Mechanics of contact between bi-material elastic solids perturbed by a flexible interface†

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This paper generalizes the classical result of R.D. Mindlin for the axisymmetric problem related to the action of a concentrated force at the interior of an isotropic elastic half-space to include a bi-material region that is perturbed by an interface exhibiting flexural behaviour. The flexural behaviour of the interface is described by a Germain–Poisson–Kirchhoff thin plate theory. A Hankel integral transform technique is used to obtain an explicit result for the deflection of the flexural interface. The reduction of the solution to conventional results associated with both half-space problems and infinite space problems is indicated. Formal results are also presented for the case where the flexural behaviour of the plate is modelled by the plate theory proposed by E. Reissner.

Keywords: Mindlin’s problem; Kelvin’s problem; bi-material Infinite space; integral transforms; embedded plate; thin plate theory; flexural deflections of a stiffened interface.

1. Introduction

The problem of an isotropic elastic solid that is internally loaded by a concentrated force was first examined by Kelvin (1848). It represents a turning point in the development of fundamental solutions to problems in the classical theory of elasticity. The exact closed form nature of Kelvin’s solution made it possible to conveniently apply the result to examine a variety of other problems of importance to the classical theory of elasticity. For example, Lamé (1866) used a combination of aligned Kelvin forces to generate a double force and a combination of double forces to create a centre of compression, which was used to develop solutions to the problem of pressurized spherical containers. Kelvin’s solution, together with a distribution of centres of compression along an axis, serves as the basis for the development of the solution by Boussinesq (1885) for the problem of an isotropic elastic half-space, which is subjected to a concentrated normal force. The solution to the problem of a spherical cavity located in an extended elastic solid that is subjected to a far-field uniaxial stress field, developed by Southwell & Gough (1926), also uses concepts derived from Kelvin’s solution. Similarly, the solution to the problem of a concentrated force acting at the interior of a half-space developed by Mindlin (1936) uses Kelvin’s solution as the starting point and successfully generalizes the internal force problem to include both Kelvin’s and Boussinesq’s solutions as special limiting cases. Since these seminal studies, the problem of the Kelvin force and its extension to a considerable range of applications has been quite extensive. Comprehensive accounts of these developments are given by Love (1927), Westergaard (1938), Sternberg (1960), Lur’e (1965), Timoshenko & Goodier (1970), Gurtin (1972), Gladwell (1980), Mura (1987),...
Ting (1993), Davis & Selvadurai (1996) and Selvadurai (2000a). Some studies of interest in connection with the present paper are due to Dundurs & Guell (1965) who examined the load transfer between half-spaces, which are subjected to centres of dilatation. In their study, there is complete continuity of displacements and tractions at the interface. Thermal effects of a centre of dilation at a joined bi-material half-space region were examined by Yu & Sanday (1991). Selvadurai (2001) and Selvadurai & Willner (2006) examined problems related to surface-reinforced half-spaces loaded internally by localized forces that are axisymmetric and non-axisymmetric. Of related interest are investigations that deal with the mechanics of rigid disc-shaped inclusions embedded at the interface of bi-material elastic regions, where either kinematic constraints are invoked to develop convenient analytical solutions or the governing integral equations are solved in a numerical fashion (Selvadurai, 1984a, 1994a,b,c, 2000b,c, 2009a; Selvadurai & Au, 1986). In this paper, we examine a generalization of the Kelvin force problem to include a bi-material region, which is separated by a bonded flexural constraint in the form of either a Germain–Poisson–Kirchhoff thin plate or a Reissner thick plate that accounts for shear deformation effects. The problem is of some interest to the study of load transfer in reinforced solids and in the evaluation of substrates and coatings that are subjected to nuclei of thermoelastic strain (Selvadurai, 2009b) and internally reinforced solids subject to indentation (Aleynikov, 2011; Selvadurai et al., 2008). Other applications of both the bonded surface plate and the embedded plate models used in the field of geosciences in connection with the mechanics of geological barriers that are encountered during pressurized geological disposal of fluids into geological formations and approximate solutions are given by Segall et al. (1994) and Selvadurai (2009c, 2012). Of related interest are several studies of surface reinforcement, surface elasticity and mechanics of coatings presented in the literature and examples of which are given by Steigmann & Ogden (1997), Schiavone & Ru (1998), Selvadurai (2007), Selvadurai & Dumont (2011) and Sapsathiyarn & Rajapakse (2013). In this paper, the focus is specifically on the evaluation of the flexural stresses that are developed in the thin plate at the bonded interface between two dissimilar elastic half-space regions and subjected to Mindlin forces. Attention is restricted to the study of the axisymmetric problem, which is facilitated by the application of a Hankel transform technique. Explicit numerical results are presented both for the Germain–Poisson–Kirchhoff thin plate and the Reissner thick plate embedded in a bi-material infinite space. The role of the mismatch in Young’s modulus and Poisson’s ratio are illustrated.

