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Undrained behaviour of a non-homogeneous elastic medium: the influence of variations in the elastic shear modulus with depth

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This paper examines the axisymmetric interior loading problem for an incompressible isotropic elastic half-space where the linear elastic shear modulus varies with depth. In particular, the non-homogeneity has an exponential variation either over the entire depth of the half-space, or over a finite depth beyond which it is constant. The mathematical formulation of the traction boundary value problem is developed through the application of integral transform techniques, and the numerical results obtained are compared with results derived from a computational procedure involving a finite-element approach.

KEYWORDS: deformation; elasticity; numerical modelling; settlement; stiffness

INTRODUCTION

The mechanics of non-homogeneous elastic media has been a topic of continuing interest to theoretical and applied solid mechanics. Extensive reviews of the subject are given by Mossakovskii (1958), Olszak (1959), Popov (1959), Rakov & Rvachev (1961), Kassir & Chuapresert (1974), Korenev (1975), Selvadurai (1979, 1996, 2007), Gladwell (1980) and Aleynikov (2011). The application of the theory of elasticity of non-homogeneous media to problems in geomechanics commences with the seminal paper by Gibson (1967), who examined the undrained elastic behaviour of a non-homogeneous elastic medium where the shear modulus varied linearly with depth. Gibson's research provided a definitive explanation for a Winkler medium, which consists of a discrete set of springs with identical stiffness. The elastic stiffness for the spring elements can be interpreted in terms of the linear variation of the shear modulus with depth for the specific case where the surface shear modulus is zero. The elastic half-space with this particular variation in shear modulus is referred to as a 'Gibson soil', and has been extensively studied by a number of Gibson's co-workers, including Brown & Gibson (1972), Awojobi & Gibson (1972), Gibson & Kalsi (1974) and Gibson & Sills (1974).

The geotechnical investigations conducted by Skempton & Henkel (1957), Ward *et al.* (1965), Burland & Lord (1970), Butler (1974), Hobbs (1974) and Abbiss (1979) refer to undrained shear moduli values that vary linearly with depth. The majority of investigations prior to Gibson's studies focused on the exponential variations in the elastic shear modulus; this led to considerable simplification in the mathematical modelling of the elasticity problems associated with non-homogeneous elastic media. The exponential variation in the shear modulus, however, leads to unbounded variations in elastic stiffness, when the formulations are applied to examine the mechanics of a half-space region. This limitation was addressed in a paper by Selvadurai *et al.* (1986), who examined the problem of the torsional indentation of a half-space region where the shear modulus has a bounded exponential variation with depth. Selvadurai (1996) also applied the bounded exponential variation in the shear modulus to examine the surface indentation of a half-space region. The accurate representation of both the near-surface non-homogeneity and far-field variation are important for estimating the undrained displacements of the inhomogeneous soils that are subjected to surface and interior loads.

This paper examines the axisymmetric problem of the interior loading of an inhomogeneous isotropic incompressible elastic half-space by a uniform circular load of finite radius. The problem is an approximation for the loading induced by an embedded foundation, such as an end-bearing pile (Poulos & Davis, 1975), an anchor plate (Selvadurai, 1989, 1993, 1994; Rajapakse & Selvadurai, 1991), or a test device such as a screw plate (Selvadurai *et al.*, 1980). The interior loading of an isotropic homogeneous elastic half-space was first examined by Mindlin (1936). The problem of the interior loading of a non-homogeneous medium with a linear variation in the shear modulus was examined by Rajapakse (1990) and Rajapakse & Selvadurai (1991), who extended the analysis to include circular footings and anchor plates in non-homogeneous elastic media. This paper examines two types of elastic non-homogeneity in the shear modulus: (a) exponential variation with depth over the entire half-space region; and (b) exponential variation with depth over a finite depth, beyond which the elastic shear modulus is assumed to be constant. The solution of these problems is accomplished using an integral transform formulation of the resulting equations of elasticity. An alternative approach for the modelling of depth-dependent non-homogeneity in a half-space is to use a layered system approach to represent the variation in the elastic moduli (Yue *et al.*, 1999). Other approaches for modelling elastic non-homogeneity include the harmonic spatial variation in the elastic modulus investigated by Selvadurai & Lan (1997, 1998) or a more generalised approach that was presented by Spencer & Selvadurai (1998).

In the case of the discrete variation in the elastic non-homogeneity, the demarcation point between the variations is assumed to be the point of application of the interior circular loads. The procedure can, however, be extended to include the case where the demarcation point is located at an arbitrary position in relation to the plane of application of the axisymmetric interior loading. The numerical results for the surface displacement of the half-space region are used to compare the influences of the bounded and unbounded values in the linear elastic shear modulus. The results derived from the current study and a previous investigation

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by Selvadurai & Katebi (2012) are compared with equivalent results for the problem of the undrained interior loading of a half-space with a linear variation in the elastic shear modulus (Rajapakse, 1990). The analytical results are also used to establish the accuracy of finite-element results for the analogous problems.

GOVERNING EQUATIONS

The development of the partial differential equations governing the elastic non-homogeneity problem is relatively straightforward, and the governing equations are presented for completeness. Details of the development are given in Popov (1959), Gibson (1967) and Korenev (1975). The displacement field in the elastic continuum is defined by $\mathbf{u}(\mathbf{x})$, where $\mathbf{x}$ is a position vector, and the linearised strain tensor $\boldsymbol{\varepsilon}(\mathbf{x})$ is defined by

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right] \tag{1}$$

The Cauchy stress tensor is $\boldsymbol{\sigma}(\mathbf{x})$, and the constitutive relationship for a non-homogeneous elastic material in which the linear elastic shear modulus varies with the coordinate direction z takes the form

$$\boldsymbol{\sigma} = 2G(z)[\alpha e\mathbf{I} + \boldsymbol{\varepsilon}] \tag{2}$$

where $G(z)$ is the linear elastic shear modulus, $\mathbf{I}$ is the unit matrix

$$e = \mathrm{tr}(\boldsymbol{\varepsilon});\ \alpha = \frac{\nu}{1-2\nu} \tag{3}$$

and ν is Poisson's ratio, which is assumed to be a constant. Attention is specifically restricted to incompressible elastic materials, for which isochoric deformations give

$$\mathrm{tr}(\boldsymbol{\varepsilon}) = 0;\ \nu = \tfrac{1}{2} \tag{4}$$

The constraints in equations (4) imply that the constitutive equations (2) are indeterminate to within an isotropic stress state $f(r,z)$ (e.g. Selvadurai & Spencer, 1972; Spencer, 1980); this needs to be determined from the solution of the equations of equilibrium, which, in the absence of body forces and for axial symmetry, reduce to

$$\nabla.\boldsymbol{\sigma} = 0 \tag{5}$$

Attention is now restricted to a state of axisymmetric deformation where the displacement components are $u_r(r,z)$ and $u_z(r,z)$, and the displacement equation of equilibrium can be reduced to

$$\nabla^2 u_r + \frac{\partial f}{\partial r} - \frac{u_r}{r^2} + \frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}z}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = 0 \tag{6}$$

$$\nabla^2 u_z + \frac{\partial f}{\partial z} + \frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}z}\left(f + 2\frac{\partial u_z}{\partial z}\right) = 0 \tag{7}$$

where ∇^2 is the axisymmetric form of Laplace's operator given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{8}$$

