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ORIGINAL ARTICLE

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Contact problems for a finitely deformed incompressible elastic halfspace

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Abstract This paper examines the class of problems related to the interaction between a finitely deformed incompressible elastic halfspace and contacting elements that include smooth, flat rigid indenters with elliptical and circular shapes and a thick plate of infinite extent. The contact between the finitely deformed elastic halfspace and the contacting elements is assumed to be bilateral. The interaction between both the rigid circular indenter and the finitely deformed halfspace is induced by a Mindlin force that acts at the interior of the halfspace regions and by exterior loads. Similar considerations apply for the contact between the flexible plate of infinite extent and the finitely deformed elastic halfspace. The theory of small deformations superposed on large deformations proposed by Green et al. (Proc R Soc Ser A 211:128–155, 1952) is used as the basis for the formulation of the problem, and results of potential theory and integral transform techniques are used to develop the analytical results. In particular, explicit results are presented for the displacement of the rigid elliptical indenter and the maximum deflection of the flexible plate induced by the Mindlin forces, when the finitely deformed halfspace region has a strain energy function of the Mooney–Rivlin form.

Keywords Small deformations superposed on large · Elliptical and circular indenters · Mindlin's problem · Plate-halfspace interaction · Reissner thick plate theory ·

1 Introduction

Although the history of finite elasticity can be traced back to the works of Cauchy [1], Green [2], Piola [3] and others, the modern developments in the theory of hyperelasticity with applications to rubber-like elastic solids have been greatly advanced by the theoretical and experimental research of R.S. Rivlin (see, e.g. Barenblatt and Joseph [4]). Contributions to the theory of finite elasticity subsequent to Rivlin's work are too extensive to be cited individually. Complete accounts of these developments are given in excellent review and survey articles, by, among others, Rivlin [5], Adkins [6], Doyle and Ericksen [7], Truesdell and Noll [8], Green and Zerna [9], Spencer [10], Green and Adkins [11], Ogden [12], Lur'e [13], Carroll and Hayes [14], Fu and Ogden [15] and Selvadurai [16]. An important consideration in the mechanics of hyperelastic materials is the state of stress that such materials could be subjected to in their wide range of applications. Components made of hyperelastic materials and natural biological materials including arteries and veins can be subjected to initial stress states through internal pressurization and stretching. Furthermore, the hyperelastic components that are in such a prestressed state can be subjected to further incremental deformations, resulting from pulsatile

Dedicated to Professor Felix Darve on the occasion of his retirement.

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flow that can initiate small strain elastic phenomena. The theory of small deformations superposed on large can be considered as a suitable theory to model this class of problems. Such theories have been proposed by Trefftz [17], Biot [18], Neuber [19] and Green et al. [20] among others. Accounts of developments that focus on small elastic deformations superposed on large are given by Truesdell and Noll [8], Green and Zerna [9], Green and Adkins [11], Eringen and Suhubi [21], Woo and Shield [22], Corneliussen and Shield [23], Fosdick and Shield [24], Beatty and Usmani [25] and Selvadurai [26–29].

In this paper, we apply the theory of small deformations superposed on large deformations developed by Green et al. [20] to examine indentation problems related to an incompressible elastic halfspace region that is subjected to a finite radial stretch. We first consider the problem of the indentation of a radially stretched halfspace by a flat rigid indenter with a smooth base and an elliptical plan form, and an explicit result is obtained for the indentational stiffness of the indenter interacting with a finitely deformed incompressible elastic halfspace with a strain energy function of the Mooney–Rivlin type. A formal procedure is also outlined for extending the analysis to include the interaction between the axially loaded elliptical indenter and an internal Mindlin force [30]. [As indicated by a reviewer, the notion of a concentrated force that acts at a point (in a mathematical sense) gives rise to singular displacement and stress fields, which negates the assumption of infinitesimal strains implicit in the classical theory. The paradox, while mathematically correct in terms of finite strain energy requirements, spatial decay of displacement and stress fields, can be resolved within the context of classical continuum mechanics by allowing the loads to act over finite regions. Alternative approaches for overcoming this criticism are derived from gradient or higher-order theories [31–33] that allow the application of concentrated forces without developing unbounded displacements and stresses. The introduction of gradient effects changes the concept of a classical continuum and brings to bear an entirely different mathematical and physical structure to the problem. As a further physical explanation to the introduction of gradient effects, consider the action of the classical Boussinesq force, at the surface of a halfspace region [34]. An alternative development of a solution to Boussinesq's problem is given in [35]. As is well known, at the point of application of the force, Boussinesq's classical result gives rise to singular displacement and stress fields. Now, consider the problem where the action of the Boussinesq force on the halfspace is diffused by a bonded, classical Poisson–Kirchhoff thin plate, which ensures compatibility of stresses and displacements at the plate-elastic halfspace interface. The solution to this problem is also classical (see, e.g. [36–40]). In this case, the stresses and displacements in the halfspace region are finite and bounded. What this implies is that the introduction of higher-order constraints certainly renders the action of localized loads physically and mathematically tractable, but the constraints introduced through the higher-order theories changes the basic problem in classical elasticity involving singular states. Furthermore, there is arbitrariness associated with the description of the higher-order constraint, and this becomes meaningful if and only if there is a specific physical description that can be identified as providing the higher-order constraint. The developments presented in the paper therefore follow the classical approach, while recognizing the limitations of the solutions developed.]

The paper also develops analogous results for the axisymmetric interaction between a smooth circular rigid indenter and a Mindlin force located within the elastic halfspace subjected to a radial stretch. A Hankel transform technique is used to reduce the indentation problem to a system of dual integral equations, which are solved using the approaches available in the literature and well documented by Sneddon [41]. It is shown that the result for the displacement of the rigid circular indenter can be obtained in exact closed form. The paper also deals with the interaction between an incompressible elastic halfspace that is initially subjected to a radial stretch and then externally indented by a thick flexible plate and internally loaded by a Mindlin force. In particular, the incremental deformations are induced by the indentation of the surface of the halfspace by a Reissner-type [42] thick plate that exhibits shear deformations. The bilateral interaction between the finitely deformed elastic halfspace and the Reissner plate is induced by an axisymmetric external load and an internal Mindlin force. The class of problems examined in the paper has some applications relevant to the characterization of material properties of hyperelastic materials and technological applications that involve interaction between indenting regions.

2 Theoretical developments

The fundamental equations governing small elastic deformations of an incompressible isotropic material subjected to an initial finite deformation are given by Green et al. [20], and the salient results are summarized for completeness. We define material points in an isotropic elastic medium by a general curvilinear coordinate system $\theta_i(\theta_1 = x; \theta_2 = y; \theta_3 = z)$, which moves with the body as it deforms. The covariant and contravariant metric tensors associated with the undeformed and deformed states are given by g_{ij}, G_{ij} and g^{ij}, G^{ij}, respectively. We restrict attention to incompressible hyperelastic materials with a strain energy function of the Mooney–Rivlin type, defined by

$$W(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{1}$$

In (1), I_1 and I_2 are the principal invariants

$$I_1 = g^{rs} G_{rs}; \qquad I_2 = g_{rs} G^{rs} \tag{2}$$

and for an incompressible elastic material $I_3 = 1$. For this class of materials, we can define a contravariant stress tensor σ^{ij}, measured per unit area of the deformed body and referred to θ_i coordinates of the deformed body, and the hyperelastic constitutive equation governing the incompressible material is given by

$$\sigma^{ij} = \Phi g^{ij} + \Psi B^{ij} + pG^{ij} \tag{3}$$

where p is an isotropic stress to be determined by satisfying the boundary conditions of a problem and

$$B^{ij} = I_1 g^{ij} - g^{ir} g^{js} G_{rs} \tag{4}$$

$$\Phi = 2\frac{\partial W}{\partial I_1}; \quad \Psi = 2\frac{\partial W}{\partial I_2} \tag{5}$$