2. Problem formulation

We consider the problem of an infinite space region, which is composed of two isotropic elastic half-space regions that occupy, \( r \in (0, \infty); z \in (0, \infty) \) (region 1) and \( r \in (0, \infty); z \in (0, -\infty) \) (region 2). The interface between the half-space regions contains a thin elastic plate of thickness \( t \), which satisfies the Germain–Poisson–Kirchhoff thin plate theory. It should be noted that, although the embedded plate has a finite thickness, the solutions for the surface loading of the adjacent elastic regions are developed by assuming that these regions are semi-infinite. The half-space regions 1 and 2 are also subjected to localized forces of magnitude \( P_1 \) and \( P_2 \), which both act in the positive \( z \)-direction and are placed at locations \((0, h_1)\) and \((0, -h_2)\) in the half-space regions 1 and 2, respectively (Fig. 1). In the classical plate theory, it is assumed that the plate experiences no in-plane extension. This places an inextensibility constraint at the bonded interface, provided that the thickness of the plate is small in relation to a characteristic dimension in the problem. Such a dimension could be the distances \( h_1 \) or \( h_2 \). The objective of the analysis is to develop a formal integral expression for the deflection of the thin plate, which could be used to determine the flexural stresses in the embedded element and the displacements of the half-space regions.
FLEXURE OF EMBEDDED PLATES

We first examine the problem of the half-space region 1, which is subjected to the combined action of a concentrated force $P_1$ and a tensile traction $q^{(1)}(r)$ at the surface. The solution to this problem can be obtained by adopting a formulation of the problem in terms of the strain potential function of Love (1927), which satisfies the bi-harmonic equation (Selvadurai, 2000a)

$$\nabla^2 \nabla^2 \Phi(r, z) = 0, \quad (1)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (2)$$

is Laplace’s operator in axisymmetric cylindrical polar coordinates. Under the combined action of the surface traction $q^{(1)}(r)$ and the concentrated force $P_1$, the boundary conditions governing the half-space problem are

$$u_r(r, 0) = 0; \quad \sigma_{zz}(r, 0) = -q^{(1)}(r). \quad (3)$$

In addition, the displacement and stress fields should satisfy the regularity conditions applicable to 3D problems in the theory of elasticity. By adopting a Hankel integral transform solution of (1) such that the zeroth-order Hankel transform of $\Phi(r, z)$ is defined by Sneddon (1951)

$$\Phi^0(\xi, z) = H_0[\Phi(r, z); \xi] = \int_0^\infty r\Phi(r, z) J_0(\xi r) \, dr, \quad (4)$$

we can obtain a relationship between the transformed values of the surface displacement $[u^{(1)}_z(r, 0)]^{q_1}$ and the applied normal contact stress $q_1(r)$ for a half-space with a zero radial displacement constraint defined by (3). Similarly the expression for the surface displacement $[u^{(1)}_z(r, 0)]^{P_1}$ of a half-space region with an inextensibility constraint at the surface and loaded by an concentrated force $P_1$ at the location $(0, h_1)$ and acting in the axial direction $z$, can be obtained by considering the superposition of equal Kelvin forces of magnitude $P_1$ that act within the infinite space region at locations $(0, \pm h_1)$. The Hankel transform of the axial surface displacement of half-space region 1 due to the combined action of $q^{(1)}(r)$

