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Mindlin's problem for an incompressible elastic half-space with an exponential variation in the linear elastic shear modulus

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ABSTRACT

This paper examines the axisymmetric problem of the internal loading of an incompressible elastic half-space where the linear elastic shear modulus varies exponentially with depth. The mathematical formulation of the traction boundary value problem is developed through the application of integral transform techniques and numerical results are obtained from the integral transform technique. The numerical results obtained from the analytical approach are used to verify the accuracy of finite element results for the analogous problems.

1. Introduction

The study of the mechanics of non-homogeneous elastic media has always occupied a prominent position in the literature in mechanics. Quite apart from the intrinsic mathematical interest, the non-homogeneity problem in elasticity has applications to many problems of technological importance. In the context of geomechanics, the inhomogeneous medium serves as a model for the study of soil and rock media which exhibit spatial variations in their elastic properties. Studies of specific interest to elastomechanics of non-homogeneous materials date back to the early studies by Klein (1956), Korenev (1975), Mossakowskii (1958), Popov (1959) and Rostovtsev (1961, 1964). Reviews of the subject are also given by Rakov and Rvachev (1961), Olszak (1959) and Popov (1973). The type of problems discussed in these developments largely focused on elastic contact problems referred to half-space regions, where the elastic modulus varied with the axial coordinate and Poisson's ratio was assumed to be constant. Also, the specific forms of elastic non-homogeneity dealt with either a linear, an exponential or a power law variation in the elastic modulus with the axial coordinate. The resulting solutions have been applied quite extensively in contact and indentation problems with application to the engineering sciences.

The renewal of interest in the application of the theory of elasticity for a non-homogeneous elastic medium commenced with a classic study by Gibson (1967), who examined the problem of the surface loading of an incompressible isotropic elastic half-space where the elastic shear modulus varied linearly with the axial coordinate. Gibson (1967) showed that when the shear modulus of the incompressible medium varied linearly from zero at the surface of the halfspace, the surface displacement profile exhibited a discontinuous form with displacements restricted only to the loaded region. Furthermore, the magnitude of the deflection within the loaded region was directly proportional to the intensity of stress at the loading point and inversely proportional to the linear increase in the elastic shear modulus with depth. Gibson's study provided a precise

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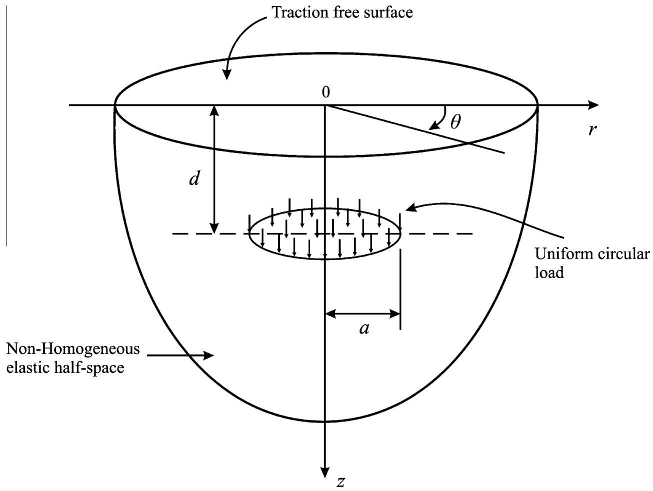

Fig. 1. Axisymmetric Mindlin loads acting at the interior of an incompressible non-homogeneous elastic half-space.

continuum definition to the discrete "*spring model of elastic support*", referred to as the "Winkler" medium (Hetényi, 1946; Selvadurai, 1979; Winkler, 1867), and used extensively in the analysis of soil-foundation interaction problems in geomechanics, The range of application of Gibson's formulation to surface loading problems has been discussed in a number of papers and these are documented in the reviews by Selvadurai (1979, 2007), Gladwell (1980) and Aleynikov (2011). The idealization of the elastic non-homogeneity in either a linear or an exponential form restricts its applicability to the study of boundary value problems relevant to half-space regions. In particular, the elastic modulus becomes unbounded as the axial coordinate approaches infinity. To overcome this deficiency, Selvadurai, Singh, and Vrbik (1986) introduced an exponential variation of the linear elastic shear modulus, with a bounded value at infinity, to examine the torsional indentation of the surface of a non-homogeneous elastic half-space. The approach was subsequently extended by Selvadurai (1996) to examine the mixed boundary value problem of the axisymmetric indentation of an isotropic elastic half-space with a constant Poisson's ratio and an elastic shear modulus that varied with a bounded exponential variation with depth. The studies by Selvadurai and Lan (1997, 1998) consider contact and crack problems where the elastic shear modulus exhibits a harmonic variation. A more generalized approach to the formulation of spatial inhomogeneity is described by Spencer and Selvadurai (1998), who examine the problem of anti-plane shear in cracks and edge dislocations in a non-homogeneous elastic solid.

In this paper we examine the problem of the axisymmetric interior loading of an incompressible elastic half-space where the elastic shear modulus varies with depth in an exponential manner. The problem of the application of an axisymmetric load at the interior of a homogeneous isotropic elastic half-space was first developed by Mindlin (1936); here we extend the study to include the influence of elastic non-homogeneity. The application of a concentrated load to the interior of an elastic half-space where the linear elastic shear modulus varies linearly with depth was examined by Rajapakse (1990) and Rajapakse and Selvadurai (1991) examined the mixed boundary value problem related to the interior loading of a non-homogeneous isotropic elastic half-space by a flexible plate. In this paper we consider the interior loading of an incompressible isotropic elastic half-space (Fig. 1). The problem of the interior loading of a half-space region has applications in geomechanics, where the interior load can be visualized as an external load that is transmitted to the interior of the half-space region through a structural element such as a pile or an anchorage (Selvadurai, 1980, 1994a, 1994b, 2000a; Yue & Selvadurai, 1995). We specifically consider the axisymmetric problem of the internal circular loading of an incompressible elastic half-space, with an exponential variation in the shear modulus. A Hankel transform development of the governing equations of elasticity is used to solve the interior loading problem. The adaptive numerical quadrature technique is used to evaluate the integrals obtained from the integral transform technique. Numerical results are presented in order to show the influence of the non-homogeneity of the responses of an incompressible elastic half-space. The numerical results are also used to establish the accuracy of finite element results for the analogous problems.