Eliminating the function $f(r,z)$ between equations (6) and (7), and based on the assumption that the differentiations commute, one obtains

$$\begin{aligned} & g(z)\nabla^2 u_r + \nabla^2\frac{\partial u_r}{\partial z} - \frac{\partial}{\partial r}\left(\nabla^2 u_z\right) \\ & \quad + g(z)\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{\partial^2 u_z}{\partial r\partial z}\right) - \frac{1}{r^2}\frac{\partial u_r}{\partial z} \\ & \quad + \left[g^2(z) + \frac{\mathrm{d}g(z)}{\mathrm{d}z}\right]\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = 0 \end{aligned} \tag{9}$$

where

$$g(z) = \frac{1}{G}\frac{\mathrm{d}G}{\mathrm{d}z} \tag{10}$$

By restricting attention to a variation in the elastic shear modulus of the form

$$G_1(z) = G_0 \mathrm{e}^{\lambda z},\ z \leqslant d \tag{11a}$$

$$G_2(z) = G_0 \mathrm{e}^{\lambda d},\ z \geqslant d \tag{11b}$$

equation (9), applicable to the separate domains, takes the form

$$\begin{aligned} & \lambda\nabla^2 u_r + \nabla^2\frac{\partial u_r}{\partial z} - \frac{\partial}{\partial r}\left(\nabla^2 u_z\right) \\ & \quad + \lambda\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{\partial^2 u_z}{\partial r\partial z}\right) \\ & \quad - \frac{1}{r^2}\frac{\partial u_r}{\partial z} + [\lambda^2]\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = 0,\ z \leqslant d \end{aligned} \tag{12a}$$

$$\nabla^2\frac{\partial u_r}{\partial z} - \frac{\partial}{\partial r}\left(\nabla^2 u_z\right) - \frac{1}{r^2}\frac{\partial u_r}{\partial z} = 0,\ z \geqslant d \tag{12b}$$

The results in equations (12), together with the incompressibility condition

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0 \tag{13}$$

constitute the set of coupled partial differential equations governing the displacement field.

In order to solve equations (12) and (13) Hankel transform representations are introduced (Sneddon, 1951; Selvadurai, 2000a), of the form

$$u_r(r,z) = \int_0^\infty \xi U_r(\xi,z)J_1(\xi r)\mathrm{d}\xi \tag{14}$$

$$u_z(r,z) = \int_0^\infty \xi U_z(\xi,z)J_0(\xi r)\mathrm{d}\xi \tag{15}$$

where ξ is the Hankel transform parameter, and J_n is the Bessel function of the first kind of order n. Using equations (13), (14) and (15), equations (12) can be reduced to

$$\begin{aligned} & \frac{\mathrm{d}^4 U_z}{\mathrm{d}z^4} + 2\lambda\frac{\mathrm{d}^3 U_z}{\mathrm{d}z^3} + \left(\lambda^2 - 2\xi^2\right)\frac{\mathrm{d}^2 U_z}{\mathrm{d}z^2} \\ & \quad - 2\lambda\xi^2\frac{\mathrm{d}U_z}{\mathrm{d}z} + \xi^2\left(\xi^2 + \lambda^2\right)U_z = 0,\ z \leqslant d \end{aligned} \tag{16a}$$

$$\frac{\mathrm{d}^4 U_z}{\mathrm{d}z^4} - 2\xi^2\frac{\mathrm{d}^2 U_z}{\mathrm{d}z^2} + \xi^4 U_z = 0,\ z \geqslant d \tag{16b}$$

The stress components relevant to the paper can be expressed in the form

$$\sigma_{zz}(r,z) = G(z)\int_0^\infty \xi\left[F + 2\frac{\mathrm{d}U_z}{\mathrm{d}z}\right]J_0(\xi r)\mathrm{d}\xi \tag{17}$$

$$\sigma_{rz}(r,z) = G(z)\int_0^\infty \xi\left[-\xi U_z + \frac{\mathrm{d}U_r}{\mathrm{d}z}\right]J_1(\xi r)\mathrm{d}\xi \tag{18}$$

where F is the Hankel transform of f, defined by

$$f(r, z) = \int_0^\infty \xi F(\xi, z) J_0(\xi r) \mathrm{d}\xi \tag{19}$$

AXISYMMETRIC INTERNAL LOADING OF A NON-HOMOGENEOUS ELASTIC HALF-SPACE

The problem is considered of an incompressible non-homogeneous elastic half-space, which is loaded internally by an axially directed circular load of radius a with stress intensity p_0 and situated at a depth $z = d$ from the traction-free surface of the half-space (Fig. 1). The most convenient approach for formulating boundary value problems of this type (Selvadurai *et al.*, 1991; Selvadurai, 2000b, 2000c) is to consider that the original half-space region is composed of (a) a layer region (superscript (1)) occupying the domain $r \in (0, \infty)$ and $z \in (d, \infty)$, and (b) a half-space region (superscript (2)) occupying the domain $r \in (0, \infty)$ and $z \in (d, \infty)$.

The elastic layer region and the elastic half-space region that are subjected to the following boundary and interface conditions are considered.

$$\sigma_{zz}^{(1)}(r, 0) = 0 \tag{20}$$

$$\sigma_{rz}^{(1)}(r, 0) = 0 \tag{21}$$

$$u_r^{(1)}(r, d) = u_r^{(2)}(r, d) \tag{22}$$

$$u_z^{(1)}(r, d) = u_z^{(2)}(r, d) \tag{23}$$

$$\sigma_{zz}^{(1)}(r, d) - \sigma_{zz}^{(2)}(r, d) = \begin{cases} p(r), & r \leqslant a \\ 0, & r > a \end{cases} \tag{24}$$

$$\sigma_{rz}^{(1)}(r, d) = \sigma_{rz}^{(2)}(r, d) \tag{25}$$

In equation (24), $p(r)$ represents the intensity of internally applied pressure over the circular area. In addition, the displacement and stress fields should satisfy the regularity conditions applicable to the layer and half-space regions as $r, z \to \infty$. Consistent with the regularity conditions at infinity, the transformed solution for the displacement components $u_r(r, z)$ and $u_z(r, z)$ can be written as

$$u_z(r, z) = \int \left[A_1 \mathrm{e}^{-k_1 z} + B_1 \mathrm{e}^{-k_2 z} + C_1 \mathrm{e}^{k_3 z} + D_1 \mathrm{e}^{k_4 z}\right] \xi J_0(\xi r) \mathrm{d}\xi; \quad z < d \tag{26a}$$

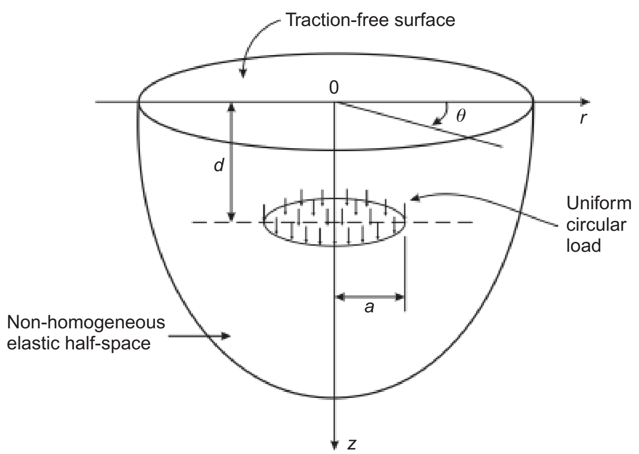

Fig. 1. Mindlin loads at the interior of an undrained non-homogeneous elastic half-space

$$u_z(r, z) = \int \left[A_2 \mathrm{e}^{-\xi z} + B_2 z \mathrm{e}^{-\xi z}\right] \xi J_0(\xi r) \mathrm{d}\xi; \; z > d \tag{26b}$$

and

$$u_r(r, z) = \int \left[\frac{A_1 k_1}{\xi} \mathrm{e}^{-k_1 z} + \frac{B_1 k_2}{\xi} \mathrm{e}^{-k_2 z} - \frac{C_1 k_3}{\xi} \mathrm{e}^{k_3 z} - \frac{D_1 k_4}{\xi} \mathrm{e}^{k_4 z}\right] \xi J_1(\xi r) \mathrm{d}\xi; \quad z < d \tag{27a}$$