We now consider the special case where the finite deformation in the incompressible material is maintained by equal stresses acting in the x- and y-directions, which give rise to stretches $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \mu$ that satisfy the incompressibility constraint

$$\lambda^2 \mu = 1; \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

The stress state corresponding to the finite deformation is given by the contravariant stress tensor

$$\begin{aligned} \sigma^{11} &= \sigma^{22} = \Phi\lambda^2 + \Psi\lambda^2\left(\lambda^2 + \mu^2\right) + p \\ \sigma^{33} &= \Phi\mu^2 + 2\Psi\lambda^2\mu^2 + p \end{aligned} \tag{7}$$

and the scalar invariant p is determined from the boundary conditions of the problem. It has been shown by Green et al. [20] that the stress components $\tilde{\sigma}^{ij}$ governing the incremental deformations can be expressed in terms of two functions $\phi_i(x, y, z)$ $(i = 1, 2)$, which are solutions of

$$\left(\nabla_1^2 + k_i \frac{\partial^2}{\partial z^2}\right)\phi_i(x, y, z) = 0; \qquad (i = 1, 2) \tag{8}$$

In (8)

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{9}$$

and k_1 and k_2 are roots of the equation

$$k^2 d_{55} + k(d_{44} + d_{55} - a - c) + d_{44} = 0 \tag{10}$$

where

$$\begin{aligned} a &= 4\lambda^2\left(C_1 + C_2\lambda^2\right); \qquad c = 4\mu^2\left(C_1 + C_2\lambda^2\right) \\ d_{44} &= 2\mu^2\left(C_1 + C_2\lambda^2\right); \qquad d_{55} = 2\lambda^2\left(C_1 + C_2\lambda^2\right) \end{aligned} \tag{11}$$

and the quadratic Eq. (10) for k has the roots

$$k_1 = 1; \qquad k_2 = \lambda^{-6} = \mu^3 \tag{12}$$

The components of the incremental displacement field can be expressed in terms of $\phi_i(x, y, z)$ $(i = 1, 2)$ in the forms

$$u = \frac{\partial}{\partial x}(\phi_1 + \phi_2); \qquad v = \frac{\partial}{\partial y}(\phi_1 + \phi_2); \qquad w = \frac{\partial}{\partial z}(k_1\phi_1 + k_2\phi_2) \tag{13}$$

The incremental stress tensor $\tilde{\sigma}^{ij}$ can also be expressed in terms of $\phi_i(x, y, z)$ $(i = 1, 2)$. Since attention will be restricted to examining the mechanics of the smooth indentation of a halfspace region occupying $0 \le r < \infty$ and $0 \le z < \infty$, the stress components of interest are

$$\begin{aligned} \tilde{\sigma}^{33} &= k_1\left(k_1 d_{55} + d_{44} - 2\sigma^{33}\right)\frac{\partial^2\phi_1}{\partial z^2} + k_2\left(k_2 d_{55} + d_{44} - 2\sigma^{33}\right)\frac{\partial^2\phi_2}{\partial z^2} \\ \tilde{\sigma}^{23} &= c_{44}\left\{(1+k_1)\frac{\partial^2\phi_1}{\partial y\partial z} + (1+k_2)\frac{\partial^2\phi_2}{\partial y\partial z}\right\} \\ \tilde{\sigma}^{31} &= c_{44}\left\{(1+k_1)\frac{\partial^2\phi_1}{\partial x\partial z} + (1+k_2)\frac{\partial^2\phi_2}{\partial x\partial z}\right\} \end{aligned} \tag{14}$$

3 Indentation of a radially stretched halfspace by a flat elliptical indenter

We now apply the theory of small deformations superposed on a homogeneous finite deformation to examine the problem of the indentation of a halfspace subjected to equal bi-axial stretches, by a flat indenter with a smooth base and an elliptical plan form (Fig. 1). The classical elasticity solution to this problem in the absence of the effects of the initial finite deformation was given by Green and Sneddon [43] and is well documented in classical texts on contact and inclusion problems (Galin [44], Lur'e [45], Selvadurai [36], Gladwell [46], Johnson [47], Willner [48], Aleynikov [49]). Since the halfspace region is subjected to an initial equal biaxial stretch, σ^{33} is zero and the scalar function p is given by

$$p = -\mu^2\left(\Phi + 2\lambda^2\Psi\right) \tag{15}$$

The halfspace region occupies $-\infty < x < \infty, -\infty < y < \infty$ and $0 \le z < \infty$, and, after initial finite deformation, the bounding plane remains a plane. The rigid indenter with a smooth base has a plan form defined by the ellipse with semi-major axis a and semi-minor axis b. The contacting region between the

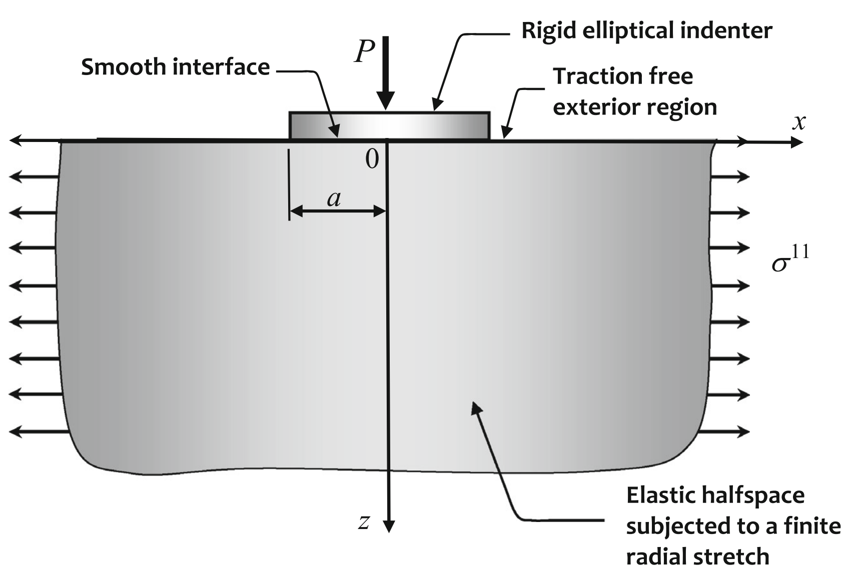

Fig. 1 Indentation of a finitely deformed halfspace by a rigid elliptical indenter

surface $z = 0$ of the finitely deformed elastic halfspace and the elliptical rigid indenter is denoted by S_{i}, and the region of the halfspace exterior to the elliptical indenter is denoted by S_{e}, such that

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \leq 1 \in S_{\mathrm{i}} \quad \text{and} \quad \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \geq 1 \in S_{\mathrm{e}} \tag{16}$$

The mixed boundary value problem related to the indentation of the finitely deformed halfspace by the elliptical indenter with a flat base is defined by the mixed boundary conditions

$$w(x, y, 0) = \Delta; \qquad (x, y) \in S_{\mathrm{i}} \tag{17}$$

$$\tilde{\sigma}^{33} = 0; \qquad (x, y) \in S_{\mathrm{e}} \tag{18}$$

$$\tilde{\sigma}^{13}(x, y, 0) = \tilde{\sigma}^{23}(x, y, 0) = 0; \qquad (x, y) \in S_{\mathrm{i}} \cup S_{\mathrm{e}} \tag{19}$$

where Δ is the indentational displacement in the axial direction. The boundary and interface conditions on the indented continuum are classical, and detailed accounts of the formulation of boundary and interface conditions are discussed in [50] and [51].