2. Governing equations

The development of the partial differential equations governing the elastic non-homogeneity problem is relatively straightforward and the essential steps are presented for completeness. Accounts of the developments are given in the references (Gibson, 1967; Korenev, 1975; Popov, 1973). We consider the class of axisymmetric problems in the theory of elasticity, referred to the cylindrical polar coordinate system (r,θ,z), where the displacement vector is

$$\mathbf{u} = \{u_r(r,z), 0, u_z(r,z)\} \tag{1}$$

and the strain tensor $\boldsymbol{\varepsilon}$ is defined by

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & 0 & \frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) \\ 0 & \frac{u_r}{r} & 0 \\ \frac{1}{2}\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\right) & 0 & \frac{\partial u_z}{\partial z} \end{pmatrix} \tag{2}$$

For axial symmetry, the non-zero components of the Cauchy stress tensor $\boldsymbol{\sigma}$ are

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{rz} & 0 & \sigma_{zz} \end{pmatrix} \tag{3}$$

and the constitutive relationship for a non-homogeneous elastic material in which the linear elastic shear modulus varies with the coordinate direction z, takes the form

$$\boldsymbol{\sigma} = 2G(z)[\alpha e\mathbf{I} + \boldsymbol{\varepsilon}] \tag{4}$$

where $G(z)$ is the linear elastic shear modulus, $\mathbf{I}$ is the unit matrix and

$$e = \mathrm{tr}(\boldsymbol{\varepsilon}); \quad \alpha = \frac{v}{(1-2v)} \tag{5}$$

and v is Poisson's ratio, which is assumed to be a constant. We specifically restrict attention to incompressible elastic materials for which isochoric deformations give

$$\mathrm{tr}(\boldsymbol{\varepsilon}) = 0; \quad v = 1/2 \tag{6}$$

The constraints (6) imply that the constitutive Eq. (4) are indeterminate to within an isotropic stress state $f(r,z)$, which needs to be determined from the solution of the equations of equilibrium, which in the absence of body forces and for axial symmetry, reduce to

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = 0 \tag{7}$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} = 0 \tag{8}$$

Using constitutive equations applicable to an incompressible elastic material with a spatial variation in the linear elastic shear modulus as defined by (4), the equations of equilibrium (7) and (8) can be expressed in terms of the displacements as follows:

$$\nabla^2 u_r + \frac{\partial f}{\partial r} - \frac{u_r}{r^2} + \frac{1}{G}\frac{dG}{dz}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = 0 \tag{9}$$

$$\nabla^2 u_z + \frac{\partial f}{\partial z} + \frac{1}{G}\frac{dG}{dz}\left(f + 2\frac{\partial u_z}{\partial z}\right) = 0 \tag{10}$$

where ∇^2 is the axisymmetric form of Laplace's operator given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{11}$$

Eliminating the function $f(r,z)$ between (9) and (10) and based on the assumption that the differentiations commute, we obtain,

$$g(z)\nabla^2 u_r + \nabla^2\frac{\partial u_r}{\partial z} - \frac{\partial}{\partial r}(\nabla^2 u_z) + g(z)\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{\partial^2 u_z}{\partial r \partial z}\right) - \frac{1}{r^2}\frac{\partial u_r}{\partial z} + \left[g^2(z) + \frac{dg(z)}{dz}\right]\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = 0 \tag{12}$$

where

$$g(z) = \frac{1}{G}\frac{dG}{dz} \tag{13}$$

By restricting attention to a variation of the elastic shear modulus to the form

$$G(z) = G_0 \exp(\lambda z) \tag{14}$$

Eq. (12) can be reduced to

$$\lambda\nabla^2 u_r + \nabla^2\frac{\partial u_r}{\partial z} - \frac{\partial}{\partial r}(\nabla^2 u_z) + \lambda\left(\frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{\partial^2 u_z}{\partial r \partial z}\right) - \frac{1}{r^2}\frac{\partial u_r}{\partial z} + [\lambda^2]\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = 0 \tag{15}$$

The result (15) together with the incompressibility condition

$$\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}=0 \tag{16}$$

constitute the set of coupled partial differential equations governing the displacement field (1).

In order to solve Eqs. (15) and (16) we introduce Hankel transform representations (Selvadurai, 2000b; Sneddon, 1951) of the following form

$$u_r(r,z)=\int_0^\infty \xi U_r(\xi,z)J_1(\xi r)d\xi \tag{17}$$

$$u_z(r,z)=\int_0^\infty \xi U_z(\xi,z)J_0(\xi r)d\xi \tag{18}$$

where ξ is the Hankel transform parameter and J_n is the Bessel function of the first kind of order n. Substituting (17) and (18) in (16) and (15) gives rise to the following:

$$\frac{dU_z}{dz}+\xi U_r=0 \tag{19}$$

$$\frac{d^3U_r}{dz^3}+2\lambda\frac{d^2U_r}{dz^2}+\xi\frac{d^2U_z}{dz^2}+(\lambda^2-\xi^2)\frac{dU_r}{dz}+\xi\lambda\frac{dU_z}{dz}-\lambda\xi^2U_r-\xi(\xi^2+\lambda^2)U_z=0 \tag{20}$$

By substitution of (19) in (20) we obtain

$$\frac{d^4U_z}{dz^4}+2\lambda\frac{d^3U_z}{dz^3}+(\lambda^2-2\xi^2)\frac{d^2U_z}{dz^2}-2\lambda\xi^2\frac{dU_z}{dz}+\xi^2(\xi^2+\lambda^2)U_z=0 \tag{21}$$

Considering the expressions for the stress–strain relationships (4), the strain–displacement relations (2) and the integral expressions (17) and (18) for the displacement components, the transformed expressions for the stress components $\sigma_{zz}(r,z)$ and $\sigma_{rz}(r,z)$ can be written as

$$\tilde{\sigma}^0_{zz}(\xi,z)=G(z)\left[F+2\frac{dU_z}{dz}\right] \tag{22}$$

$$\tilde{\sigma}^1_{rz}=G(z)\left[-\xi U_z+\frac{dU_r}{dz}\right] \tag{23}$$

where F is the Hankel transform of f defined by

$$f(r,z)=\int_0^\infty \xi F(\xi,z)J_0(\xi r)d\xi \tag{24}$$

The inverse of these stress components are given by

$$\sigma_{zz}(r,z)=G(z)\int_0^\infty \xi\left[F+2\frac{dU_z}{dz}\right]J_0(\xi r)d\xi \tag{25}$$

$$\sigma_{rz}(r,z)=G(z)\int_0^\infty \xi\left[-\xi U_z+\frac{dU_r}{dz}\right]J_1(\xi r)d\xi \tag{26}$$

