International Journal of Fracture, Vol. 16, No. 4, August 1980

Alphen aan den Rijn, The Netherlands

The penny-shaped crack problem for a finitely deformed incompressible elastic solid

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(Received December 15, 1978; in revised form December 14, 1979)

ABSTRACT
In this paper the theory of small deformations superposed on large is used to examine the axisymmetric problem of a penny-shaped crack located in an incompressible elastic infinite solid which is subjected to a uniform finite radial stretch. The small axisymmetric deformations are due to a uniform stress applied in the axial direction. Formal integral expressions are derived for the displacements and stresses in the elastic solid. An exact expression is developed for critical stress necessary for the propagation of a penny-shaped crack in a finitely deformed elastic solid.

1. Introduction

Theories of incremental deformations are concerned with the infinitesimal deformation of an elastic medium which is subjected to a known finite deformation. Such incremental theories have been proposed, among others, by Trefftz [1], Neuber [2], Biot [3] and Green *et al.* [4]. The general theory of small deformations superposed on large developed by Green *et al.* [4] has received wide application in the analysis of this class of problems. This theory provides exact solutions for problems of infinitesimal strains superimposed on an assigned and exact large initial deformation. Comprehensive accounts of the method of Green *et al.* [4] together with references to further studies which involve small deformations superposed on large are given by Green and Zerna [5] and Eringen and Suhubi [6].

In this paper we apply the general theory of Green *et al.* [4] to the analysis of a penny-shaped crack in a finitely deformed elastic solid. An infinite elastic solid contains a penny-shaped crack which occupies the region $z = 0; r \leq a_0$. The infinite elastic medium is subjected to a uniform finite radial deformation. The superposed small deformation consists of a uniform stress which is applied in the axial direction. The analysis of the penny-shaped crack problem employs a Hankel transform formulation which reduces the problem to a pair of dual integral equations. Explicit solutions for these dual integral equations are obtained by using the general results of Sneddon [7, 8]. Formal integral results presented for the incremental displacement and stress fields are in a form applicable to an incompressible elastic medium with an arbitrary form of the strain energy function. The critical superposed axial stress necessary for the propagation of the penny-shaped crack in the finitely deformed medium is evaluated in exact closed form. This critical stress is found to be dependent on the magnitude of the finite radial stretch. Therefore in contrast to the classical elasticity problem, the presence of a large radial stress influences the critical axial stress necessary for the propagation of the penny-shaped crack.

2. Notation and formulae

We consider an incompressible elastic medium of infinite extent containing a penny-shaped crack occupying the region $z = 0, r \leq a_0$ (Fig. 1). The medium is subjected to a finite radial stretch μ, maintaining the axial stress $\tau^{33} = 0$. For this particular finite deformation problem, the non-zero components of the contravariant stress tensor τ^{ij} are

$$\tau^{11} = r^2\tau^{22} = \left\{\mu^2 - \frac{1}{\mu^4}\right\}(\Phi + \mu^2\Psi) \tag{1}$$

where

$$\Phi = \frac{2\partial W}{\partial I_1}; \quad \Psi = \frac{2\partial W}{\partial I_2} \tag{2}$$

and $W = W(I_1, I_2)$ is the strain energy function for the incompressible elastic material.

We now superpose on the finitely deformed infinite medium a further infinitesimal axially symmetric deformation characterized by the following displacement field:

$$u_r = u(r, z); \quad u_\theta = 0; \quad u_z = w(r, z). \tag{3}$$

The components of the stress tensor τ'^{ij} governing the superposed deformation can be expressed as (see Green and Zerna [5])

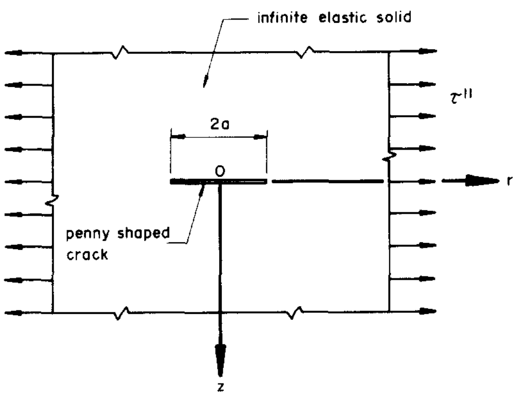

Fig. 1. The geometry of the penny-shaped crack

$$\tau'^{11} = p' + \alpha_1\frac{\partial w}{\partial z} + \alpha_2\frac{u}{r}; \quad r^2\tau'^{22} = p' + \alpha_1\frac{\partial w}{\partial z} + \alpha_2\frac{\partial u}{\partial r}$$