Considering the expressions for the stress components $\tilde{\sigma}^{13}$ and $\tilde{\sigma}^{23}$ given by (14), the boundary condition (19) will be satisfied if

$$(1 + k_1)\frac{\partial \phi_1}{\partial z} + (1 + k_2)\frac{\partial \phi_2}{\partial z} = 0 \tag{20}$$

on the plane $z = 0$. The constraint (20) in turn will be satisfied by selecting $\phi_i(x, y, z)$ $(i = 1, 2)$ as

$$\phi_1 = \frac{\sqrt{k_1}}{(1 + k_1)}\chi\left(x, y, z/\sqrt{k_1}\right); \qquad \phi_2 = \frac{\sqrt{k_2}}{(1 + k_2)}\chi\left(x, y, z/\sqrt{k_2}\right) \tag{21}$$

and

$$\nabla^2\chi(x, y, z) = 0 \tag{22}$$

The mixed boundary conditions (17) and (18) can be written as

$$\left(\frac{k_1}{1 + k_1} - \frac{k_2}{1 + k_2}\right)\frac{\partial \chi}{\partial z} = \Delta; \qquad (x, y) \in S_{\mathrm{i}} \tag{23}$$

$$\kappa\frac{\partial^2 \chi}{\partial z^2} = 0; \qquad (x, y) \in S_{\mathrm{e}} \tag{24}$$

where

$$\kappa = \left(\frac{(k_1 d_{55} + d_{44})\sqrt{k_1}}{1 + k_1} - \frac{(k_2 d_{55} + d_{44})\sqrt{k_2}}{1 + k_2}\right) \tag{25}$$

The solution of the mixed boundary value problem can be approached in a variety of ways, the most straightforward of which is to utilize the solution to the analogous hydrodynamic problem developed by Lamb [52], which suggests that, in the mixed boundary value problem posed by (23) and (24), $\partial\chi/\mathrm{d}z$ represents the velocity potential of the motion of a perfect fluid through an elliptical aperture in a thin boundary, for which

$$\frac{\partial \chi}{\partial z} = \frac{\Delta(1 + k_1)(1 + k_2)a}{2(k_1 - k_2)K(e_0)}\int_{\xi}^{\infty}\frac{\mathrm{d}s}{\left[s(a^2 + s)(b^2 + s)\right]^{1/2}} \tag{26}$$

and (ξ, η, ζ) are the ellipsoidal coordinates of the point (x, y, z) and are the roots of

$$\frac{x^2}{a^2 + \theta} + \frac{y^2}{b^2 + \theta} + \frac{z^2}{\theta} - 1 = 0 \tag{27}$$

The result by Lamb [52] represents not only an important development in the area of flow of ideal fluids through apertures but the results have been applied by a number of investigators to study crack, contact and inclusion problems related to elastic media [53–59] and for the analysis of Darcy flow in stratified orthotropic porous media [60] as well as flow through defects in barriers separating dissimilar porous media [61]. Although there is no known explicit solution for the function $\chi(x, y, z)$, its use in the analysis of crack, contact, inclusion and porous media fluid flow problems requires only a knowledge of its derivatives with respect to z. In the ellipsoidal coordinate system, S_{i} corresponds to $\xi = 0$ and S_{e} corresponds to $\eta = 0$. In (26), $K(e_0)$ is the complete elliptic integral of the first kind, where $e_0 = (a^2 - b^2)/a^2$. The contact stress beneath the elliptical indenter is given by

$$\tilde{\sigma}^{33}(x, y, 0) = \kappa \frac{\partial^2 \chi}{\partial z^2}; \qquad (x, y) \in S_{\mathrm{i}} \tag{28}$$

Using (26), the expression (28) gives the compressive stress at the contact zone

$$\tilde{\sigma}^{33}(x, y, 0) = -\frac{\kappa \Delta (1 + k_1)(1 + k_2)}{(k_1 - k_2) b K(e_0)} \frac{1}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}; \qquad (x, y) \in S_{\mathrm{i}} \tag{29}$$

The compressive force exerted by the indentation Δ is given by

$$P = -\frac{\kappa \Delta (1 + k_1)(1 + k_2)}{(k_1 - k_2) b K(e_0)} \int\int_{(x,y)\in S_{\mathrm{i}}} \frac{\mathrm{d}x\mathrm{d}y}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}} \tag{30}$$

Using (29) in (30), we obtain the following exact closed form result for the force indentation relationship for the elliptical indenter at the surface of a radially stretched incompressible elastic halfspace:

$$P = \frac{4\pi \kappa a \Delta (1 + k_1)(1 + k_2)}{(k_1 - k_2) K(e_0)} \tag{31}$$

In the particular case of an incompressible elastic material with a strain energy function of the Mooney–Rivlin form (1), the expression (31) gives

$$P = \frac{4\pi a \Delta (C_1 + \lambda^2 C_2)(\lambda^9 + \lambda^6 + 3\lambda^3 - 1)}{\lambda^4 (\lambda^3 + 1) K(e_0)} \tag{32}$$

The result (32) has a form similar to that obtained by Green et al. [20] for the indentation of a radially stretched halfspace by a smooth rigid spherical indenter. Also, the expression (32) contains the term $(\lambda^9 + \lambda^6 + 3\lambda^3 - 1)$, which would indicate that, for a finite load P, the indentational displacement would become unbounded as λ approaches a value close to 2/3. This result implies that the surface of the halfspace that is acted upon by a radial compression will become unstable for a certain value of the compressive stress. This observation is consistent with the original observation of Green et al. [20]. In the absence of a radial stretch, $\lambda = 1$, (32) reduces to

$$P = \frac{4\pi a G \Delta}{K(e_0)} \tag{33}$$

where $G (= 2(C_1 + C_2))$ is the linear elastic shear modulus. In the particular case when $a = b$, (32) reduces to $P = 8aG\Delta$, which is the classical result given by Boussinesq [34] for the indentation of an incompressible elastic halfspace by a frictionless smooth rigid circular indenter with a flat base, obtained using results of potential theory and confirmed by the classical result of Harding and Sneddon [62] that reduces the contact problem to the solution of a system of dual integral equations [36,41,46,63,64].