Fig. 1. A flexible plate at the interface of a bi-material elastic infinite space.
and the concentrated force $P_1$ can be evaluated in the following form:

$$
\tilde{u}_z^{(1)}(\xi) = \frac{(3 - 4v_1)}{4\xi \mu_1 (1 - v_1)} q^{(1)}(\xi) + \frac{P_1}{8\pi \mu_1 (1 - v_1) \xi} [(3 - 4v_1 + \xi h_1) \exp(-\xi h_1)] . \tag{5}
$$

Similarly, considering the problem of half-space region 2 with the inextensibility surface constraint, the action of a tensile surface traction $q^{(2)}(r)$ acting on the surface and a concentrated force of magnitude $P_2$ that acts in the $z$-direction at the point $(0, -h_2)$, we can show that

$$
\tilde{u}_z^{(2)}(\xi) = -\frac{(3 - 4v_2)}{4\xi \mu_2 (1 - v_2)} q^{(2)}(\xi) + \frac{P_2}{8\pi \mu_2 (1 - v_2) \xi} [(3 - 4v_2 + \xi h_2) \exp(-\xi h_2)]. \tag{6}
$$

3. Axisymmetric flexure of the bonded embedded plate

We now consider the problem of the flexure of the thin elastic plate, which is embedded in bonded contact with both half-space regions. The flexure of the plate is induced by the action of concentrated forces $P_1$ and $P_2$ placed at locations $(0, h_1)$ and $(0, -h_2)$, and acting, respectively, at half-space regions 1 and 2. In view of the axial symmetry of the problem, the flexural deflection of the plate in the positive $z$-direction, denoted by $w(r)$, is governed by the ordinary differential equation (Timoshenko & Woinowsky-Krieger, 1959; Selvadurai, 1979, 2000a; Constanda, 1990)

$$
D\tilde{\nabla}^2 \tilde{\nabla}^2 w(r) + q^{(1)}(r) = q^{(2)}(r), \tag{7}
$$

where

$$
\tilde{\nabla}^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}, \tag{8}
$$

$D = E_p r^3 / 12(1 - v_p^2)$ is the flexural rigidity of the plate, $E_p$ and $v_p$ are, respectively, Young's modulus and Poisson's ratio of the plate material, $q^{(1)}(r)$ is the contact normal stress at the constrained surface, which is bonded to half-space region 1, and $q^{(2)}(r)$ is the contact normal stress at the constrained surface, which is bonded to half-space region 2. In view of the assumed compatibility of axial displacements at the bonded plate interface, we have

$$
w(r) = u_z^{(1)}(r) = u_z^{(2)}(r). \tag{9}
$$

Applying the zeroth-order Hankel transforms to (7), we obtain

$$
D\xi^4 \tilde{w}(\xi) + \tilde{q}^{(1)}(\xi) = \tilde{q}^{(2)}(\xi). \tag{10}
$$

Considering the zeroth-order Hankel transform of (9) and the expressions (5), (6) and (10), we can obtain an expression for $\tilde{w}(\xi)$; applying the Hankel inversion theorem, we obtain the following integral expression for the plate deflection in bonded contact with the half-space regions:

$$
w^b(r) = \frac{1}{2\pi} \int_0^\infty \sum_{i=1}^{2} \left[ \frac{P_i (3 - 4v_i + \xi h_i) \exp(-\xi h_i)}{(3 - 4v_i)} \right] \left[ D\xi^3 + \sum_{i=1}^{2} \frac{4\mu_i (1 - v_i)}{(3 - 4v_i)} \right]^{-1} J_0(\xi r) \, d\xi. \tag{11}
$$

This solution represents a convenient generalization of Kelvin’s classical solution to include a bi-material elastic domain and the presence of a flexible structural element at the interface. Certain
limiting cases can be readily recovered from the above solution. In the particular case when

\[ D = 0; \quad h_i \to 0; \quad \mu_i \to \mu; \quad v_i \to v; \quad P_i \to P \]

the problem reduces to that of Kelvin’s classical problem of a concentrated force of magnitude \( 2P \) acting at the interior of a homogeneous elastic half-space with elastic constants \( \mu \) and \( v \). The result (11) thus reduces to

\[
w(r) = \frac{P(3 - 4v)}{8\pi \mu(1 - v)} \int_0^\infty J_0(\xi r) \frac{d\xi}{1 + \xi^2(D/C_b)}
\]

which is in agreement with the classical result for the axial displacement of the elastic solid at the plane of application of the axial concentrated force \( 2P \). The solution also exhibits the correct order in the singularity in the displacement field. Similarly, the constrained surface displacement associated with a Mindlin (1936) problem for the axisymmetric internal loading of a half-space region is also recovered from the result (11).