3. The axisymmetric internal loading of a non-homogeneous elastic half-space

We consider the problem of an incompressible non-homogeneous elastic half-space, which is loaded internally by an axially directed circular load of radius a with stress intensity p_0 and situated at a depth $z = d$ from the traction free surface of the half-space (Fig. 1). The most convenient approach for formulating boundary value problems of this type (Selvadurai, Singh, & Au, 1991; Selvadurai, 1993, 2000c, 2000d) is to consider that the original half-space region is composed of (i) a layer region (superscript $^{(1)}$) occupying the domain $r\in(0,\infty)$ and $z\in(0,d)$ and (ii) a half-space region (superscript $^{(2)}$) occupying the domain $r\in(0,\infty)$ and $z\in(d,\infty)$.

We consider the elastic layer region and the elastic half-space region that are subjected to the following boundary, interface and regularity conditions:

$$\sigma^{(1)}_{zz}(r,0)=0 \tag{27}$$

$$\sigma^{(1)}_{rz}(r,0)=0 \tag{28}$$

$$u^{(1)}_r(r,d)=u^{(2)}_r(r,d) \tag{29}$$

$$u^{(1)}_z(r,d)=u^{(2)}_z(r,d) \tag{30}$$

$$\sigma_{zz}^{(1)}(r,d) - \sigma_{zz}^{(2)}(r,d) = \begin{cases} p(r), & r \leqslant a \\ 0, & r > a \end{cases} \tag{31}$$

$$\sigma_{rz}^{(1)}(r,d) = \sigma_{rz}^{(2)}(r,d) \tag{32}$$

In Eq. (31), $p(r)$ represents the intensity of internally applied pressure over the circular area. In addition to these boundary and continuity conditions, it is assumed that the displacements and stresses should satisfy the appropriate regularity conditions in the layer and half-space regions as $r, z \to \infty$.Consistent with the regularity condition at infinity, the general solutions of Eq. (21) can be written as

$$U_z(\xi,z) = \begin{cases} A_1 e^{-k_1 z} + B_1 e^{-k_2 z} + C_1 e^{k_3 z} + D_1 e^{k_4 z}; & z < d \\ A_2 e^{-k_1 z} + B_2 e^{-k_2 z}; & z > d \end{cases} \tag{33}$$

where

$$\begin{aligned} k_1 &= \frac{1}{2}\left[\lambda + \sqrt{\lambda^2 + 4i\lambda\xi + 4\xi^2}\right] \\ k_2 &= \frac{1}{2}\left[\lambda + \sqrt{\lambda^2 - 4i\lambda\xi + 4\xi^2}\right] \\ k_3 &= \frac{1}{2}\left[-\lambda + \sqrt{\lambda^2 + 4i\lambda\xi + 4\xi^2}\right] \\ k_4 &= \frac{1}{2}\left[-\lambda + \sqrt{\lambda^2 - 4i\lambda\xi + 4\xi^2}\right] \end{aligned} \tag{34}$$

From Eqs. (19) and (33), the general solution of $U_r(\xi,z)$ can be expressed as

$$U_r(\xi,z) = \begin{cases} \frac{A_1 k_1}{\xi} e^{-k_1 z} + \frac{B_1 k_2}{\xi} e^{-k_2 z} - \frac{C_1 k_3}{\xi} e^{k_3 z} - \frac{D_1 k_4}{\xi} e^{k_4 z}, & z < d \\ \frac{A_2 k_1}{\xi} e^{-k_1 z} + \frac{B_2 k_2}{\xi} e^{-k_2 z}, & z > d \end{cases} \tag{35}$$

where A, B, C and D in Eqs. (33) and (35) are arbitrary functions of ξ to be determined from appropriate boundary and continuity conditions.

The Eq. (10) can be reduced to

$$\frac{d^2 U_r}{dz^2} - \xi^2 U_r - \xi F + \lambda\left(\frac{dU_r}{dz} - \xi U_z\right) = 0 \tag{36}$$

The substitution of (33) and (35) in (36) results in

$$F(z) = \begin{cases} A_1 q_1 e^{-k_1 z} + B_1 q_2 e^{-k_2 z} + C_1 q_3 e^{k_3 z} + D_1 q_4 e^{k_4 z}, & z < d \\ A_2 q_1 e^{-k_1 z} + B_2 q_2 e^{-k_2 z}, & z > d \end{cases} \tag{37}$$

where

$$\begin{aligned} q_i &= \frac{k_i^3}{\xi^2} - k_i - \frac{\lambda k_i^2}{\xi^2} - \lambda \quad i = 1,2 \\ q_{i+2} &= -\frac{k_{i+2}^3}{\xi^2} + k_{i+2} - \frac{\lambda k_{i+2}^2}{\xi^2} - \lambda \quad i = 1,2 \end{aligned} \tag{38}$$

