2 + k_1^2)} \right) J_1(r_{mi}\xi) \frac{r_1}{a} J_1(r_1\xi)\mathrm{d}\xi, \quad \text{(A5)}$$

and for $j = 2, 3, \ldots, n$ and $i = 1, 2, \ldots, n$:

$$u_{ij}(r_{mi}, 0) = \int_0^\infty \left(\frac{k_2(\xi^2 + k_1^2) - k_1(\xi^2 + k_2^2)}{\xi(2k_1 - q_1)(\xi^2 + k_2^2) + \xi(q_2 - 2k_2)(\xi^2 + k_1^2)} \right) J_1(r_{mi}\xi) \left[\frac{r_j}{a} J_1(r_j\xi) - \frac{r_{j-1}}{a} J_1(r_{j-1}\xi) \right] \mathrm{d}\xi; \quad \text{(A6)}$$

u'_{ij}: by using Eq. (47), we have
for $j = 1$ and $i = 1, 2, \ldots, n$:

$$u'_{i1}(r_{mi}, 0) = \int_0^\infty \left(\frac{k_2 q_1 - k_1 q_2}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \xi J_1(r_{mi}\xi) \left[\int_0^{r_1} \frac{r}{a} J_1(\xi r)\mathrm{d}r \right] \mathrm{d}\xi, \quad \text{(A7)}$$

and for $j = 2, 3, \ldots, n$ and $i = 1, 2, \ldots, n$:

$$u'_{ij}(r_{mi}, 0) = \int_0^\infty \left(\frac{k_2 q_1 - k_1 q_2}{(2k_1 - q_1)(\xi^2 + k_2^2) + (q_2 - 2k_2)(\xi^2 + k_1^2)} \right) \xi J_1(r_{mi}\xi) \left[\int_{r_{j-1}}^{r_j} \frac{r}{a} J_1(r\xi)\mathrm{d}r \right] \mathrm{d}\xi. \quad \text{(A8)}$$

References

1. Hertz, H.: On the contact of firm elastic bodies (in German: Ueber die Berührung fester elastischer Körper). J. Reine Angewandte Mathematik **92**, 156–171 (1882)
2. Boussinesq, J.: Application des Potentiels à l'Etude de l'Equilibre et du Mouvement des Solides. Gauthier-Villars, Paris (1885)
3. Galin, L.A.: Contact Problems in the Theory of Elasticity. North Carolina State College (1961) (Translated from Russian by H. Moss)
4. Sneddon, I.N.: Fourier Transforms. McGraw-Hill, New York (1951)
5. Sneddon, I.N.: The relation between load and penetration in the axisymmetric Boussinesq problem for a punch of arbitrary profile. Int. J. Eng. Sci. **3**, 45–47 (1965)
6. Ufliand, Ia.S.: Survey of Applications of Integral Transforms in the Theory of Elasticity, (Engl. Trans. Edited by I.N. Sneddon) Tech. Rep. 65-1556. North Carolina State College, Raleigh, NC (1965)
7. Selvadurai, A.P.S.: Elastic Analysis of Soil-Foundation Interaction. Developments in Geotechnical Engineering, vol. 17. Elsevier, Amsterdam (1979)
8. Selvadurai, A.P.S.: The analytical method in geomechanics. Appl. Mech. Rev. **60**, 87–107 (2007)
9. Gladwell, G.M.L.: Contact Problems in the Classical Theory of Elasticity. Sijthoff and Noordhoff, The Netherlands (1980)
10. Willner, K.: Kontinuums- und Kontaktmechanik. Synthetische und Analytische Darstellung. Springer, Berlin (2003)
11. Selvadurai, A.P.S.: The interaction between a uniformly loaded circular plate and an isotropic elastic halfspace: a variational approach. J. Struct. Mech. **7**, 231–246 (1979)
12. Selvadurai, A.P.S.: Elastic contact between a flexible circular plate and a transversely isotropic elastic halfspace. Int. J. Solid. Struct. **16**, 167–176 (1980)
13. Selvadurai, A.P.S.: The flexure of an infinite strip of finite width embedded in an isotropic elastic medium of infinite extent. Int. J. Numer. Anal. Meth. Geomech. **8**, 157–166 (1984)
14. Selvadurai, A.P.S.: A contact problem for a Reissner plate and an isotropic elastic halfspace. J. Theor. Appl. Mech. **3**, 181–196 (1986)