$$\tau'^{33} = p' + \alpha_3\frac{\partial w}{\partial z}; \quad \tau'^{13} = \alpha_4\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) \tag{4}$$

where $\alpha_i (i = 1, 2, 3, 4)$ are defined in Appendix 1.

As has been pointed out by Green *et al.* [4] the solution of the requisite equations governing the superposed deformation is facilitated by the use of potential function techniques which have been developed for the analysis of the classical problem in anisotropic elasticity. Briefly, the solution to the superposed displacement (3) can be expressed in terms of two functions $\varphi_n (n = 1, 2)$ such that

$$u = \frac{\partial}{\partial r}(\varphi_1 + \varphi_2); \quad w = \frac{\partial}{\partial z}(k_1\varphi_1 + k_2\varphi_2) \tag{5}$$

where k_1 and k_2 are the roots of the equation

$$k^2\{\alpha_4+\tau^{11}\}+k\{\alpha_1-\alpha_3+2\alpha_4-\tau^{11}\}+\alpha_4=0. \tag{6}$$

The functions φ_n are solutions of the equations

$$\left\{\frac{\partial^2\varphi_n}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_n}{\partial r}+k_n\frac{\partial^2\varphi_n}{\partial z^2}\right\}=0 \quad (n=1,2). \tag{7}$$

For the analysis of the penny-shaped crack problem we employ a Hankel transform development. The nth-order Hankel transform of $\varphi_i(r,z)(i=1,2)$ is defined by

$$\bar{\varphi}_i^n(\xi,z)=\mathscr{H}_n\{\varphi_i(r,z);\xi\}=\int_0^\infty r\varphi_i(r,z)J_n(\xi r/a)\,\mathrm{d}r \tag{8a}$$

and the length parameter 'a' is taken as the radius of the penny-shaped crack in its finitely deformed state (i.e. $a=\mu a_0$). The corresponding Hankel inversion theorem is

$$\varphi_i(r,z)=\mathscr{H}_n^{-1}\{\bar{\varphi}_i^n(\xi,z);r\}=\frac{1}{a^2}\int_0^\infty \xi\bar{\varphi}_i^n(\xi,z)\,J_n(\xi r/a)\mathrm{d}\xi. \tag{8b}$$

Since the penny-shaped crack problem will be formulated in relation to a halfspace region ($z\geq 0$), the relevant solutions of (7) are given by

$$[\varphi_1(r,z);\varphi_2(r,z)]=\frac{1}{a^2}\int_0^\infty \xi\,[\bar{A}(\xi)\mathrm{e}^{-(\xi z/a\sqrt{k_1})};\bar{B}(\xi)\mathrm{e}^{-(\xi z/a\sqrt{k_2})}]\,J_0(\xi r/a)\mathrm{d}\xi \tag{9}$$

where $\bar{A}(\xi)$ and $\bar{B}(\xi)$ are arbitrary functions.

The isotropic stress p' can be expressed in terms of $\varphi_i(r,z)$ in the forms

$$p'=\begin{cases}\{-\alpha_4-k_1(\alpha_1+\alpha_4-\tau^{11})\}\dfrac{\partial^2\varphi_1}{\partial z^2}+\{-k_2(\alpha_3-\alpha_4)+k_2^2(\alpha_4+\tau^{11})\}\dfrac{\partial^2\varphi_2}{\partial z^2}.\\ \{-k_1(\alpha_3-\alpha_4)+k_1^2(\alpha_4+\tau^{11})\dfrac{\partial^2\varphi_1}{\partial z^2}+\{-\alpha_4-k_2(\alpha_1+\alpha_4-\tau^{11})\}\dfrac{\partial^2\varphi_2}{\partial z^2}\end{cases} \tag{10}$$