Substituting Eqs. (33), (35) and (37) into the boundary and continuity equations defined by (27)–(32) results in a system of linear simultaneous equations for the arbitrary functions $A_1(\xi), A_2(\xi), B_1(\xi), B_2(\xi), \ldots$, etc., as follows:

$$A_1\theta_1 + B_1\theta_2 + C_1\theta_3 + D_1\theta_4 = 0 \tag{39}$$

$$A_1\eta_1 + B_1\eta_2 + C_1\eta_3 + D_1\eta_4 = 0 \tag{40}$$

$$[A_1 - A_2]\beta_1 e^{-k_1 d} + [B_1 - B_2]\beta_2 e^{-k_2 d} - C\beta_3 e^{k_3 d} - D\beta_4 e^{k_4 d} = 0 \tag{41}$$

$$[A_1 - A_2]e^{-k_1 d} + [B_1 - B_2]e^{-k_2 d} + C_1 e^{k_3 d} + D_1 e^{k_4 d} = 0 \tag{42}$$

$$[A_1 - A_2]\eta_1 e^{-k_1 d} + [B_1 - B_2]\eta_2 e^{-k_2 d} + C_1\eta_3 e^{k_3 d} + D_1\eta_4 e^{k_4 d} = 0 \tag{43}$$

$$[A_1 - A_2]\theta_1 e^{-k_1 d} + [B_1 - B_2]\theta_2 e^{-k_2 d} + C_1\theta_3 e^{k_3 d} + D_1\theta_4 e^{k_4 d} = \frac{\tilde{p}(\xi)}{G(d)} \tag{44}$$

where

$$\beta_i = \frac{k_i}{\xi}; \quad \eta_i = \xi + \frac{k_i^2}{\xi}; \quad i = 1,2,3,4 \tag{45}$$

$$\theta_i = q_i - 2k_i; \quad \theta_{i+2} = q_{i+2} + 2k_{i+2}; \quad i = 1,2$$

$$G(d) = G_0 e^{\lambda d} \tag{46}$$

$$\tilde{p}(\xi) = \int_0^\infty r p(r) J_0(\xi r) dr \tag{47}$$

The substitution of the explicit results for arbitrary functions A_1, B_1, C_1, D_1, A_2 and B_2 in Eqs. (33), (35), (25) and (26) results in explicit solutions for displacements and stresses at an arbitrary point within the domain of the non-homogeneous elastic half-space. The expressions for stresses and displacements involve infinite integrals with integrands decaying exponentially with increasing values of the Hankel transform parameter ξ. This completes the formal analysis of the axisymmetric internal loading of an incompressible elastic half-space region with an exponential variation of the linear elastic shear modulus with depth.

4. Modulus of elasticity

In this paper the half-space is assumed to be elastic and non-homogeneous with a linear elastic shear modulus that varies with depth. The justification for considering elastic non-homogeneity was prompted as a result of experimental evidences related to measurement of elastic properties of geomaterials where depositional effects can give rise to increased elastic stiffness with depth (see e.g. Abbiss, 1979; Atkinson, 1975; Burland & Lord, 1969; Burland, Longworth, & Moore, 1977; Butler, 1974; Cooke & Price, 1973; Costa Filch & Vaughan, 1980; Cripps & Taylor, 1981, 1986; Hobbs, 1974; Hooper & Butler, 1966; Marsland, 1973; Simons & Som, 1969; Simpson, O'Riordan, & Croft, 1979; Skempton & Henkel, 1957; Ward, Marsland, & Samuels, 1965). All the investigations mentioned above show that the modulus of elasticity in soils generally increases with depth.

In order to provide an example application of the variation of the modulus of elasticity, a simple linear fit and an exponential fit shown in Fig. 2 have been completed for the data provided by Burland et al. (1977), who investigated the depth variation of the geotechnical properties of Oxford clay. Burland et al. (1977) showed that the vertical Young's modulus increases with depth, although the variation is not necessarily linear. The rate of increase in the Young's modulus was higher at

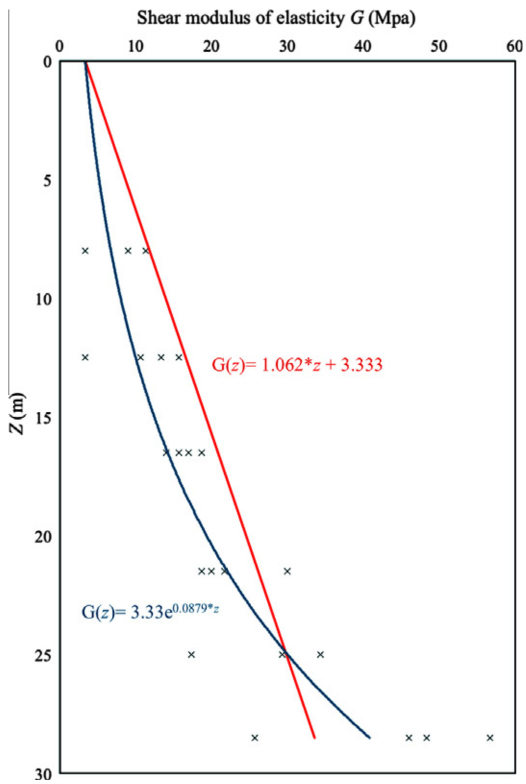

Fig. 2. Linear and exponential fit for the data provided by Burland et al. (1977).

greater depths in Oxford clay. He reported that the undrained (or elastic behavior at a Poisson's ratio of 0.5 and applicable to fluid saturated media) vertical Young's modulus, E_v, increases with depth from 10 MPa to 160 MPa.

$$G = \frac{E_{\text{secant}}}{2(1+\nu)} = \frac{E_{\text{secant}}}{3} \tag{48}$$

$$G(z) = 3.33e^{0.0879*z} \tag{49}$$

$$G(z) = 3.33 + 1.062*z \tag{50}$$

In the above equations, the SI unit of modulus of the elasticity (shear modulus G and Secant Young's modulus E_{secant}) is megaPascals (MPa) and the SI unit of z is meters (m). The numerical results for these two variations are presented in the following sections.

5. Finite element model

A finite element analysis of the analogous problem was also performed using the COMSOL Multiphysics™ software. The axisymmetric half-space region is represented by a finite domain where the outer boundaries extend to 1000 times the radius of the loaded area in both the r- and z-directions. In order to obtain this ratio (l/a = 1000), a calibration was performed between computational results with different (l/a) ratios and known analytical solutions for the classical contact problem (Gladwell, 1980). A mixed $U-P$ formulation was employed in the COMSOL Multiphysics™ software in order to model an incompressible material. Fig. 3a shows the finite element representation of the classical problem of the indentation of the surface of a half-space by a rigid circular disc of radius "a" subjected to an axial load P. As can be seen from Fig. 3b, the analytical and computational results are virtually identical beyond l/a = 1000. The same (l/a) ratio was used to develop computational results for the non-homogeneous case. The computation results are presented in the next section. The analytical–numerical results developed in this paper are used to verify the accuracy of the finite element analysis.