$$u_r(r, z) = \int \left[A_2 \mathrm{e}^{-\xi z} + B_2 \left(z - \frac{1}{\xi}\right) \mathrm{e}^{-\xi z}\right] \xi J_1(\xi r) \mathrm{d}\xi; \; z > d \tag{27b}$$

where

$$\begin{aligned} k_1 &= \tfrac{1}{2}\left(\lambda + \sqrt{\lambda^2 + 4i\lambda\xi + 4\xi^2}\right) \\ k_2 &= \tfrac{1}{2}\left(\lambda + \sqrt{\lambda^2 - 4i\lambda\xi + 4\xi^2}\right) \\ k_3 &= \tfrac{1}{2}\left(-\lambda + \sqrt{\lambda^2 + 4i\lambda\xi + 4\xi^2}\right) \\ k_4 &= \tfrac{1}{2}\left(-\lambda + \sqrt{\lambda^2 - 4i\lambda\xi + 4\xi^2}\right) \end{aligned} \tag{28}$$

and A, B, C and D are arbitrary functions of ξ to be determined from appropriate boundary and continuity conditions.

Using equations (26), (27) and (7), $f(r, z)$ takes the form

$$f(r, z) = \int \left[A_1 q_1 \mathrm{e}^{-k_1 z} + B_1 q_2 \mathrm{e}^{-k_2 z} + C_1 q_3 \mathrm{e}^{k_3 z} + D_1 q_4 \mathrm{e}^{k_4 z}\right] \xi J_0(\xi r) \mathrm{d}\xi; \quad z < d \tag{29a}$$

$$f(r, z) = \int \left[-2B_2 \mathrm{e}^{-\xi z}\right] \xi J_0(\xi r) \mathrm{d}\xi; \; z < d \tag{29b}$$

where

$$\begin{aligned} q_i &= \frac{k_i^3}{\xi^2} - k_i - \frac{\lambda k_i^2}{\xi^2} - \lambda; \; i = 1, 2 \\ q_{i+2} &= -\frac{k_{i+2}^3}{\xi^2} + k_{i+2} - \frac{\lambda k_{i+2}^2}{\xi^2} - \lambda; \; i = 1, 2 \end{aligned} \tag{30}$$

Substituting equations (26), (27) and (29) into the boundary and continuity equations defined by equations (20)–(25) results in a system of linear simultaneous equations for the arbitrary functions $A_1(\xi)$, $A_2(\xi)$, $B_1(\xi)$, $B_2(\xi)$, …, etc. (see the Appendix). The substitution of the explicit results for arbitrary functions A_1, B_1, C_1, D_1, A_2 and B_2 in equations (17), (18), (26) and (27) results in explicit solutions for displacements and stresses at an arbitrary point within the domain of the non-homogeneous elastic half-space. The expressions for stresses and displacements involve infinite integrals with integrands decaying exponentially with increasing values of the Hankel transform parameter ξ. This completes the formal analysis of the axisymmetric internal loading of an incompressible elastic half-space region with an exponential variation of the linear elastic shear modulus with depth.

MODULUS OF ELASTICITY

In this paper the half-space is considered to be elastic non-homogeneous. The justification for considering elastic non-homogeneity stems from results of experimental investigation (e.g. Skempton & Henkel, 1957; Ward *et al.*, 1965; Hooper & Butler, 1966; Simons & Som, 1969; Burland & Lord, 1970; Cooke & Price, 1973; Marsland, 1973; Butler, 1974; Hobbs, 1974; Atkinson, 1975; Burland *et al.*, 1977; Abbiss, 1979; Simpson *et al.*, 1979; Costa Filho & Vaughan, 1980), in which it is shown that the shear modulus of many soils increases with depth. It is worth mentioning that elastic moduli in these studies were determined using triaxial compression, in situ pressure meter and plate loading tests.

The early work by Skempton & Henkel (1957) on London Clay suggested that the modulus of elasticity increases with depth. Similar observations by Ward *et al.* (1965) and Marsland (1973) on London Clay using both triaxial and plate loading tests show that the elastic modulus increases with depth, but it can be observed from these and other studies that the triaxial test results are very variable. However, the results from large in situ plate load tests are more consistent, and are also typically higher than those obtained from laboratory triaxial tests. By comparing with laboratory values, Marsland (1973) noted that modulus values given by the plate tests are much closer to those derived from the analysis of settlement records (Hooper & Butler, 1966). Instrumented pile test results presented by Cooke & Price (1973) also seem to support this view.

Burland *et al.* (1977) studied the depth variation of the geotechnical properties of Oxford Clay and Kellaways Beds; the results showed that the vertical Young's modulus increases with depth, although the variation is not necessarily linear. The rate of increase in the Young's modulus was higher at greater depths in Oxford Clay. Burland *et al.* (1977) reported that the undrained vertical Young's modulus, E_v, increases with depth from 10 MPa to 160 MPa. In order to provide an example application of the variation of the modulus of elasticity, a simple linear fit and an exponential fit have been completed for the data provided by Burland *et al.* (1977).

$$G = \frac{E_{\text{secant}}}{2(1+\upsilon)} = \frac{E_{\text{secant}}}{3} \tag{31}$$

$$G(z) = 3{\cdot}33\mathrm{e}^{0{\cdot}0879z} \tag{32}$$

$$G(z) = 3{\cdot}33 + 1{\cdot}062z \tag{33}$$

In the above equations, the SI unit of the modulus of elasticity (shear modulus G and secant Young's modulus E_{secant}) is expressed in megapascals (MPa), and the SI unit for z is given in metres (m). The numerical results for these two variations are presented in the next section.

Studies of the variation in the shear modulus with depth are not limited to clay; the results of large plate tests at various depths by Abbiss (1979) show that the stiffness of chalk and also its static Young's modulus increase with depth. Cripps & Taylor (1981, 1986) investigated the engineering properties of overconsolidated clays and mudrocks. There are also a few tests that considered the measurement of the modulus of elasticity under drained conditions (Atkinson, 1975; Hooper & Wood, 1977).

All the investigations mentioned above show that the modulus of elasticity in soils generally increases with depth, although the variation is not necessarily linear or exponential.

NUMERICAL RESULTS

The procedure outlined in the previous section leads to explicit infinite integral expressions for the displacement and stress fields within the non-homogeneous elastic half-space under a buried circular load. The integrands of these integrals cannot be expressed in explicit forms. Consequently, results of interest for practical applications can be developed only through a numerical integration of the infinite integrals. Eason *et al.* (1955) developed a special numerical procedure to evaluate such infinite integrals, and further applications are investigated by Selvadurai & Rajapakse (1985) and Oliveira *et al.* (2012). Owing to the oscillatory and singular nature of these integrands, an adaptive numerical procedure is used to enhance the accuracy of the numerical results; examples of such an application are given by Rahimian *et al.* (2007) and Katebi *et al.* (2010). For numerical evaluation of the integrals, the upper limit of integration is replaced by a finite value ξ_0; this limit is increased until a convergent result is obtained. The approach outlined in this section was applied to evaluate the influence of the non-homogeneity on the displacements of a non-homogeneous elastic half-space under uniform internal loading over a circular area. It should be pointed out that all numerical results are presented in non-dimensional forms.

In the following, numerical results are presented that illustrate the influence of the segmental non-homogeneity on results of engineering interest. Further, in order to have a better understanding of the influence of non-homogeneity on displacements and stresses, a comparison has been made between these results and those obtained by Selvadurai & Katebi (2012) and Rajapakse (1990). Selvadurai & Katebi (2012) considered the problem with an exponential variation of the shear modulus, while Rajapakse (1990) considered a linear variation in the shear modulus. Segmental, exponential and linear variations are given by equation (11) and

$$G(z) = G_0\mathrm{e}^{\lambda z} \tag{34}$$

$$G(z) = G_0 + mz \tag{35}$$

respectively.