4 Combined interaction between a flat indenter and a Mindlin force for a radially stretched halfspace

We consider the problem of a radially stretched elastic halfspace simultaneously loaded by a smooth rigid elliptical indenter and an internal axisymmetric force of the Mindlin type [31] that is located at a finite distance c from the plane $z = 0$ (Fig. 2). Since the contact between the rigid indenter and radially stretched elastic halfspace is assumed to be smooth, it is implicitly assumed that the contact is bilateral and the contacting surface does not experience separation. Unilateral contact problems in classical elasticity have examined the cases where separation occurs at the interface (Selvadurai [36], Gladwell [46], Panagiotopoulos [65], Villaggio [66], Selvadurai [67–72]), and these approaches can also be used to examine the problem related to the incremental

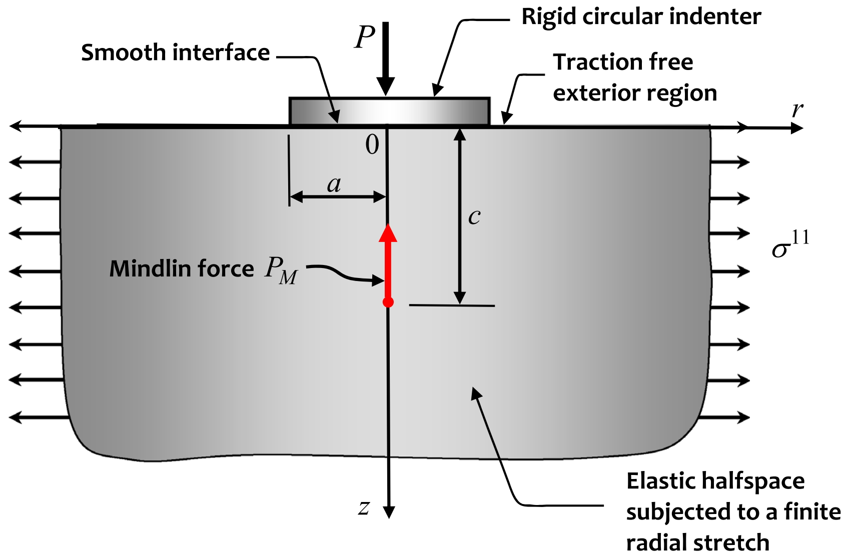

Fig. 2 Interaction of a rigid circular indenter and a Mindlin force for a radially stretched elastic halfspace

indentation. The incremental surface displacement of the finitely deformed elastic halfspace due to the action of the Mindlin force (P_{M} acting in the negative z-direction) can be obtained from the results given by Woo and Shield [22] (see also Selvadurai [28]) and takes the form

$$w_{\mathrm{M}}(x, y, 0) = \frac{P_{\mathrm{M}}(k_1 - k_2)}{2\pi\Theta(h_1 - h_2)(1 + k_1)\sqrt{k_2}}\left(\frac{h_1}{(x^2 + y^2 + c_1^2)^{1/2}} - \frac{h_2}{(x^2 + y^2 + c_2^2)^{1/2}}\right) \tag{34}$$

where

$$h_\alpha = \frac{\left(\Phi + \Psi\lambda^2\right)\left(\lambda^6 k_\alpha + 1\right)}{\lambda^4}; \qquad c_\alpha = \frac{c}{\sqrt{k_\alpha}}; \qquad (\alpha = 1, 2) \tag{35}$$

and the parameter Θ is defined in "Appendix". The notion of a Mindlin force is an idealization in as much as the Kelvin force [73,74] or a Boussinesq force [34,35,75,76] are indeed idealizations that give rise to singular stress and displacement fields at the point of application of the concentrated force. The global equations of equilibrium are satisfied, and the regularity conditions for displacement and stress fields appropriate for infinite and semi-infinite domains ensure boundedness of the elastic strain energy. The objection to singular fields can be overcome by ensuring that the localized force is distributed over a finite surface area ensuring that the resultant of tractions is equipollent to the localized force. The embedded rigid sphere or spheroid for the Kelvin force [77–79] is a useful mechanical analogue with bounded fields. The problems that result are considerably more complicated and are more conveniently examined by appeal to computational approaches. A further alternative is to consider the use of higher-order theories of continua [31–33,80], which offer possible approaches to moderating the effects of singular fields.

The mixed boundary value problem governing the bilateral interaction between the rigid elliptical indenter with a smooth flat base and the Mindlin force P_{M} is given by

$$w(x, y, 0) = \Delta + w_{\mathrm{M}}(x, y, 0); \qquad (x, y) \in S_{\mathrm{i}} \tag{36}$$

and the traction free boundary conditions are identical to (18) and (19), representing traction free boundary conditions on regions S_{e} and $S_{\mathrm{i}} \cup S_{\mathrm{e}}$, respectively.

Considering the form of $w_{\mathrm{M}}(x, y)$ given by (34), it appears that the mixed boundary value problem cannot be directly solved using the techniques presented previously. It is, however, possible to develop a formal solution to the axial displacement of the indenter using Betti's reciprocal theorem. Successful applications of Betti's reciprocal theorem, particularly to contact and inclusion problems, are given by Shield [81], Shield and Anderson [82], Selvadurai [83–86] and Selvadurai and Dumont [87]. The solution to the problem of the action of an incremental concentrated normal force P on the surface of a radially stretched halfspace can be obtained from the results given by Woo and Shield [22] and can be written in a general form

$$w(x,y,z)=\frac{F(x,y,z)}{(\Phi+\lambda^2\Psi)} \tag{37}$$

Considering the contact stress distribution (29), the displacement at the location $z=c$ (the point of application of the Mindlin force) can be evaluated from the expression

$$w=\frac{\kappa\Delta(1+k_1)(1+k_2)}{(k_1-k_2)(\Phi+\lambda^2\Psi)bK(e_0)}\int\int_{S_i}\frac{F(\bar{x},\bar{y},c)\mathrm{d}\bar{x}\mathrm{d}\bar{y}}{\sqrt{1-(\bar{x}/a)^2-(\bar{y}/b)^2}} \tag{38}$$

Applying Betti's reciprocal theorem to the problem, the displacement of the elliptical indenter in bilateral smooth bilateral contact with the finitely deformed elastic halfspace is given by

$$P_{\mathrm{M}}w=Pw_{\mathrm{M}} \tag{39}$$

This result is again a formal approach for obtaining the rigid displacement of the elliptical indenter w_{M} due to the Mindlin force P_{M}. The integrations in (38) can only be performed numerically. The value of w_{M} can be combined with the result for the displacement due to the directly loaded elliptical indenter to obtain the net displacement due to the combined action of the external load P and the internal Mindlin force P_{M}.

The axisymmetric problem of the indentation of a radially stretched elastic halfspace by a circular rigid indenter and a internal Mindlin force can, however, be formulated in relation to the quasi-harmonic functions $\phi_\alpha(r,z)$, $(\alpha=1,2)$, which are solutions of the partial differential equations

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+k_\alpha\frac{\partial^2}{\partial z^2}\right)\phi_\alpha(r,z)=0 \tag{40}$$

where $k_\alpha(\alpha=1,2)$ are the roots of the equation

$$k^2(\alpha_4+\sigma^{11})+k(\alpha_1-\alpha_3+2\alpha_4-\sigma^{11})+\alpha_4=0 \tag{41}$$

in which

$$\begin{aligned}\alpha_1&=\zeta_1-\frac{2C_0}{\mu^4}; &\quad \alpha_2&=\frac{2}{\mu^4}\left(\Phi+\mu^2\Psi\right)\\ \alpha_3&=\zeta_2+\frac{2C_0}{\mu^4}; &\quad \alpha_4&=\frac{C_0}{\mu^4}\end{aligned} \tag{42}$$

The displacements and stresses can be uniquely expressed in terms of $\phi_\alpha(r,z)$. The results of interest to the formulation of the boundary value problems are given by

$$\begin{aligned}u(r,z)&=\frac{\partial}{\partial r}(\phi_1+\phi_2); \qquad w(r,z)=\frac{\partial}{\partial z}(k_1\phi_1+k_2\phi_2)\\ \tilde{\sigma}^{33}&=\tilde{p}+\alpha_3\frac{\partial^2}{\partial z^2}(k_1\phi_1+k_2\phi_2)\\ \tilde{\sigma}^{13}&=\alpha_4\frac{\partial^2}{\partial r\partial z}[(1+k_1)\phi_1+(1+k_2)\phi_2]\end{aligned} \tag{43}$$