6. Numerical results from analytical solutions and computational estimates

The analysis presented in the previous section leads to explicit infinite integral expressions for the displacement and stress fields within the non-homogeneous but isotropic elastic half-space region subjected to an axisymmetric circular load of constant stress intensity p_0. The integrands of these integrals are such that they cannot be evaluated in explicit forms in terms of familiar special functions. Consequently, results of interest to practical applications of the developments can only be developed through a numerical integration of the infinite integrals. Special numerical procedures for the evaluation of such infinite integrals were developed by Eason, Noble, and Sneddon (1955) and further applications are given by Selvadurai and Rajapakse (1985) and Oliveira, Dumont, and Selvadurai (2012). Since the integrands have oscillatory forms, the accuracy of

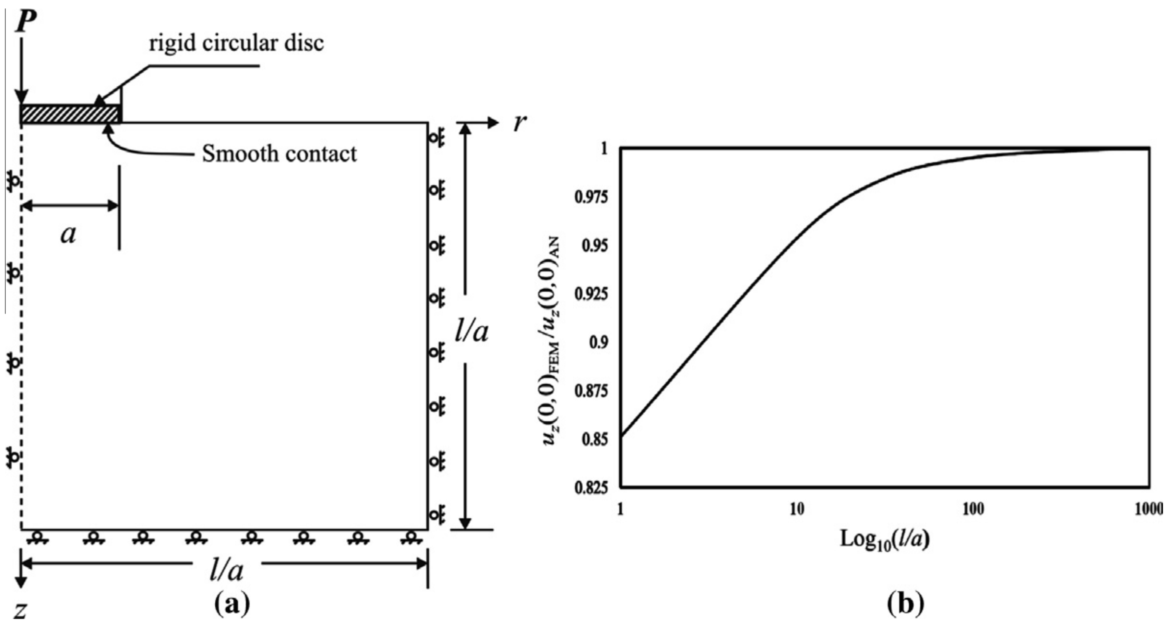

Fig. 3. (a) Finite element representation of the classic problem of the indentation of the surface of a rigid circular disc. (b) Vertical displacement for different l/a.

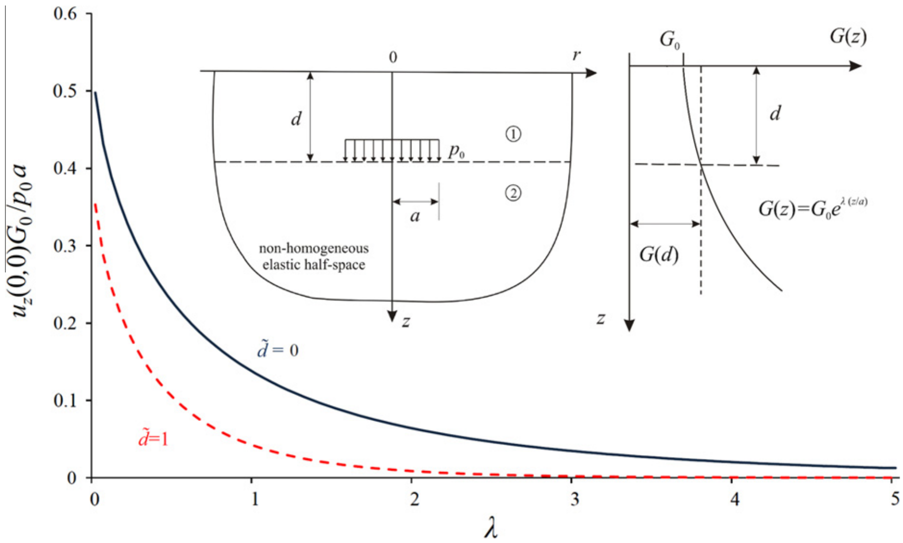

Fig. 4. Variation of the vertical displacement for different λ.

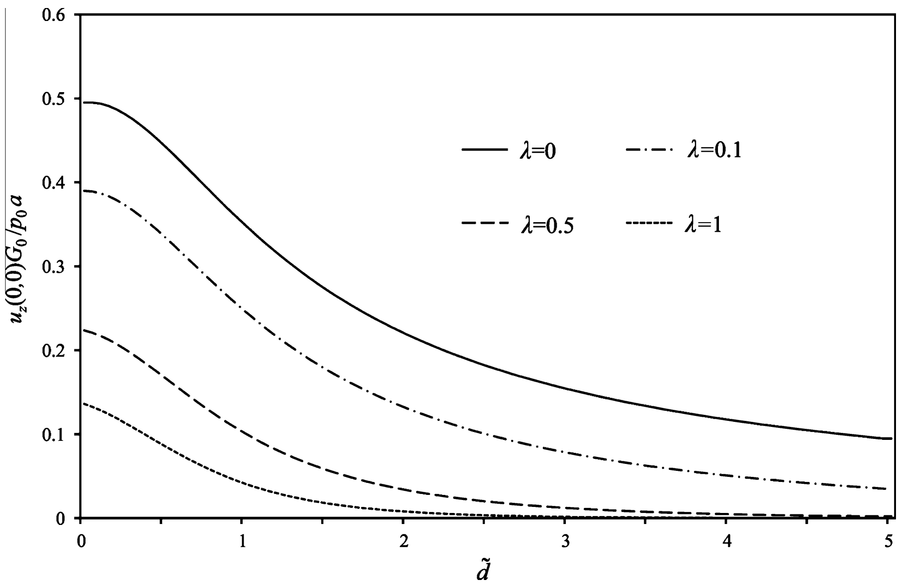

Fig. 5. Variation of the vertical displacement for different depth of loading ($\tilde{d}$).

the numerical evaluation procedure can also be enhanced by using an adaptive numerical quadrature technique and an example of such an application is given by Katebi, Rahimian, Khojasteh, and Pak (2010).For numerical evaluation of the integrals, the upper limit of integration is replaced by a finite value ξ_0; this limit is increased until a convergent result is obtained. In the ensuing, we present numerical results that illustrate the influence of the exponential inhomogeneity on results of engineering interest.