Since there are different variations of the shear modulus, these variations must be related in order to make comparisons. These can be related through the equation

$$\left[\frac{\mathrm{d}G_{\text{Exp}}(z)}{\mathrm{d}z}\right]_{z=0} = \left[\frac{\mathrm{d}G_{\text{Linear}}(z)}{\mathrm{d}z}\right]_{z=0} \tag{36}$$

Equation (33) yields

$$\lambda = \frac{m}{G_0} \tag{37}$$

Figure 2 shows the surface displacement of a non-homogeneous incompressible elastic half-space for different λ, which is directly related to the shear modulus. The vertical surface displacement for different depths and diameters of the loading $\tilde{d}$ ($\tilde{d} = d/a$) are shown in Fig. 3. Also, the variations of vertical displacement along the z-axis for different depths of the loading $\tilde{d}$ ($\tilde{d} = d/a$) are shown in Fig. 4. It is evident that the presence of non-homogeneity has a significant effect on the maximum displacement of the half-space.

To provide a better estimate of the relative influence of the elastic non-homogeneity on the displacements of the medium, the ratio of the displacement in a non-homogeneous medium to that in a homogeneous medium has been evaluated for different values of λ, and is presented in Fig. 5 for both exponential and segmental variations. Fig. 6 shows the variation of the vertical displacement of a non-homogeneous incompressible half-space with segmental variation of the

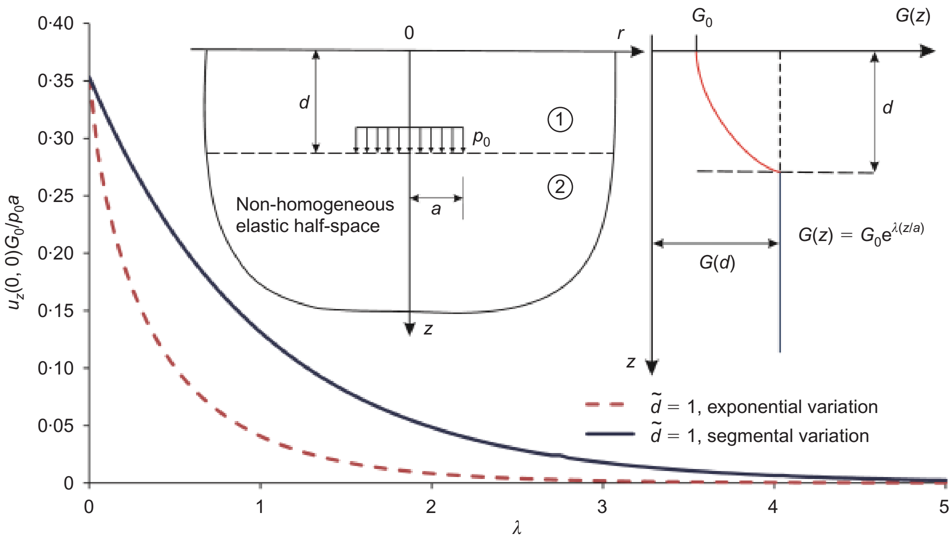

Fig. 2. Variation of vertical displacement for different λ (λ is directly related to shear modulus defined by equation (11))

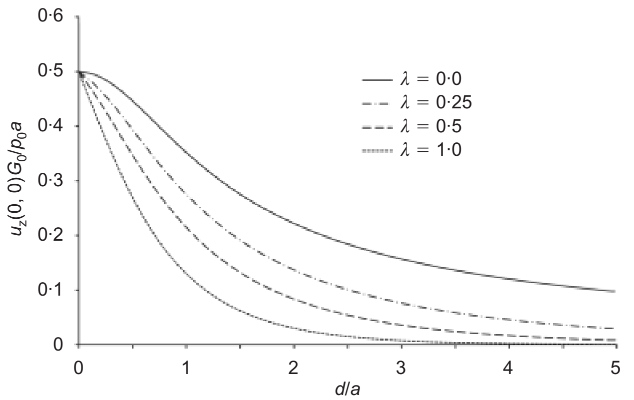

Fig. 3. Vertical surface displacement for different depths and diameters of loading

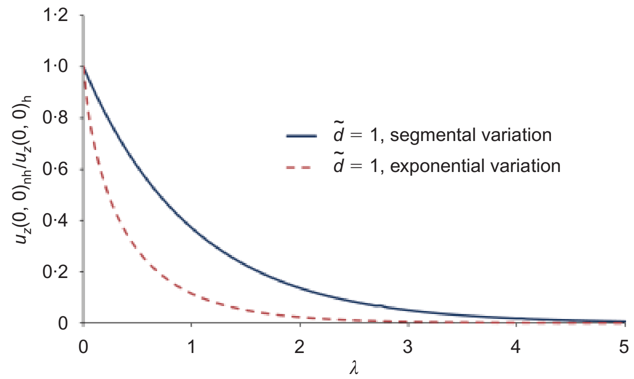

Fig. 5. Ratio of displacement in a non-homogeneous medium to a homogeneous medium for different λ

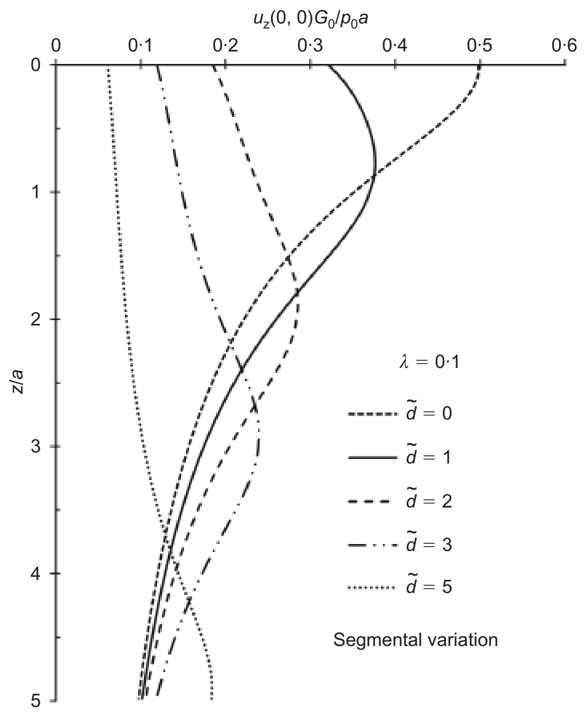

Fig. 4. Variations of vertical displacement along z-axis for different depths of loading

shear modulus in the z-direction for different λ at a depth $\tilde{d} = 1$. The comparison of the vertical displacement along the z-direction for different shear modulus variations is shown in Fig. 7. As would be expected, the vertical displacement is much lower for the exponential variation than for the linear or segmental variation of the shear modulus.

Furthermore, the variation of vertical displacement of a non-homogeneous incompressible half-space with segmental variation along the r-axis at depth $\tilde{d} = 1$ is shown in Fig. 8. Fig. 9 shows the comparison of vertical displacement along the r-axis for exponential, segmental and linear variation of the shear modulus. As can be seen in the results presented in Fig. 9, the slope of the curve should be zero where $r = 0$; however, this is not correctly addressed by the results of Rajapakse (1990).

The presented figures show that the response of the medium is influenced by the degree of non-homogeneity. As would be expected, the vertical displacement decreases as the shear modulus increases, if all other parameters are kept constant. (i.e. as G increases, the stiffness of the half-space also increases).

The numerical results for the linear and exponential fit mentioned in the previous section are shown in Fig. 10. For a better understanding of the influence of the degree of non-homogeneity on the response, the variation of vertical displacements along the z-axis has been plotted in Fig. 11 for different depths and diameters of loading.