For the solution of the boundary value problems related to the indentation of the finitely deformed incompressible elastic halfspace, we employ solutions of (40) based on Hankel transform developments [63]. Considering the zeroth-order Hankel transform development of (40), it can be shown that the relevant solutions applicable to the halfspace region $z\geq 0$ can be written as

$$\phi_\alpha(r,z)=\int_0^\infty \xi A_\alpha(\xi)\exp\left(-\frac{\xi z}{\sqrt{k_\alpha}}\right)J_0(\xi r)\mathrm{d}\xi; \qquad (\alpha=1,2) \tag{44}$$

where A_α are arbitrary functions. The traction free boundary condition (19) gives

$$A_1(\xi) = -\sqrt{\frac{k_1}{k_2}}\left(\frac{1+k_2}{1+k_1}\right)A_2(\xi) = A(\xi) \tag{45}$$

The mixed boundary conditions of the type (17) and (18) applicable, respectively, to the regions $0 \le r \le a$ and $a < r < \infty$ reduce to a pair of dual integral equations

$$\int_0^\infty \xi^2 A(\xi) J_0(\xi r)\mathrm{d}\xi = \frac{\Delta + w_\mathrm{M}(r)}{\left(\sqrt{\frac{k_1}{k_2}}\left(\frac{1+k_2}{1+k_1}\right) - \sqrt{k_2}\right)}; \qquad 0 \le r \le a \tag{46}$$

$$\int_0^\infty \xi^3 A(\xi) J_0(\xi r)\mathrm{d}\xi = 0; \qquad a < r < \infty \tag{47}$$

where $w_\mathrm{M}(r)$ is defined by (34). The solution of the dual system is given by Sneddon [63] and Selvadurai [64] and will not be repeated here. Avoiding details of the method of solution, it can be shown that for an incompressible elastic material, the relationship between the displacement Δ, the load P applied to the circular rigid indenter and P_M, the Mindlin force acting at a finite depth c from the surface of the halfspace region, can be evaluated in the form

$$\Delta = \frac{P}{4\Theta a}\left(\sqrt{\frac{k_1}{k_2}}\left(\frac{1+k_2}{1+k_1}\right) - \sqrt{k_2}\right)\left(1 - \frac{2P_\mathrm{M}\Lambda}{\pi P}\left\{\frac{h_1}{(h_1-h_2)}\tan^{-1}\left(\frac{a}{c_1}\right) - \frac{h_2}{(h_1-h_2)}\tan^{-1}\left(\frac{a}{c_2}\right)\right\}\right) \tag{48}$$

where

$$\Lambda = \frac{(k_1-k_2)}{\sqrt{k_2}(1+k_1)\left(\sqrt{\frac{k_1}{k_2}}\left(\frac{1+k_2}{1+k_1}\right) - \sqrt{k_2}\right)} \tag{49}$$

In the particular case of a Mooney–Rivlin material, (48) reduces to

$$\frac{\Delta}{P/8aG} = \psi\left(1 - \frac{2P_\mathrm{M}}{\pi P}\left\{\frac{(\lambda^6+1)}{(\lambda^6-1)}\tan^{-1}\left(\frac{a}{c}\right) - \frac{2}{(\lambda^6-1)}\tan^{-1}\left(\frac{a}{\lambda^3 c}\right)\right\}\right) \tag{50}$$

where

$$\psi = \frac{2\lambda^4(\lambda^3+1)(1+\Gamma)}{(\lambda^9+\lambda^6+3\lambda^3-1)\left(1+\lambda^2\Gamma\right)}; \qquad \Gamma = \frac{C_2}{C_1} \tag{51}$$

In the absence of an initial finite deformation, $\lambda = 1$ and (50) reduces to the result obtained by Selvadurai [88,89] for the classical elasticity problem of the interaction between a rigid circular indenter and a Mindlin force when the material is incompressible: i.e.

$$\frac{\Delta}{P/8aG} = \left(1 - \frac{2P_\mathrm{M}}{\pi P}\left\{\tan^{-1}\left(\frac{a}{c}\right) + \frac{ac}{(a^2+c^2)}\right\}\right) \tag{52}$$

from which the other classical results can be recovered.

5 Contact between a Reissner plate and a Mindlin force for a radially stretched halfspace

The developments presented previously can easily be extended to other types of problems involving flexural contact similar to those that can be induced by a plate or other flexible structural element. The interaction described by either the Mindlin plate theory [90,91] or the Reissner plate theory [36,92,93] takes into consideration the role of shear deformations present in moderately thick plates. We consider the problem of the contact between an externally loaded thick plate whose flexural behaviour can be described by a Reissner plate theory and a radially stretched elastic halfspace, which contains a Mindlin force that is located at a finite distance from the interface between the contacting solids. Both the external loading of the Reissner plate and

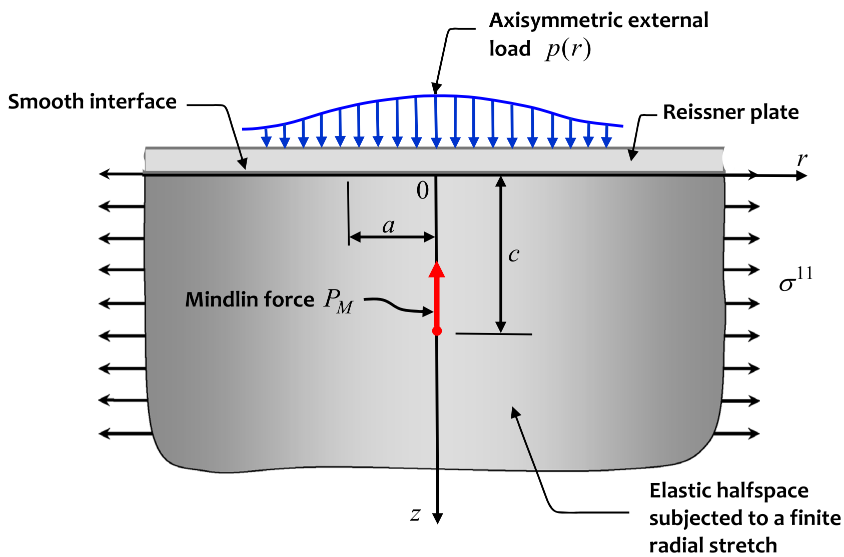

Fig. 3 Interaction between a Reissner plate and a Mindlin force for a radially stretched elastic halfspace

the internal loading of the finitely deformed elastic halfspace result in a state of axial symmetry in the problem (Fig. 3). The basic problem of the interaction between a thick plate and the initially stressed elastic halfspace is relevant to the geosciences, specifically problems dealing the interaction between the earth's lithosphere (the crustal plate) and the asthenosphere (the mantle). For short-term timescales of a geological nature (10^3–10^4 years), the earth's lithosphere and the asthenosphere are usually modelled, respectively, as a thin elastic plate and a dense fluid, which is modelled by a Winkler spring support (Nadai [94], Brotchie and Silvester [95], Walcott [96], Cathles [97] (see also Selvadurai [36]). The constitutive behaviour of geomaterials encountered in geomechanics and geosciences in general is much more complicated [98–102], but the elasticity model provides a suitable first approximation for the treatment of the problem and the solution lends itself to be extended to consider viscoelastic effects. The flexural rigidity of the lithosphere is usually deduced from observations of the wavelength and amplitude of crustal flexure in the vicinity of the supercrustal loads. The thickness of the crustal plates can be deduced from such calculations. The accuracy of the estimates will depend on the type of models used to represent both the crustal plate and the asthenosphere. The thick Reissner plate model is an improvement to the Germain–Poisson–Kirchhoff plate model used extensively in such interaction studies [36]. Also, the finitely deformed elastic halfspace region is an improvement both in terms of the incorporation of the geostatic stress states and incorporating continuum characteristics in the asthenosphere. The constitutive relationship used to model the elastic behaviour of the asthenosphere is that of an incompressible elastic material with a strain energy function of the Mooney–Rivlin form. The analysis can be extended to other forms of strain energy functions, but the Mooney–Rivlin form captures the incompressible behaviour essential to model the asthenosphere and offers the possibility of extending the modelling of the interaction problem to include Newtonian viscosity of the asthenosphere.