We note that in the case of homogeneous plates, the classical Germain–Poisson–Kirchhoff, as well as Reissner’s plate model, suggests that the in-plane (stretching) and the out-of-plane (bending) deformations are uncoupled. The fact that the bending equation (7) is thus uncoupled from the stretching equation does not imply that in-plane stretching is necessarily absent; therefore, the flexural actions do not necessarily place an inextensibility constraint at the bonded interface. It should be noted that the application of the first boundary condition of (3) on the in-plane equilibrium equation imposes the inextensibility constraint imposed in the current problem.

Also, in the case when the loads of total value \( P_0 = P_1 + P_2 \) are applied directly on the plate embedded in bonded contact between dissimilar elastic media, the deflection of the plate is given by

\[
w^b(r) = \frac{P_0}{2\pi C_b} \int_0^\infty J_0(\xi r) \frac{d\xi}{[1 + \xi^2(D/C_b)]},
\]

where

\[
C_b = \sum_{i=1}^{2} \frac{4\mu_i(1 - v_i)}{(3 - 4v_i)}.
\]

The deflection at the centre of the plate in bonded contact with the half-space regions can be evaluated in the form

\[
w^b(0) = \frac{\sqrt{3}}{9} \left[ \frac{P_0}{\sqrt{D}} \right] \left[ \frac{(3 - 4v_1)(3 - 4v_2)}{4\mu_1(1 - v_1)(3 - 4v_2) + \mu_2(1 - v_2)(3 - 4v_1)} \right]^{2/3}.
\]

The result for the interaction between the embedded plate and the interacting internally loaded half-space regions with the bonded constraint modelled by an inextensibility constraint is an approximation of the interactive behaviour resulting from the thin-plate assumption, which considers the inextensibility of the neutral plane of the plate. At the same time shear stresses are generated at the interface between the plate and the half-space regions, which are not consistent with the model for a Germain–Poisson–Kirchhoff thin-plate theory, which has zero shear tractions on the plane boundaries of the plate. Nonetheless, the inextensibility constraint is a suitable bound for the interactive behaviour between the embedded plate and the half-space regions.
4. Axisymmetric flexure of the smoothly embedded plate

The problem examined in the previous section is a limiting case where there is perfect bonding at the interfaces between the thin plate; this bonding is modelled by imposing an inextensibility criterion at the surface of the half-space, which is derived from the inextensibility condition at the mid-plane of the thin plate. An alternative interface condition is to assume that the plate is embedded in frictionless contact with the two elastic half-space regions. In this case, the interaction between half-space region 1 and the flexible plate is examined by considering the half-space problem as being internally loaded by a concentrated load $P_1$ at the location $(0, h_1)$ that acts in the axial direction $z$ and is subjected to the boundary conditions

$$\sigma_{rz}(r, 0) = 0; \quad \sigma_{zz}(r, 0) = -q^{(1)}(r).$$

(17)

Again, we note that the replacement of the first equation of (3) with the first equation of (17) naturally transforms, in this and subsequent problems the inextensible flexible interface into a frictionless interface such that the in-plane equations of the Germain–Poisson–Kirchhoff and the Reissner plate theories are identical.

Considering these boundary conditions, the Hankel transform of the axial surface displacement of half-space region 1 due to the combined action of $q_1(r)$ and the concentrated force $P_1$ can be evaluated in the following form:

$$\bar{u}^{(1)}_z(\xi) = \frac{(1 - \nu_1)}{\xi \mu_1} q_1(\xi) + \frac{P_1}{4\pi \mu_1 \xi} [2(1 - \nu_1) + \xi h_1] \exp(-\xi h_1).$$

(18)