Figures 12 and 13 show the variation of non-dimensionalised

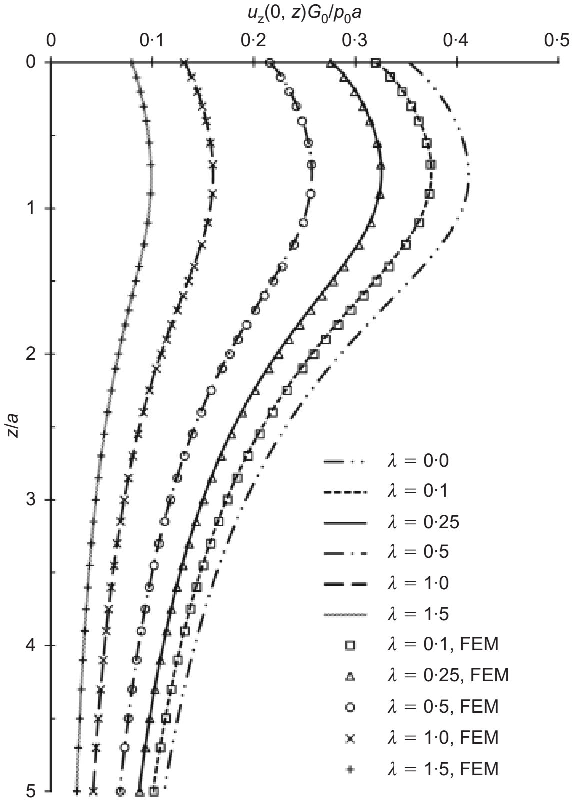

Fig. 6. Variation of vertical displacement along z-axis for different λ at depth $\tilde{d} = 1$

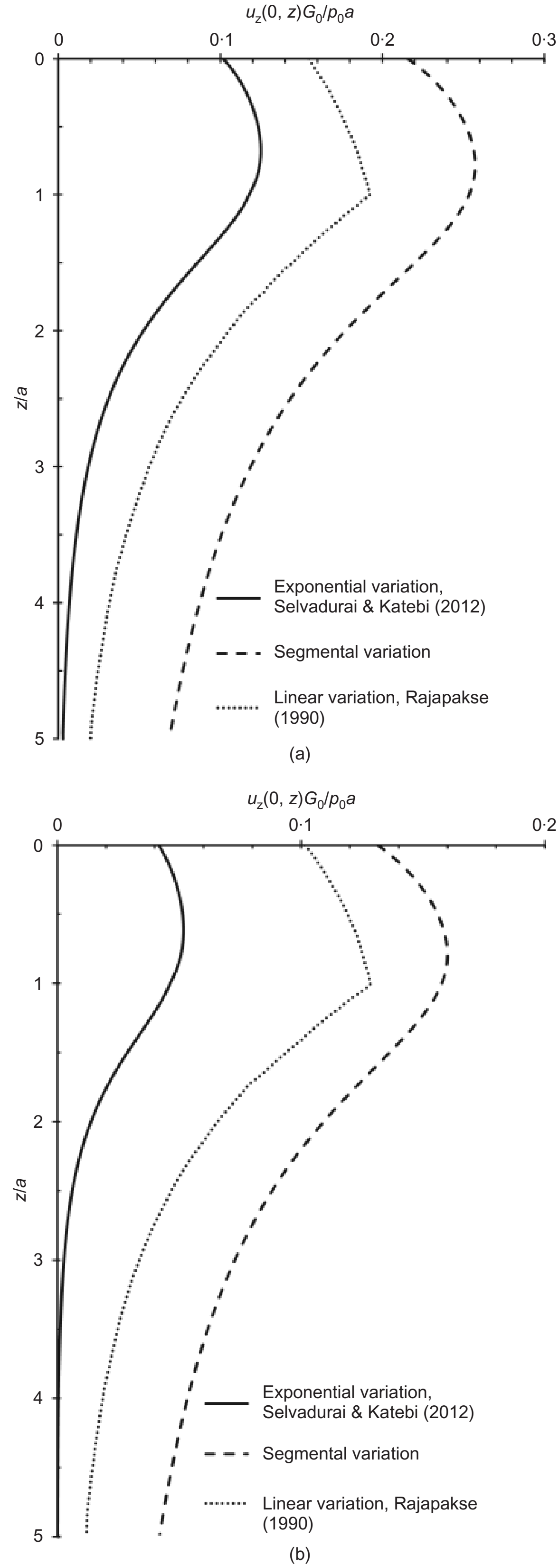

Fig. 7. Comparison of vertical displacement along z-axis at depth $\tilde{d} = 1$ for: (a) $\lambda = 0{\cdot}5$; (b) $\lambda = 1$

normal and radial stress along the z-direction for segmental non-homogeneity at a depth $\tilde{d} = 2$. As is evident from Fig. 9, σ_{zz} and σ_{rr} sustain a discontinuity at $\tilde{d} = 2$, owing to the effect of the applied load. It can be observed that σ_{zz} decreases with an increase in the shear modulus, but unlike the displacement, the influence of non-homogeneity is moderate, and limited to the vicinity of the applied load. It should be noted that $\sigma_{\theta\theta}$ and σ_{rr} are equal, owing to the symmetry of the problem.

In order to check the consistency of the results, one can use the strain–stress relation to calculate the strains and then the displacements. The variation of normal strains along the z-axis is given in Fig. 14.

Computational results

To provide a comparison with the analytical solutions, a finite-element analysis of incompressible non-homogeneous elastic half-space problem was performed using COMSOL Multiphysics^{TM} software. The axisymmetric half-space region is represented by a finite domain where the outer boundaries extend to 1000 times the radius of the loaded area in both the r- and z-directions. In order to obtain this ratio ($l/a = 1000$), a calibration was performed between computational results with different l/a ratios and known analytical solutions for the classical contact problem (Selvadurai, 1979; Gladwell, 1980). A mixed U–P formulation was employed in the COMSOL Multiphysics^{TM} software in order to model an incompressible material. Fig. 15(a) shows the finite-element representation of the classical problem of the indentation of the surface of a half-space by a rigid circular disc of radius a, subjected to an axial load P. As can be seen from Fig. 15(b), the analytical and computational results are virtually identical beyond $l/a = 1000$. The same l/a ratio was used to develop computational results for the non-homogeneous case.

Figure 6 gives a comparison of the analytical results and finite-element results for the internal loading of a non-homogeneous incompressible half-space for segmental variation of the shear modulus (the computational results are

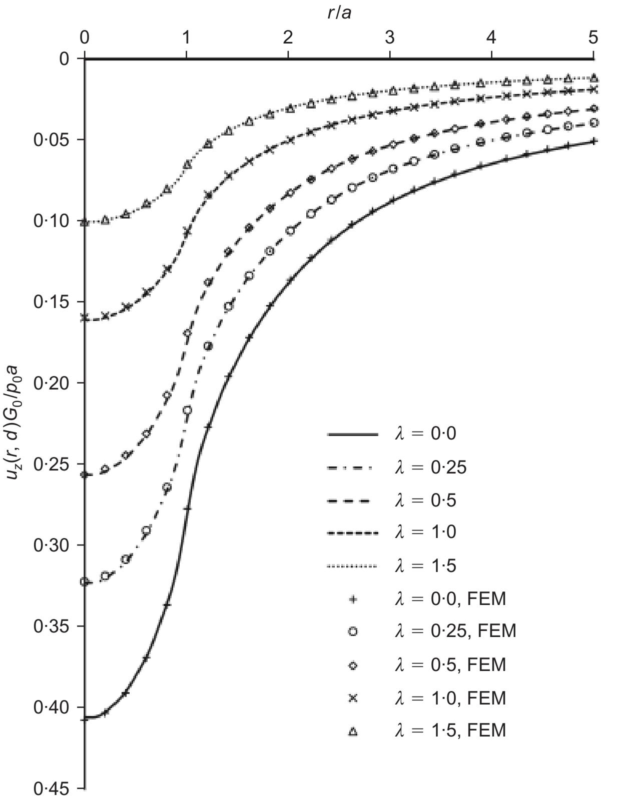

Fig. 8. Variation of vertical displacement along *r*-axis for different λ at depth $\tilde{d} = 1$

indicated by the symbols □, ○, Δ, etc.). It can be seen that there is excellent correlation between the analytical results derived from the exponential variations in the shear modulus and the computational results (accurate to within 0·3%). This almost negligible difference could have arisen either from an error in the numerical calculation of the analytical–numerical solution or through idealisation of the half-space region as a finite domain. The discrepancies are considered to be well within the range acceptable for engineering applications of the results.