15. Rajapakse, R.K.N.D., Selvadurai, A.P.S.: On the performance of Mindlin plate elements in modelling plate elastic medium interaction. In. J. Num. Meth. Eng. **23**, 1229–1244 (1986)
16. Selvadurai, A.P.S., Dumont, N.A.: Mindlin's problem for a halfspace indented by a flexible plate. J. Elast. **105**, 253–269 (2011)
17. Oliveira, M.F.F., Dumont, N.A., Selvadurai, A.P.S.: Boundary element formulation of axisymmetric problems for an elastic halfspace. Eng. An. Bound. El. **36**, 1478–1492 (2012)
18. Kassir, M.K., Chuapresert, M.F.: A rigid punch in contact with a non-homogeneous elastic solid. J. Appl. Mech. **41**, 1019–1024 (1974)
19. Clements, D.L., Atkinson, C., Rogers, C.: Antiplane crack problems for inhomogeneous elastic material. Acta Mech. **29**, 199–211 (1978)
20. Sladek, V., Sladek, J., Markechova, I.: An advanced boundary element method for elasticity problems in nonhomogeneous media. Acta Mech. **97**, 71–90 (1993)
21. Muravskii, G.B.: Response of an elastic halfspace with power-law nonhomogeneity to static loads. Arch. Appl. Mech. **78**, 965–987 (2008)
22. Clements, D.L., Kusuma, J.: Axisymmetric loading of a class of inhomogeneous transversely isotropic halfspaces with quadratic elastic moduli. Q. J. Mech. Appl. Math. (2011). doi:10.1093/qjmam/hbq021
23. Koronev, B.G.: A die resting on elastic halfspace, the modulus of elasticity of which is an exponential function of depth. Dokl. Akad. Nauk. SSSR **112**, 5 (1957)
24. Mossakovskii, V.I.: Pressure of a circular punch on an elastic halfspace whose modulus of elasticity is an exponential function. Prikl. Mat. Mekh. **22**, 168–171 (1958)
25. Gibson, R.E.: Some results concerning displacements and stresses in a non-homogeneous elastic halfspace. Geotechnique **17**, 58–67 (1967)
26. Selvadurai, A.P.S., Singh, B.M., Vrbik, J.: A Reissner–Sagoci problem for a non-homogeneous elastic solid. J. Elast. **16**, 383–391 (1986)
27. Selvadurai, A.P.S.: The settlement of a rigid circular foundation resting on a halfspace exhibiting a near surface elastic non-homogeneity. Int. J. Numer. Anal. Meth. Geomech. **20**, 351–364 (1996)
28. Selvadurai, A.P.S.: The penny-shaped crack at a bonded plane with a localized elastic non-homogeneity. Eur. J. Mech. A. **19**, 525–534 (2000)
29. Rajapakse, R.K.N.D.: A vertical load in the interior of a non-homogeneous incompressible elastic halfspace. Q. J. Mech. Appl. Math. **43**, 1–14 (1990)
30. Rajapakse, R.K.N.D., Selvadurai, A.P.S.: Response of circular footings and anchor plates in non-homogeneous elastic soils. Int. J. Numer. Anal. Meth. Geomech. **15**, 457–470 (1991)
31. Selvadurai, A.P.S., Lan, Q.: A disc inclusion at a non-homogeneous elastic interface. Mech. Res. Comm. **24**, 473–488 (1997)
32. Selvadurai, A.P.S., Lan, Q.: Axisymmetric mixed boundary value problems for an elastic halfspace with a periodic nonhomogeneity. Int. J. Solids. Struct. **35**, 1813–1826 (1998)
33. Spencer, A.J.M., Selvadurai, A.P.S.: Some generalized anti-plane strain problems for an inhomogeneous elastic halfspace. J. Eng. Math. **34**, 403–416 (1998)
34. Selvadurai, A.P.S., Katebi, A.: Mindlin's problem for an incompressible elastic halfspace with an exponential variation in the linear elastic shear modulus. Int. J. Eng. Sci. (2013). doi:10.1016/j.ijengsci.2013.01.002