An expression similar to (18) can be derived for the surface displacement of half-space region 2, which is subjected to a Mindlin force $P_2$ at location $(0, -h_2)$ that acts in the axial direction $z$ and is subjected to the boundary conditions of type (17). An expression similar to (18) can be derived for the zeroth-order Hankel transform of surface displacement $u^{(2)}_z(r)$. The interaction between the smoothly embedded Germain–Poisson–Kirchhoff thin plate and the elastic half-space region under the action of the Mindlin loads can be examined using an approach identical to that presented previously for the plate embedded in bonded contact. Omitting details, it can be shown that the deflection of the smoothly embedded plate is given by

$$w^s(r) = \frac{1}{4\pi} \int_0^{\infty} \sum_{i=1}^{2} \left[ \frac{P_i[2(1 - \nu_i) + \xi h_i]}{(1 - \nu_i)} \exp(-\xi h_i) \right] \frac{D \xi^3}{\sum_{i=1}^{2} \frac{\mu_i}{(1 - \nu_i)}} J_0(\xi r) \, d\xi.$$  

(19)

It should be noted that when the elastic half-space regions are incompressible, the interface bonding conditions have no influence on the solution to the interaction and both (11) and (19) reduce to the same result. Also, a result of some practical interest relates to the maximum deflection of the embedded plate, for the case when loads of total value $P_0(= P_1 + P_2)$ are applied directly on the plate embedded in smooth contact between dissimilar elastic media, which is given by the integral

$$w^s(r) = \frac{P_0}{2\pi C_s} \int_0^{\infty} \frac{J_0(\xi r) \, d\xi}{[1 + \xi^3(D/C_s)]^3},$$

(20)

where

$$C_s = \sum_{i=1}^{2} \frac{\mu_i}{(1 - \nu_i)}.$$  

(21)
The deflection at the centre of the plate can be evaluated in the form

\[ w^r(0) = \frac{\sqrt{3}}{9} \left( \frac{P_0}{\sqrt{D}} \right) \left[ \frac{(1 - \nu_1)(1 - \nu_2)}{\mu_1(1 - \nu_2) + \mu_2(1 - \nu_1)} \right]^{2/3}. \]  

(22)

The relative influence of the interface conditions on the plate deflections can be obtained by comparing the results (16) and (22) for the plate contained between two dissimilar half-space regions and subjected to a concentrated force \( P_0 \), i.e.

\[ \frac{w^s(0)}{w^b(0)} = \left[ \left( \frac{1 - \nu_1(1 - \nu_2)}{3(1 - 4\nu_1)(3 - 4\nu_2)} \right) \left( \frac{4[\Gamma(1 - \nu_1)(3 - 4\nu_2) + (1 - \nu_2)(3 - 4\nu_1)]}{\Gamma(1 - \nu_2) + (1 - \nu_1)} \right) \right]^{2/3}, \]

(23)

where \( \Gamma = \mu_1 / \mu_2 \) is a modular ratio. In the limit of incompressibility of both half-space regions (i.e. \( \nu_1 = \nu_2 = 1/2 \)), \( [w^s(0)/w^b(0)] = 1 \), and when \( \nu_1 = \nu_2 = 0 \), \( [w^s(0)/w^b(0)] \approx 1.211 \). Figure 2 illustrates the relative influences of \( \nu_1, \nu_2 \) and \( \Gamma \). As is evident, when \( \Gamma < 0 \), the plate deflection ratio given by (23) is virtually independent of \( \nu_1 \). Similarly, when \( \Gamma > 0 \), the plate deflection ratio is virtually independent of \( \nu_2 \).

Similarly, for the case where frictionless contact is imposed on the interface between the half-spaces and the thin plate, the analysis presented here is applicable only when the contact is bilateral. This presupposes that there is a normal stress acting on the interface separation between the half-space regions and the plate. The effect of this normal stress will be additive and does not affect the contact problem. If such a normal stress is not present, the contact between the plate and the elastic half-space regions is unilateral and this requires an alternative formulation of the contact problem (Selvadurai, 1994a,b, 2003).