CONCLUDING REMARKS

In this paper, a mathematical treatment is presented for the displacements and stresses corresponding to axisymmetric interior loading of a non-homogeneous incompressible isotropic elastic half-space where the linear elastic modulus varies exponentially over a finite depth, beyond which it is constant. These results have been compared with two existing solutions for both linear and exponential variations of the shear modulus. The influence of non-homogeneity on the response of the half-space is clearly shown by the numerical results presented. The interior loading of an elastic half-space can serve as a useful model for examining the interior loading of geologic media with predominantly isochoric deformations. Experimental investigations of geological media such as London Clay deposits show that the modulus of elasticity in soils generally increases with depth, although the variation is not necessarily linear or exponential. The analysis of the traction boundary value problem related to the interior loading of a non-homogeneous elastic half-space can be obtained in a form where results of practical interest can be derived through the evaluation of infinite integrals. The study can also be used as a benchmarking solution for examining the accuracy of computational approaches that can ultimately be used to examine more complicated variations of the shear modulus with depth.

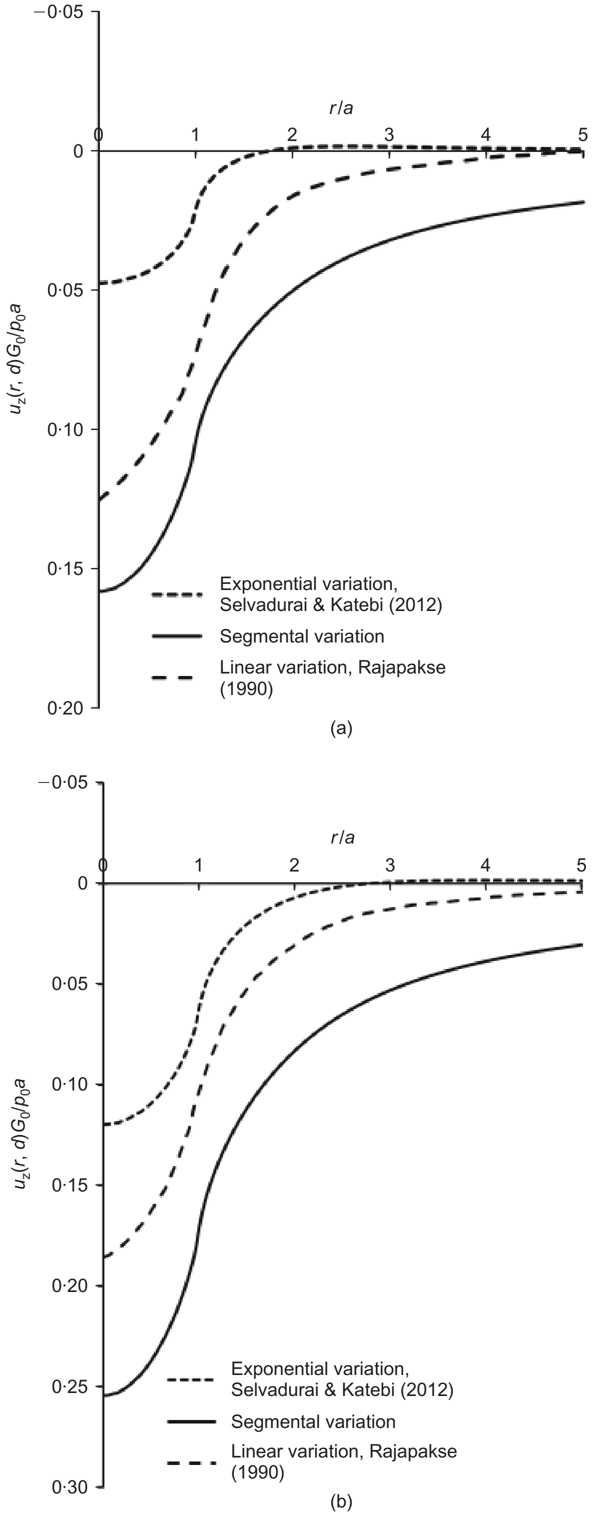

Fig. 9. Comparison of vertical displacement along *r*-axis at depth $\tilde{d} = 1$ for: (a) $\lambda = 0{\cdot}5$; (b) $\lambda = 1$

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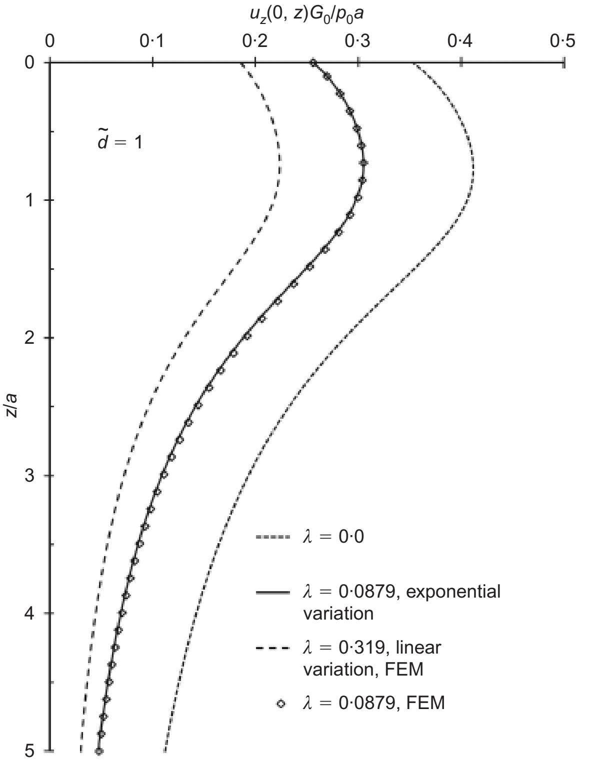

Fig. 10. Variations of vertical displacement along *z*-axis for linear and exponential variation of shear modulus fitted to data provided by Burland *et al.* (1977)