35. Mossakovskii, V.I.: The fundamental mixed problem of the theory of elasticity for a halfspace with a circular line separating the boundary conditions. J. Appl. Math. Mech. **18**, 187–196 (1954); (in Russian)
36. Ufliand Ia., S.: The contact problem of the theory of elasticity for a die, circular in its plane, in the presence of adhesion. J. Appl. Math. Mech. **20**, 578–587 (1956)
37. Selvadurai, A.P.S.: Elastostatic bounds for the stiffness of an elliptical disc inclusion embedded at a transversely isotropic bimaterial interface. J. Appl. Math. Phys. (ZAMP) **35**, 13–23 (1984)
38. Selvadurai, A.P.S.: Rotational stiffness of a rigid elliptical disc inclusion embedded at a bi-material elastic interface. Solid. Mech. Arch. **10**, 3–16 (1985)
39. Selvadurai, A.P.S.: An inclusion at a bi-material elastic interface. J. Eng. Math. **37**, 155–170 (2000)
40. Selvadurai, A.P.S.: On certain bounds for the in-plane translational stiffness of a disc inclusion at a bi-material elastic interface. Mech. Res. Comm. **30**, 227–234 (2003)
41. Selvadurai, A.P.S.: On an invariance principle for unilateral contact at a bi-material elastic interface. Int. J. Eng. Sci. **41**, 721–739 (2003)
42. Selvadurai, A.P.S.: Boussinesq indentation of an isotropic elastic halfspace reinforced with an inextensible membrane. Int. J. Eng. Sci. **47**, 1339–1345 (2009)
43. Selvadurai, A.P.S., Au, M.C.: Generalized displacements of a rigid elliptical anchor embedded at a bi-material geological interface. Int. J. Numer. Anal. Meth. Geomech. **10**, 633–652 (1986)
44. Westergaard, H.M.: Theory of Elasticity and Plasticity. Wiley, New York (1952)
45. Selvadurai, A.P.S.: Partial Differential Equations in Mechanics, vol. 2 The Biharmonic Equation, Poisson's Equation. Springer, Berlin (2000)
46. Selvadurai, A.P.S.: On Boussinesq's problem. Int. J. Eng. Sci. **39**, 317–322 (2001)
47. Labaki, J., Mesquita, E., Rajapakse, R.K.N.D.: Coupled horizontal and rocking vibrations of a rigid circular plate on a transversely isotropic bi-material interface. Eng. An. Bound. El. **37**, 1367–1377 (2013)
48. Selvadurai, A.P.S.: The influence of a boundary fracture on the elastic stiffness of a deeply embedded anchor plate. Int. J. Num. Anal. Meth. Geomech. **13**, 159–170 (1989)
49. Selvadurai, A.P.S.: The axial loading of a rigid circular anchor plate embedded in an elastic halfspace. Int. J. Num. Anal. Meth. Geomech. **17**, 343–353 (1993)
50. Selvadurai, A.P.S.: On the problem of a detached anchor plate embedded in a crack. Int. J. Solids Struct. **31**, 1279–1290 (1994)
51. Selvadurai, A.P.S.: Mindlin's problem for a halfspace with a bonded flexural surface constraint. Mech. Res. Comm. **28**, 157–164 (2001)

52. Selvadurai, A.P.S.: Mechanics of a rigid circular disc bonded to a cracked elastic halfspace. Int. J. Solids Struct. **39**, 6035–6053 (2002)
53. Selvadurai, A.P.S., Rajapakse, R.K.N.D.: On the load transfer from a cylindrical inclusion into an elastic halfspace. Int. J. Solids Struct. **21**, 1213–1229 (1985)
54. Selvadurai, A.P.S.: Fracture evolution during indentation of a brittle elastic solid. Mech. Cohes. Frict. Mat. **5**, 325–339 (2000)
55. Selvadurai, A.P.S., Karpurapu, R.: Composite infinite element for modeling unbounded saturated soil media. J. Geotech. Eng. ASCE **115**, 1633–1646 (1989)