Since plate theories have been developed with the assumption of zero shear tractions on the bounding planes of the plate, it is appropriate and consistent to consider the mechanics of a smooth contact between a Reissner thick plate and an initially stressed halfspace region. The plate is subjected to an axisymmetric external stress $p(r)$, and the contact stress induced at the smooth interface between the thick plate and the halfspace due to the external load $p(r)$ and the Mindlin force P_M is denoted by $q(r)$ (Fig. 3).

Since the flexure of the Reissner plate is axisymmetric, the ordinary differential equation governing the plate deflection $w(r)$ is given by

$$D\tilde{\nabla}^2\tilde{\nabla}^2 w(r) + (1 - T\tilde{\nabla}^2)[q(r) - p(r)] = 0 \tag{53}$$

where

$$\tilde{\nabla}^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{dr}}; \qquad T = \frac{h^2}{10}\left(\frac{2-\nu_p}{1-\nu_p}\right); \qquad D = \frac{G_p h^3}{6(1-\nu_p)} \tag{54}$$

and G_p, ν_p and h are, respectively, the elastic constants and the plate thickness. It should be noted that in addition to the flexural deflection $w(r)$, the shear deformations are defined in relation to a stress function $S(r)$ that gives the second fundamental equation

$$\tilde{\nabla}^2 S(r) - \frac{10}{h^2} S(r) = 0 \tag{55}$$

This equation, however, is not needed for the discussion that follows. Since the incremental deformations of contact between the Reissner plate and the finitely deformed halfspace are assumed to be bilateral, we can obtain a relationship between the incremental surface displacements of the finitely deformed halfspace $w(r)$ and the loading actions of $q(r)$ and P_{M}. Considering a Hankel transform development and using the result (34), we obtain the zeroth-order Hankel transform for the surface deflection $\bar{w}(\xi)$ as follows:

$$\bar{w}(\xi) = \frac{(k_2 - k_1)}{(1 + k_1)\Theta\xi\sqrt{k_2}} \left(\bar{q}(\xi) - \frac{P_{\mathrm{M}}(h_1 e^{-\eta_1} - h_2 e^{-\eta_2})}{2\pi(h_1 - h_2)} \right); \qquad \eta_i = \frac{\xi c}{\sqrt{k_i}} \tag{56}$$

The zeroth-order Hankel transform of the plate bending Eq. (53) can be written as

$$D\xi^4 \bar{w}(\xi) + (1 + \xi^2 T)\bar{q}(\xi) = (1 + \xi^2 T)\bar{p}(\xi) \tag{57}$$

The transformed equivalent of the contact stress $\bar{q}(r)$ can be eliminated between (56) and (57), which yields a result for $\bar{w}(\xi)$; inversion of the result gives the following expression for $w(r)$:

$$w(r) = \int_0^\infty \frac{\left(\bar{p}(\xi) - \frac{P_{\mathrm{M}}(h_1 e^{-\eta_1} - h_2 e^{-\eta_2})}{2\pi(h_1 - h_2)}\right)(1 + \xi^2 T)}{\left(D\xi^3 + \frac{(1+k_1)\Theta\sqrt{k_2}(1+\xi^2 T)}{(k_2 - k_1)}\right)} J_0(\xi r)\mathrm{d}\xi \tag{58}$$

The result (58) can be numerically evaluated to determine the deflection of the Reissner plate under the action of the external load $p(r)$ and the Mindlin force P_{M}. For example, for (i) a Mooney–Rivlin material, (ii) in the absence of the Mindlin force, (iii) for an external load that corresponds to a uniform circular load of radius a and stress intensity p_0 and (iv) setting a length parameter as a, (58) reduces to

$$w(r) = \frac{p_0 a^4}{D} \int_0^\infty \frac{(1 + \omega^2 \tilde{T})\ J_1(\omega) J_0(\omega\rho)}{\omega\left(\omega^3 + \frac{(1+\omega^2\tilde{T})}{\psi} R\right)} \mathrm{d}\omega \tag{59}$$

where ψ and Γ are defined by (51) and

$$E_s = 6(C_1 + C_2); \quad \rho = \frac{r}{a}; \quad \tilde{T} = \frac{1}{10}\left(\frac{h}{a}\right)^2 \left(\frac{2 - \nu_p}{1 - \nu_p}\right); \quad \rho = \frac{r}{a}; \ R = \frac{2E_s a^3}{3D} \tag{60}$$

Also, when $\lambda = 1$ (i.e. $\psi = 1$) and $\tilde{T}$ is set to zero, which should be interpreted as a plate model having no shear deformation effects, the result (59) reduces to the classical problem of a Germain–Poisson–Kirchhoff thin plate in bilateral contact with an incompressible elastic halfspace and subjected to a uniform circular load of radius a and stress intensity p_0:

$$w(r) = \frac{p_0 a^4}{D} \int_0^\infty \frac{J_1(\omega) J_0(\omega\rho)}{\omega\left(\omega^3 + \left(\frac{2E_s a^3}{3D}\right)\right)} \mathrm{d}\omega \tag{61}$$

The Hankel–Lipschitz type integrals given by (59) and (61) do not appear to have any explicit closed form solutions. They can be evaluated numerically up to any required accuracy. Since the main emphasis of the study relates to the evaluation of the influence of the initial finite deformation on the plate deflection, attention is focused on the estimation of the deflection of the plate at the centre of the circular loaded area. The influence of the shear deformations of the Reissner plate can be assessed by comparing the displacement at the circular