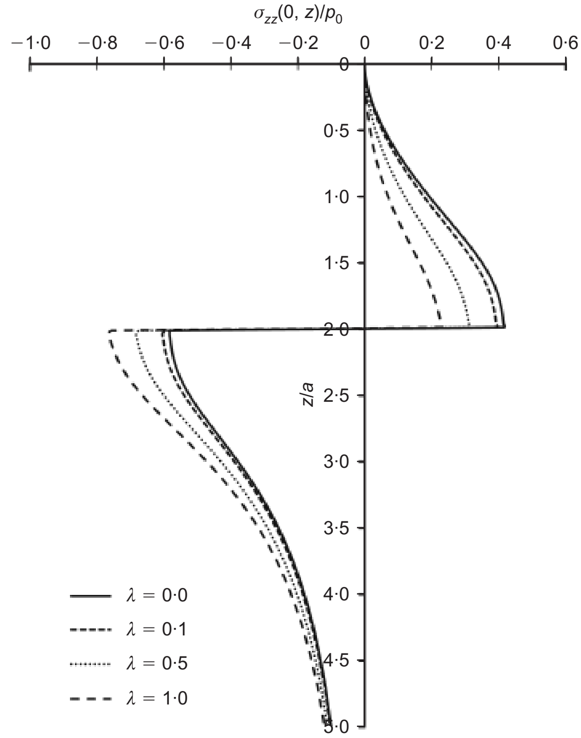

Fig. 12. Normal stress along *z*-axis for different λ at depth $\tilde{d} = 2$

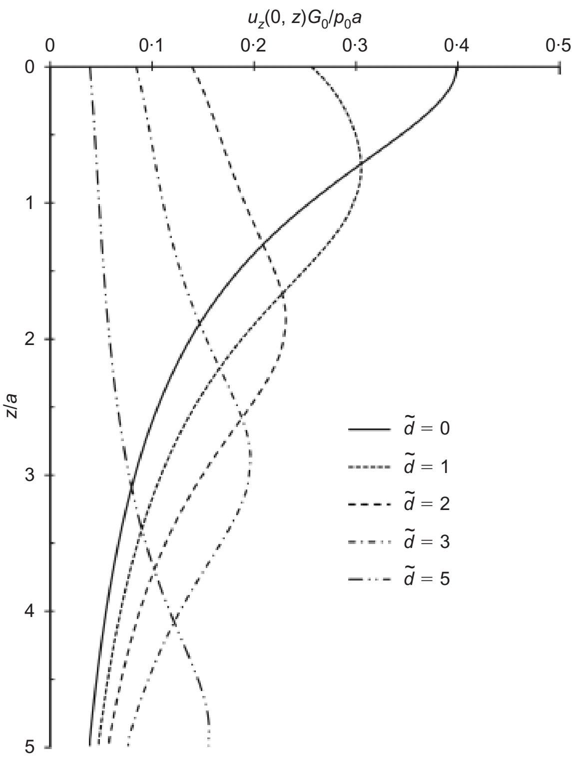

Fig. 11. Variations of vertical displacement along *z*-axis for $\lambda = 0{\cdot}0879$ and different depths of loading for exponential variation of shear modulus fitted to data provided by Burland *et al.* (1977)

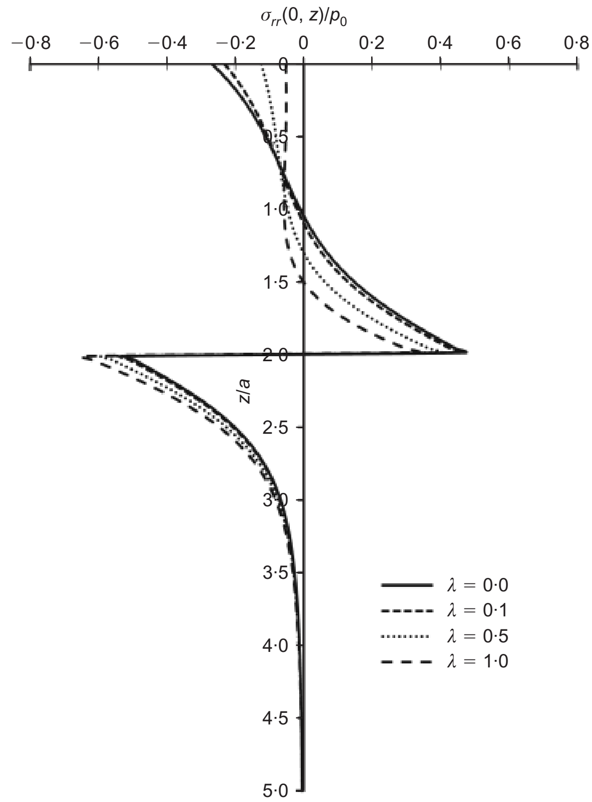

Fig. 13. Radial stress along *z*-axis for different λ at depth $\tilde{d} = 2$

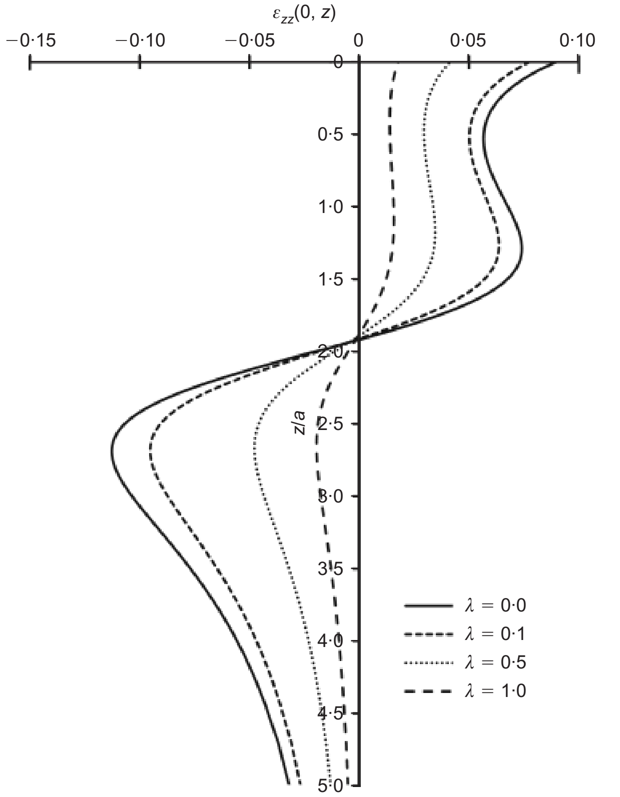

Fig. 14. Normal strain along z-axis for different λ at depth $\tilde{d} = 2$

APPENDIX

$$A_1\theta_1 + B_1\theta_2 + C_1\theta_3 + D_1\theta_4 = 0 \tag{38}$$

$$A_1\eta_1 + B_1\eta_2 + C_1\eta_3 + D_1\eta_4 = 0 \tag{39}$$

$$\begin{aligned} &A_1k_1e^{-k_1d} + B_1k_2e^{-k_2d} - C_1k_3e^{k_3d} - D_1k_4e^{k_4d} \\ &= [A_2\xi + B_2(d\xi - 1)]e^{-\xi d} \end{aligned} \tag{40}$$

$$A_1e^{-k_1d} + B_1e^{-k_2d} + C_1e^{k_3d} + D_1e^{k_4d} = (A_2 + B_2d)e^{-\xi d} \tag{41}$$

$$\begin{aligned} &A_1\eta_1e^{-k_1d} + B_1\eta_2e^{-k_2d} + C_1\eta_3e^{k_3d} + D_1\eta_4e^{k_4d} \\ &= (2\xi A_2 + 2\xi B_2d - 2B_2)e^{-\xi d} \end{aligned} \tag{42}$$

$$\begin{aligned} &A_1\theta_1e^{-k_1d} + B_1\theta_2e^{-k_2d} + C_1\theta_3e^{k_3d} + D_1\theta_4e^{k_4d} + (A_2 + B_2d)e^{-\xi d} \\ &= \frac{\tilde{p}(\xi)}{G(d)} \end{aligned} \tag{43}$$

where

$$\begin{aligned} &\eta_i = \xi + \frac{k_i^2}{\xi};\ i = 1, 2, 3, 4 \\ &\theta_i = q_i - 2k_i;\ \theta_{i+2} = q_{i+2} + 2k_{i+2};\ i = 1,\ 2 \\ &G(d) = G_0e^{\lambda d} \\ &\tilde{p}(\xi) = \int_0^\infty rp(r)J_0(\xi r)\mathrm{d}r \end{aligned} \tag{44}$$

The explicit solutions for the arbitrary functions A_1, B_1, C_1, D_1, A_2 and B_2 can be expressed as follows.