The approach outlined in this section was applied to evaluate the influence of the non-homogeneity on the displacements of a non-homogeneous elastic half-space under uniform internal loading over a circular area. It should be pointed out that all numerical results are presented in non-dimensional forms.

Fig. 4 shows the surface displacement of a non-homogeneous incompressible elastic half-space for different λ, which is directly related to the shear modulus by $G = G_0e^{\lambda z}$. The variations in vertical displacement for different locations of the loading $\tilde{d}(\tilde{d} = d/a)$ are shown in Fig. 5. It is evident that the presence of non-homogeneous conditions has a significant effect on the maximum surface displacement of the half-space.

To provide a better estimate of the relative influence of the elastic non-homogeneity on the displacements of the medium, the ratio of the displacement in a non-homogeneous medium to a homogeneous medium for different values of λ is

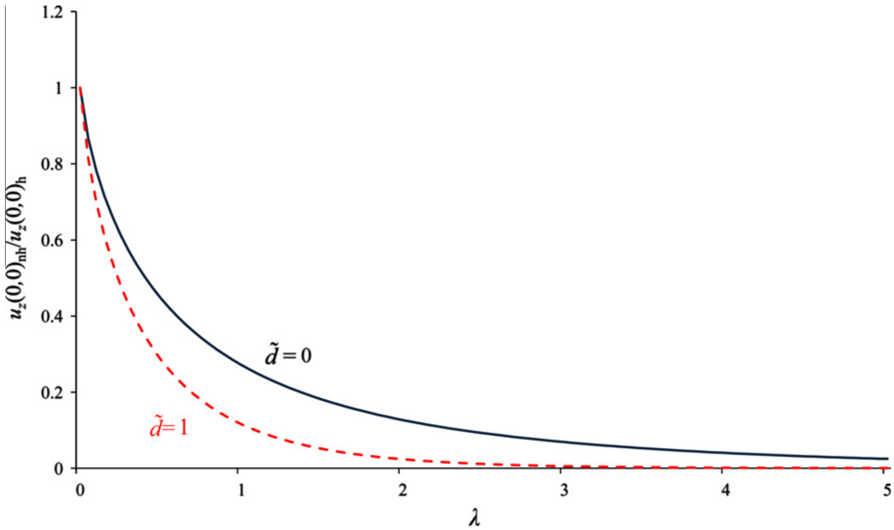

Fig. 6. Ratio of displacement in a non-homogeneous medium to a homogeneous medium for different λ.

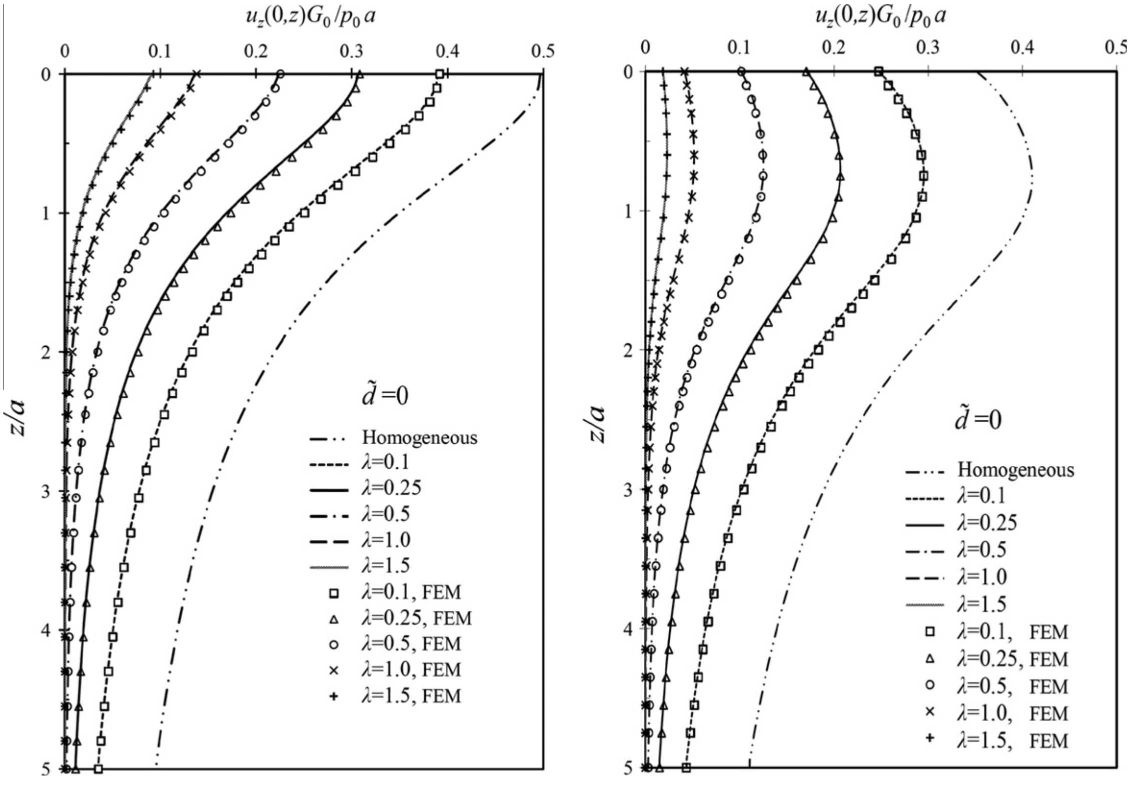

Fig. 7. Variation of the vertical displacement along the z-axis for different λ at depths $\tilde{d} = 0$ and $\tilde{d} = 1$.

illustrated in Fig. 6. Fig. 7 shows the variation of the vertical displacement of a non-homogeneous incompressible half-space in the $z-$ direction for different shear moduli at depths $\tilde{d} = 0$ and $\tilde{d} = 1$ respectively.