5. Axisymmetric flexure of the smoothly embedded Reissner plate

The developments presented here can be easily extended to other types of flexural constraints similar to those described by either the Mindlin plate theory (Mindlin, 1951; Rajapakse & Selvadurai, 1985) or the Reissner plate theory (Reissner, 1944, 1947; Timoshenko & Woinowsky-Krieger, 1959; Selvadurai, 1984b; Constanda, 1990; van Rensburg et al., 2009; Ghia & Bulgariu, 2013) that take into consideration the role of shear deformations present in moderately thick plates. Since both plate theories have been developed with the assumption of zero shear tractions on the bounding planes of the plate, it is appropriate and consistent to consider the mechanics of interaction between the half-space regions and a smoothly embedded thick plate that is subjected to tractions \( p^{(1)}(r) \) and \( p^{(2)}(r) \), which act on the plate and are directed along the positive \( z \)-direction, and the interactive stresses \( q^{(1)}(r) \) and \( q^{(2)}(r) \) that act, respectively, in the negative and positive \( z \)-directions. The differential equation governing flexure of the Reissner plate (Reissner, 1947; Selvadurai, 1979, 1984b) embedded in smooth contact between the half-space regions and subjected to axisymmetric tractions \( p^{(1)}(r) \) and \( p^{(2)}(r) \) (Fig. 3) is given by

\[ D\tilde{\nabla}^2\tilde{\nabla}^2w(r) + (1 - T\tilde{\nabla}^2)[q^{(1)}(r) - p^{(1)}(r)] = (1 - T\tilde{\nabla}^2)[q^{(2)}(r) + p^{(2)}(r)], \]

(24)

where

\[ T = \frac{h^2}{10} \left( \frac{2 - \nu_p}{1 - \nu_p} \right). \]

(25)
In addition to the flexural deflection $w(r)$, the shear deformations are defined in relation to a stress function $\Psi(r)$ that gives the second fundamental equation:

$$\tilde{\nabla}^2 \Psi(r) - \frac{10}{h^2} \Psi(r) = 0.$$  \hspace{1cm}  (26)
This equation, however, is not needed for the discussion that follows. If the axial displacements between the smoothly embedded Reissner plate and the elastic half-space regions are compatible, the relationships between the contact stresses $q^{(1)}(r)$ and $q^{(2)}(r)$ and the displacements can be obtained using the solution to the analogous half-space problem.

Considering a Hankel transform development of the governing differential equation (24) and the results

$$\bar{q}^{(1)}(\xi) = \frac{\xi \mu_1 \bar{w}(\xi)}{(1 - \nu_1)}; \quad \bar{q}^{(2)}(\xi) = -\frac{\xi \mu_2 \bar{w}(\xi)}{(1 - \nu_2)}$$

and performing a Hankel transform inversion, we obtain the expression for the deflection of the smoothly embedded Reissner plate as

$$w^R(r) = \int_0^\infty (1 + \xi^2 T) \left( \sum_{i=1}^{2} \bar{p}^{(i)}(\xi) \right) \left[ D\xi^3 + (1 + \xi^2 T) C_s \right]^{-1} J_0(\xi r) d\xi,$$

where $C_s$ is defined by (21). It is evident that when the shear effects in the Reissner plate are neglected ($T \to 0$), the result (28) converges to its counterpart for the Germain–Poisson–Kirchhoff thin plate theory. We consider the problem of a Reissner plate that is embedded in smooth contact between two dissimilar elastic half-spaces where the interaction is induced by circular loads of uniform intensity $p_i$ and radii $a_i$ ($i = 1, 2$). The expression for the deflection is given by

$$w^R(r) = \int_0^\infty (1 + \xi^2 T) \left( \sum_{i=1}^{2} \left( \frac{a_i p_i}{\xi} \right) J_1(\xi a_i) \right) \left[ D\xi^3 + (1 + \xi^2 T) C_s \right]^{-1} J_0(\xi r) d\xi.$$

The expression (29) can be evaluated numerically to obtain results for the deflections and flexural moments in the embedded plate. The influence of shear deformations on the deflections in the Reissner plate can be compared with analogous results obtained for the Germain–Poisson–Kirchhoff thin plate theory. To keep the illustrative numerical results to a minimum, we will examine the special case where the flexure of the smoothly embedded plate is initiated by uniform circular loads of radii $a_i = a$ and...
Fig. 4. Influence of the Poisson ratio $\nu_p$, shear moduli ratio $\Phi$ and thickness ratio $\Delta$ on the relative deflection of the embedded thick plate at the centre of the loaded plate.

stress intensity $p^{(i)}(r) = p_0$, for which (29) reduces to

$$w_R(r) = \frac{2p_0}{C_s} \int_0^{\infty} \left( \frac{1 + \xi^2 T}{\xi[(D/C_s)\xi^3 + 1 + \xi^2 T]} \right) J_1(\xi a)J_0(\xi r) \, d\xi.$$  \hspace{0.5cm} (30)

As $T \to 0$, and as $a \to 0$ but maintaining $p_0 \pi a^2 \to P_0$, the result (30) reduces to the expression (20) for the Germain–Poisson–Kirchhoff thin plate smoothly embedded between dissimilar half-space regions.