$$A_1 = \frac{\gamma\ell_3\chi_4}{\theta_3} \tag{45}$$

$$B_1 = -\frac{\gamma\ell_3}{\theta_2} - \frac{\gamma\ell_3\chi_4\theta_1}{\theta_2\theta_3} \tag{46}$$

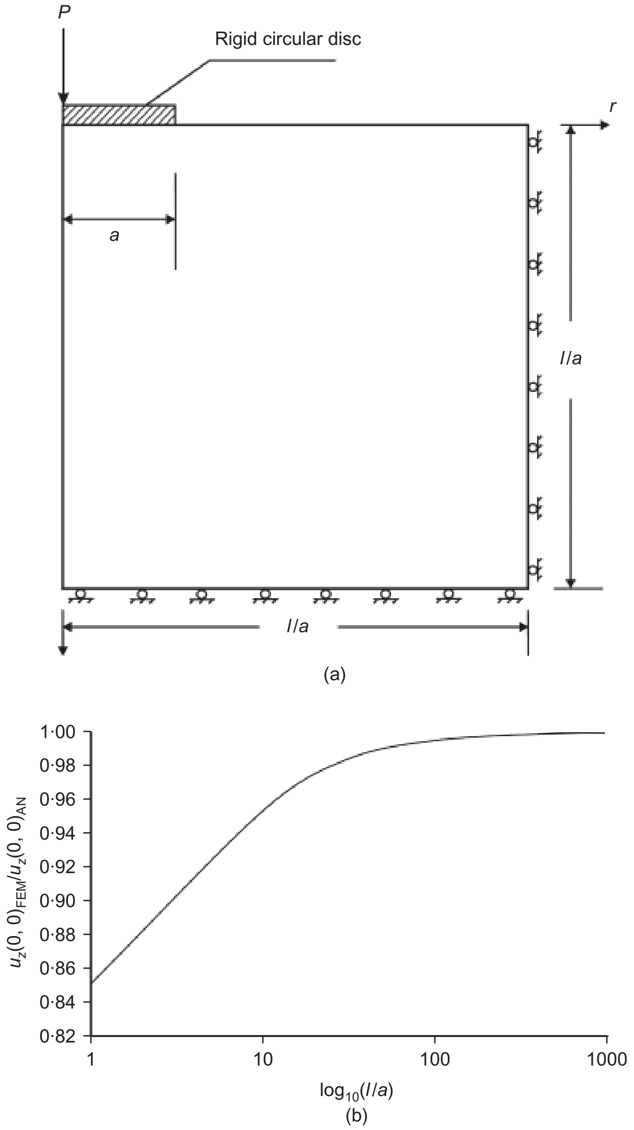

Fig. 15. (a) Finite-element representation of the classic problem of the indentation of the surface of a rigid circular disc; (b) vertical displacement for different l/a

$$C_1 = \frac{\gamma\ell_2}{\theta_2} - \frac{\gamma\ell_3\chi_4(\ell_1/\ell_3 - \ell_2\theta_1/\ell_3\theta_2)}{\theta_3} \tag{47}$$

$$D_1 = \frac{\gamma\ell_5}{\theta_2} - \frac{\gamma\ell_3\chi_4(\ell_4/\ell_3 - \ell_5\theta_1/\ell_3\theta_2)}{\theta_3} \tag{48}$$

$$A_2 = \frac{\gamma\ell_3\delta_2}{\theta_2} - \frac{\gamma\ell_3\chi_4(\delta_1 - \delta_2\theta_1/\theta_2)}{\theta_3} \tag{49}$$

$$B_2 = \frac{\gamma\ell_3R_1}{\theta_2} - \frac{\gamma\ell_3\chi_4(R_1 - R_2\theta_1/\theta_2)}{\theta_3} \tag{50}$$

where

$$\ell_i = \eta_4\theta_i - \eta_i\theta_4;\ i = 1, 2, 3$$

$$\ell_{i+3} = -\eta_3\theta_i + \eta_i\theta_4;\ i = 1, 2$$

$$I_i = e^{-d(\xi+k_i)}(\xi - k_i);\ I_{i+2} = e^{-d(\xi-k_{i+2})}(\xi + k_{i+2});\ i = 1, 2$$

$$J_i = e^{-2d\xi-d(\xi+k_i)}(2\xi - \eta_i);\ J_{i+2} = e^{-2d\xi-d(\xi-k_i)}(2\xi - \eta_{i+2});$$
$$i = 1, 2$$

$$h_i = -e^{-d(\xi+k_i)}(2\xi + \theta_i);\ h_{i+2} = -e^{-d(\xi-k_{i+2})}(2\xi + \theta_{i+2});$$
$$i = 1, 2$$

$$\omega_i = e^{d(\xi - k_i)}(-1 + d\xi - dk_i);\ \omega_{i+2} = e^{d(\xi + k_{i+2})}(-1 + d\xi + dk_{i+2});$$
$$i = 1, 2$$
$$f_i = e^{d(\xi - k_i)}\left(e^{d(k_i + k_4)}\xi\eta_1 + e^{d(k_i + k_4)}k_4\eta_1 - \xi\eta_4 - k_i\eta_4\right);\ i = 1, 2$$
$$f_3 = e^{d\xi}\left(-e^{dk_4}\xi\eta_3 - e^{dk_4}k_4\eta_3 + e^{dk_3}\xi\eta_4 + e^{dk_3}k_3\eta_4\right)$$
$$\chi_i = \left(-2e^{-2d}I_i - J_i\right)\eta_4 + \eta_i\left(-2e^{-2d}I_4 - J_4\right);\ i = 1, 2, 3$$
$$\chi_4 = -\chi_2\ell_2 + \chi_1\ell_3$$
$$\theta_i = \ell_3(\eta_4 h_i - \eta_i h_4) - \ell_i(\eta_4 h_3 - \eta_3 h_4);\ i = 1, 2$$
$$\theta_3 = -\chi_4\theta_1 + (\chi_1\ell_3 - \chi_3\ell_1)\theta_2$$
$$\delta_i = \omega_i - \left(\frac{\omega_4\eta_i}{\eta_4}\right) - \frac{\omega_3 - (\omega_4\eta_3/\eta_4)\ell_2}{\ell_3};\ i = 1, 2$$
$$R_i = \frac{f_1}{\eta_4} + \frac{f_3\ell_i}{\eta_4\ell_3}; i = 1, 2$$
$$\gamma = \frac{e^{-d\xi}\eta_4 P}{G} \tag{51}$$

NOTATION

A, B, C, D	arbitrary functions of ξ
a	radius of loading
d	embedment depth of loading
$\tilde{d}$	$= d/a$
E_{secant}	secant Young's modulus
E_{v}	undrained vertical Young's modulus
F	Hankel transform of f
f	arbitrary function of (r, z)
$G(z)$	shear modulus varying in z-direction
G_0	shear modulus at surface
$g(z)$	arbitrary function of z dependent on the variation in the shear modulus
$\mathbf{I}$	unit matrix
J_n	Bessel function of the first kind of order n
m	non-homogeneous parameter (linear)
P	axial load
$p(r)$	intensity of internally applied pressure
p_0	applied stress intensity
r	radial coordinate
U_r, U_z	Hankel transforms of $u_r(r, z)$ and $u_z(r, z)$, respectively
$\mathbf{u}$	displacement vector
$u_r(r, z)$	radial displacement component
$u_z(r, z)$	vertical displacement component
x	position vector
z	vertical coordinate
$\boldsymbol{\varepsilon}$	strain tensor
λ	non-homogeneous parameter (exponential)
ν	Poisson's ratio
ξ	Hankel transform parameter
$\boldsymbol{\sigma}$	stress tensor
σ_{rr}	radial stress
σ_{zz}	axial stress
$\sigma_{\theta\theta}$	circumferential stress
∇^2	Laplace's operator

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