The computational results are also indicated by the symbols □, ○, Δ, etc. in Figs. 6 and 7. By comparing the computational results with analytical results, we show that there is anexcellent correlation between the analytical results derived for the exponential variations in the shear modulus, and the computational results (accurate to within 0.3%). This almost negligible difference could have arisen from the idealization of the half-space region by a finite domain. The discrepancies are considered to be well within the range acceptable for engineering application of the results.

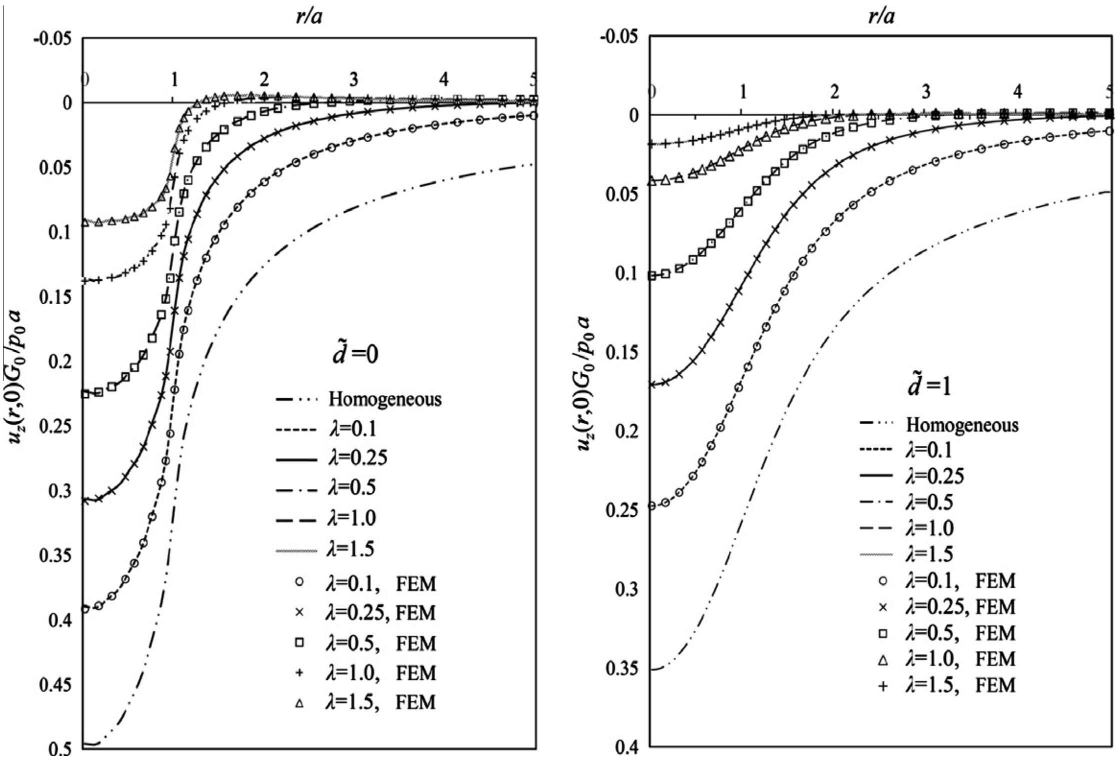

Fig. 8. Variation of the vertical displacement along the r-axis for different λ at depths $\tilde{d} = 0$ and $\tilde{d} = 1$.

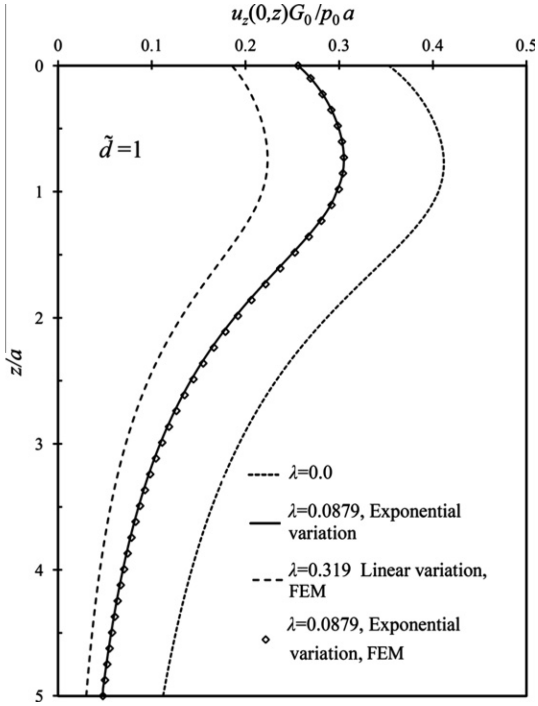

Fig. 9. Variations of vertical displacement along the z-axis for the fitted linear and exponential variation of shear modulus to the data provided by Burland et al. (1977).

Furthermore, the variation of vertical displacement of a non-homogeneous incompressible half-space along the r-axis for different λ at depths $\tilde{d} = 0$ and $\tilde{d} = 1$ are shown in Fig. 8. These Figures illustrate that the response of the medium is

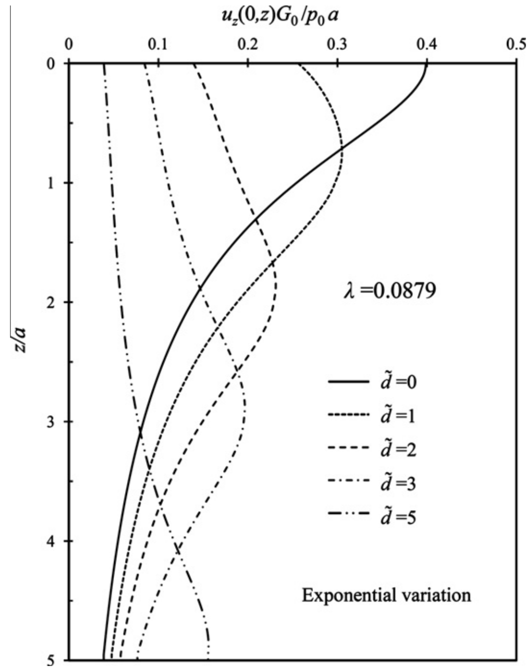

Fig. 10. Variations of vertical displacement along the z-axis for different depths of the loading for the fitted exponential variation of shear modulus to the data provided by Burland et al. (1977).

influenced by the degree of non-homogeneity. As would be expected, the vertical displacement decreases as the shear modulus increases, if all other parameters are kept constant. (i.e. as G increases, the stiffness of the half-space also increases).