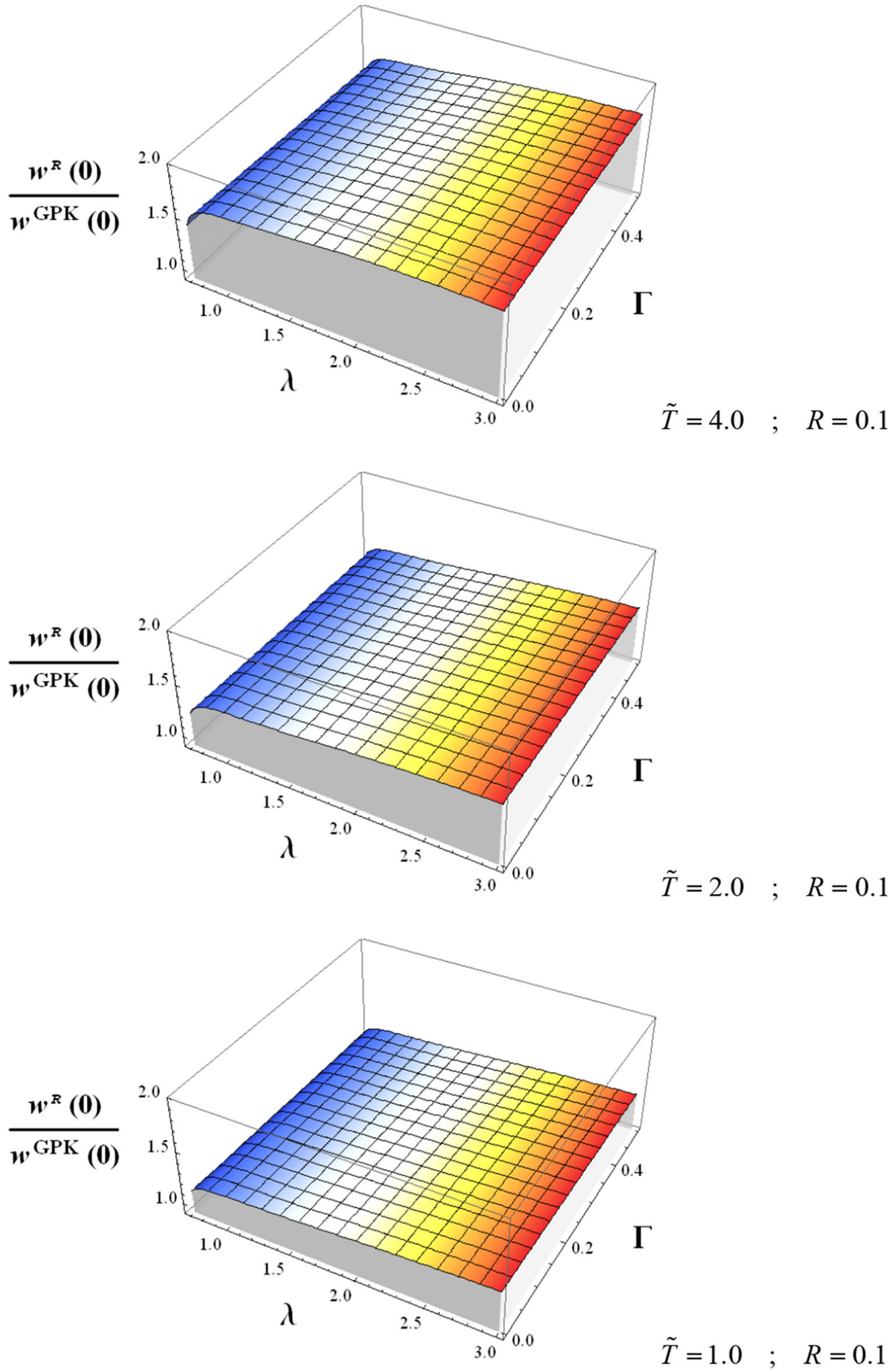

Fig. 4 Normalized deflection of the centre of a Reissner plate in smooth contact with a finitely deformed elastic halfspace and subjected to a uniform circular load

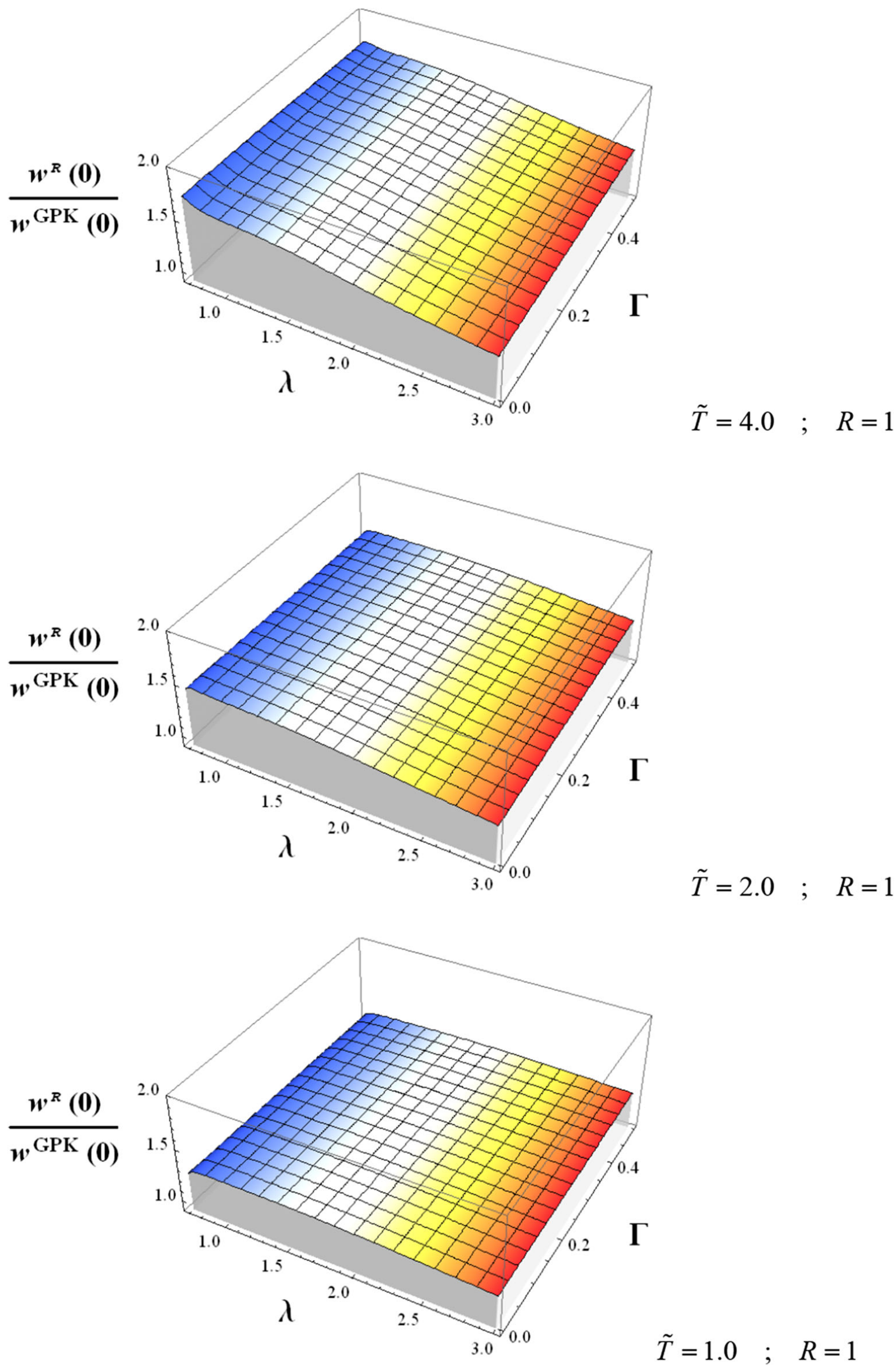

Fig. 5 Normalized deflection of the centre of a Reissner plate in smooth contact with a finitely deformed elastic halfspace and subjected to a uniform circular load