In the case of a single half-space region, the result conforms to the solution obtained by Pister & Westmann (1962) for the loading of a thick plate in contact with an isotropic elastic half-space.
The influence of plate thickness and the resulting shear deformations can be examined by considering the ratio of plate deflections at the origin for the Reissner plate and the classical Germain-Poisson-Kirchhoff thin plate theory. The result can be expressed in the form

$$\frac{w^R(0)}{w^{GPK}(0)} = \frac{\int_0^\infty \left( \frac{1}{\eta} \left[ 1 + \eta^2 \Delta^2 \Omega \right] \right) J_1(\eta) \, d\eta}{\int_0^\infty \left( \frac{1}{\eta} \left[ 1 + \eta^2 \Delta^3 \Phi \right] \right) J_1(\eta) \, d\eta}, \quad (31)$$

where

$$\Delta = \frac{h}{a}; \quad \Phi = \frac{\mu_p}{6(1 - \nu_p)C_s}; \quad \Omega = \frac{1}{10} \left( \frac{2 - \nu_p}{1 - \nu_p} \right)$$

and $\mu_p$ is the linear elastic shear modulus of the plate. The expression (31) can be evaluated for representative values of non-dimensional material parameters $\Phi$ and $\Omega$ and geometric parameter $\Delta$. For example, for an elastically stiffer plate $\Phi >> 1$ with the parameter $\Delta$ representing the thickness of the plate in relation to the radius of the circular load, $\Delta >> 1$ corresponds to a highly localized loading and $\Delta << 1$ represents a diffused loading. Figure 4 presents results for $w^R(0)/w^{GPK}(0)$ for a range of values for $\Delta$ and $\Phi$. The results are self-explanatory and clearly indicate the influence of the range of values of parameters $\Delta$, $\Phi$ and Poisson’s ratio of the plate material $\nu_p$ on the normalized deflection at the origin. It is evident that the influence of shear deformations of the plate becomes dominant only when $\Phi$ becomes small and $\Delta$ becomes large.

6. Conclusions

The paper presents certain solutions pertaining to a thin elastic plate that is embedded at the interface of two bi-material isotropic elastic half-space regions. The axisymmetric flexure of the plate is induced by concentrated forces, which act at points in the interior of the half-space regions. It is shown that integral transform techniques can be successfully applied to determine formal integral expressions for the flexural deflections of the plate, and thus the flexural stresses. These integral expressions are amenable to numerical evaluation using either symbolic mathematical evaluation procedures or quadrature techniques. The solutions developed here can be regarded as fundamental solutions, which can be used to develop, through superposition techniques, results for dipoles and other localized loading configurations that are of interest to engineering applications and computational implementations. The paper also illustrates the applicability of the procedures for the study of the mechanics of a smoothly embedded Reissner plate where shear deformations were accounted for in the interactive process. The extension of the work to include transversely isotropic elastic behaviour of the half-space regions is straightforward. In this study, we have focused on the class of problems involving the interaction between the embedded plate (Germain–Poisson–Kirchhoff or Reissner) and the dissimilar elastic half-spaces where the interface conditions have been imposed by the inextensibility and frictionless conditions on the adhering surfaces. This particular formulation has not addressed the important problem of transmission of interface/inter-laminar stress continuity, particularly in relation to the shear stresses that can develop at the bonded surfaces. The most prudent strategy in this regard is to consider the formulation of the problem in relation to an elastic layer embedded in bonded contact with the half-space regions. This will allow the exact satisfaction of interface conditions related to both tractions and displacements, which would allow the examination of the effects when the plate can be approximated as a limiting case of the elastic layer. If continuity of curvature is to be invoked, then there needs to be the consideration of higher order gradient or surface effects in the treatment of the interaction problem.
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