The numerical results for the linear and exponential fit mentioned in the pervious section are shown in Fig. 9. For a better understanding of the influence of the degree of non-homogeneity on the response, the variation of vertical displacements along the z-axis has been plotted in Fig. 10 for different depths and diameter of the loading.

7. Concluding remarks

The problem of the interior loading of a non-homogeneous incompressible elastic halfspace can serve as a useful model for examining the interior loading of geologic media with predominantly isochoric deformations. The types of non-homogeneities that can be examined are many and varied and largely governed by experimentally derived variations. The linear variation in the elastic shear modulus has been applied to geologic media such as London Clay deposits and the results have been used in geomechanics literature. The exponential variation in the shear modulus has an advantage in that the governing partial differential equations of elasticity for a non-homogeneous medium are considerably simplified through the use of this approximation. The analysis of the traction boundary value problem related to the interior loading of a non-homogeneous elastic halfspace can be obtained in a form where results of practical interest can be derived through the evaluation of infinite integrals. The study can also be used as a benchmarking solution for examining the accuracy of computational approaches that can ultimately be used to examine more complicated variations of the shear modulus with depth.

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Appendix A

The explicit solutions for the arbitrary functions A_1, B_1, C_1, D_1, A_2 and B_2 can be expressed as follows:

$$A_1 = (e^{-dk_1} f_1 \delta_1 \eta_4)/(\eta_1 \vartheta) - (e^{-dk_1} f_1 \eta_3 \delta_2)/(\eta_1 \vartheta) + \frac{\eta_2}{\eta_1}[(e^{-dk_1} f_1 \delta_2 \gamma_2)/(\gamma_1 \vartheta) - (e^{-dk_1} f_1 \delta_1 \gamma_3)/(\gamma_1 \vartheta)]$$

$$B_1 = -(e^{-dk_1} f_1 \delta_2 \gamma_2)/(\gamma_1 \vartheta) + (e^{-dk_1} f_1 \delta_1 \gamma_3)/(\gamma_1 \vartheta)$$

$$D_1 = -(e^{-dk_1}f_1\delta_1)/\vartheta$$

$$C_1 = (e^{-dk_1}f_1\delta_2)/\vartheta$$

$$\begin{aligned}A_2 = &-(e^{dk_4}f_1\delta_1)/\vartheta + (e^{-dk_1}f_1\delta_1\eta_4)/(\eta_1\vartheta) + (e^{+dk_3}f_1\delta_2)/\vartheta - (e^{-dk_1}f_1\eta_3\delta_2)/(\eta_1\vartheta) - e^{dk_1-dk_2}[(e^{-dk_1}f_1\delta_2\gamma_2)/(\gamma_1\vartheta)\\ &- (e^{-dk_1}f_1\delta_1\gamma_3)/(\gamma_1\vartheta)] + \frac{\eta_2}{\eta_1}[(e^{-dk_1}f_1\delta_2\gamma_2)/(\gamma_1\vartheta) - (e^{-dk_1}f_1\delta_1\gamma_3)/(\gamma_1\vartheta)] + e^{dk_1-dk_2}[-(e^{-dk_1}f_3\delta_1)/\vartheta + (e^{-dk_1}f_2\delta_2)/\vartheta\\ &+ (e^{-dk_1}f_1\delta_2\gamma_2)/(\gamma_1\vartheta) - (e^{-dk_1}f_1\delta_1\gamma_3)/(\gamma_1\vartheta)]\end{aligned}$$

$$B_2 = (e^{-dk_1}f_3\delta_1)/\vartheta - (e^{-dk_1}f_2\delta_2)/\vartheta - (e^{-dk_1}f_1\delta_2\gamma_2)/(\gamma_1\vartheta) + (e^{-dk_1}f_1\delta_1\gamma_3)/(\gamma_1\vartheta)$$

where

$$f_1 = (-\beta_1 e^{-dk_1-dk_2} + \beta_2 e^{-dk_1-dk_2});\quad f_2 = (\beta_1 e^{-dk_1+dk_3} + \beta_3 e^{-dk_1+dk_3});\quad f_3 = (\beta_1 e^{-dk_1+dk_4} + \beta_4 e^{-dk_1+dk_4});$$

$$f_4 = (-\eta_1 e^{-dk_1-dk_2} + \eta_2 e^{-dk_1-dk_2});\quad f_5 = (\eta_1 e^{-dk_1+dk_3} - \eta_3 e^{-dk_1+dk_3});\quad f_6 = (\eta_1 e^{-dk_1+dk_4} - \eta_4 e^{-dk_1+dk_4});$$

$$f_7 = (-\theta_1 e^{-dk_1-dk_2} + \theta_2 e^{-dk_1-dk_2});\quad f_8 = (\theta_1 e^{-dk_1+dk_3} - \theta_3 e^{-dk_1+dk_3});\quad f_9 = (\theta_1 e^{-dk_1+dk_4} - \theta_4 e^{-dk_1+dk_4});$$

$$\delta_1 = (-f_2f_4 + f_1f_5);\quad \delta_2 = (-f_3f_4 + f_1f_6); \gamma_1 = (-\eta_2\theta_1 + \eta_1\theta_2);\quad \gamma_2 = (-\eta_3\theta_1 + \eta_1\theta_3);$$

$$\gamma_3 = (-\eta_4\theta_1 + \eta_1\theta_4);\quad \vartheta = (G/P)(-\delta_2(-f_2f_7 + f_1f_8) + \delta_1(-f_3f_7 + f_1f_9))$$

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