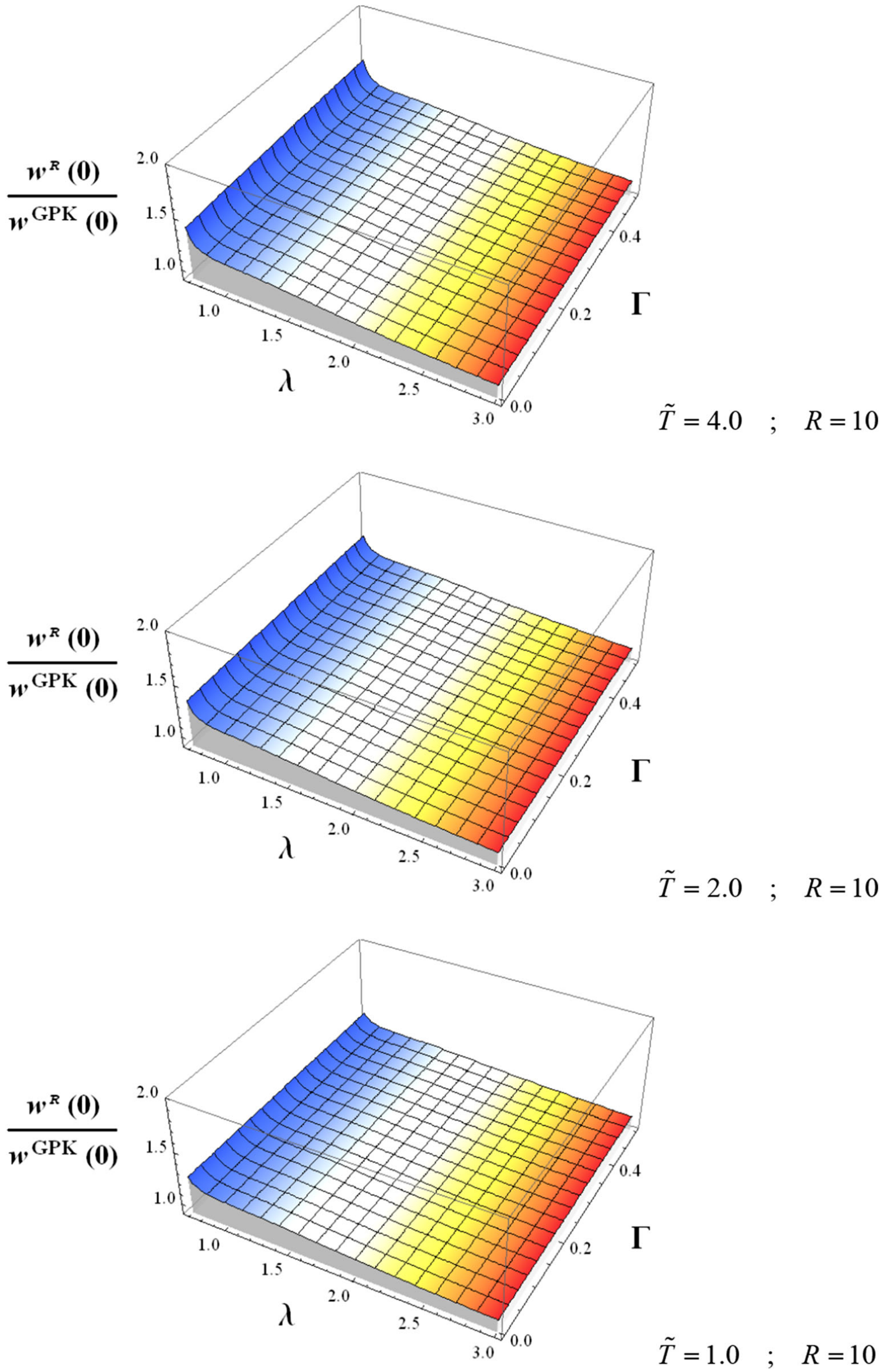

Fig. 6 Normalized deflection of the centre of a Reissner plate in smooth contact with a finitely deformed elastic halfspace and subjected to a uniform circular load

loaded region as determined from (59) with equivalent results for a Germain–Poisson–Kirchhoff thin plate theory $w^{\mathrm{GPK}}(0)$: i.e.

$$\frac{w^{R}(0)}{w^{\mathrm{GPK}}(0)} = \frac{\int_0^{\infty} \frac{(1+\omega^2\tilde{T})\, J_1(\omega)}{\omega\left(\omega^3+\frac{(1+\omega^2\tilde{T})}{\psi}R\right)}\mathrm{d}\omega}{\int_0^{\infty} \frac{J_1(\omega)}{\omega\left(\omega^3+\frac{1}{\psi}R\right)}\mathrm{d}\omega} \tag{62}$$

The result (62) has been evaluated for a range of values of the shear deformation parameter $\tilde{T}$, the finite stretch λ, the ratio of the Mooney–Rivlin constants Γ and the relative stiffness parameter R. The results of the numerical evaluations of (62) are shown in Figs. 4, 5 and 6. The influence of the shear deformations on the normalized displacements materialize when the relative rigidity between the plate and the elastic halfspace becomes small (i.e. $R \ll 1$) and when the initial finite deformations are small (i.e. $\lambda \to 1$). The Mooney parameter ratio Γ has only a marginal influence on the central deflection of the axisymmetrically loaded plate.

6 Conclusions

The paper presents certain solutions that pertain to the mechanics of contact between a finitely deformed incompressible elastic halfspace and rigid and flexible indenters, where the influence of internal Mindlin forces preserve a state of axial symmetry. In the instances when the indenter has an elliptical plan form, the indentational stiffness relationship can be evaluated in exact closed form. Although specific results are provided only for an elastic halfspace material with a strain energy function of the Mooney–Rivlin form, the procedures employed can be extend to include other forms of strain energy functions applicable to incompressible elastic materials. In the instance where the indentation takes place with the combined action of a Mindlin force, exact closed form results can be obtained for the case of a rigid circular indenter on a halfspace material with a strain energy function of a Mooney–Rivlin form. Results developed for the interaction between a Reissner plate and a finitely deformed elastic halfspace region illustrate the influence of the relative rigidity parameter and the radial stretch on the central deflection of the thick plate. The results presented in the paper have applications to material characterization aspects of hyperelastic materials that are considered to be incompressible.

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Appendix

$$\Theta = \sqrt{\frac{k_1}{k_2}}\left(\frac{1+k_2}{1+k_1}\right)\left(\frac{\alpha_4}{k_1} - \alpha_3 + \alpha_1 + \alpha_4 - \sigma^{11}\right) + \alpha_4 + k_2(\alpha_4 + \sigma^{11})$$

$$\alpha_1 = \zeta_1 - \frac{2C_0}{\lambda^4}; \quad \alpha_2 = \frac{2(\Phi + \lambda^8\Psi)}{\lambda^4} - \frac{4C_0}{\lambda^4}; \quad \alpha_3 = \zeta_2 + \frac{2C_0}{\lambda^4}; \quad \alpha_3 = \frac{C_0}{\lambda^4}$$

$$C_0 = (\Phi + \lambda^2\Psi)$$

$$\zeta_1 = 2A_0\left(\frac{1}{\lambda^2} - \lambda^4\right) + 2B_0\left(\frac{1}{\lambda^4} - \lambda^8\right) + 2F_0\left(1 + \frac{1}{\lambda^6} - 2\lambda^8\right)$$

$$\zeta_2 = 2\left(\frac{1}{\lambda^2} - \lambda^4\right)\left(\frac{A_0}{\lambda^6} + \frac{2B_0}{\lambda^2} + \frac{3F_0}{\lambda^4}\right)$$

$$A_0 = 2\frac{\partial^2 W}{\partial I_1^2}; \quad B_0 = 2\frac{\partial^2 W}{\partial I_2^2}; \quad F_0 = 2\frac{\partial^2 W}{\partial I_1 \partial I_2}$$

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