3. The penny-shaped crack problem

We first consider the problem in which the penny-shaped crack contained in the finitely deformed incompressible elastic infinite medium is deformed by an internal pressure $g(r)$ acting on the surface of the crack. The effect of the uniform axial tensile stress applied to a finitely deformed medium containing a penny-shaped crack with stress free surfaces can be deduced from the above solution. We observe that in the case of the internally loaded crack problem, the distribution of stress in the neighbourhood of the crack is the same as that produced for an elastic halfspace region ($z\geq 0$), the surface of which is subjected to the mixed boundary conditions

$$\tau'^{13}(r,0)=0;\qquad 0\leq r\leq\infty \tag{11a}$$

$$\tau'^{33}(r,0)=-g(r);\qquad r\leq a \tag{11b}$$

$$w(r,0)=0;\qquad a\leq r\leq\infty \tag{11c}$$

To satisfy the traction boundary condition (11a) we require

$$\bar{A}(\xi)=-\sqrt{\frac{k_1}{k_2}}\left\{\frac{1+k_2}{1+k_1}\right\}\bar{B}(\xi). \tag{12}$$

It can be shown that the remaining boundary conditions yield the pair of dual integral equations

$$\frac{\theta}{a^4}\int_0^\infty \xi^3\,\tilde{B}(\xi)J_0(\xi r/a)\,\mathrm{d}\xi = -g(r);\quad r\leq a$$
$$\int_0^\infty \xi^2\,\tilde{B}(\xi)J_0(\xi r/a)\,\mathrm{d}\xi = 0;\quad r\geq a \tag{13}$$

where

$$\theta = \sqrt{\frac{k_1}{k_2}}\left\{\frac{1+k_2}{1+k_1}\right\}\left\{\frac{\alpha_4}{k_1} - \alpha_3 + \alpha_1 + \alpha_4 - \tau^{11}\right\} + \alpha_4 + k_2\{\alpha_4 + \tau^{11}\}. \tag{14}$$

The solution of the dual integral equations (13) is given by (see Sneddon [8] and Sneddon and Lowengrub [9])

$$\tilde{B}(\xi) = -\frac{2a^4}{\theta\pi\xi^2}\int_0^1 \chi\sin\chi\xi\,\mathrm{d}\chi\int_0^1 \frac{tg(t\chi)\mathrm{d}t}{\sqrt{1-t^2}} \tag{15}$$

The displacements and stresses in the halfspace region due to specific forms of the superposed stress $g(r)$ can be determined by making use of the explicit forms of $\varphi_i(r, z)$ and the relationships (4) and (5). In the particular instance when the halfspace region is subjected to a uniform state of stress T over the region $r \leq a$, (15) reduces to

$$\tilde{B}(\xi) = \frac{2Ta^4}{\xi^2\pi\theta}\left\{\frac{\cos\xi}{\xi} - \frac{\sin\xi}{\xi^2}\right\} \tag{16}$$

Formal integral expressions for the corresponding displacements in the halfspace region are given by

$$u(r,z) = \frac{2Ta}{\pi\theta}\left[-\sqrt{\frac{k_1}{k_2}}\left(\frac{1+k_2}{1+k_1}\right)\left\{\frac{z}{a\sqrt{k_1}}\,{}_1S_0^1 + {}_1S_{-1}^1 - \frac{r}{a}\,{}_1S_0^0\right\} + \left\{\frac{z}{a\sqrt{k_2}}\,{}_2S_0^1 + {}_2S_{-1}^1 - \frac{r}{a}\,{}_2S_0^0\right\}\right] \tag{17a}$$

$$w(r,z) = \frac{2Ta}{\pi\theta}\left[\sqrt{\frac{k_1}{k_2}}\left(\frac{1+k_2}{1+k_1}\right)\left\{-1 + \frac{z}{a\sqrt{k_1}}\,{}_1S_0^0 + \frac{r}{a}\,{}_1S_0^1\right\} - \sqrt{k_2}\left\{-1 + \frac{z}{a\sqrt{k_2}}\,{}_2S_0^0 + \frac{r}{a}\,{}_2S_0^1\right\}\right] \tag{17b}$$

where

$${}_jS_n^m = \int_0^\infty \xi^{n-1}\,\mathrm{e}^{-(\xi z/a\sqrt{k_j})}J_m(\xi r/a)\sin\xi\,\mathrm{d}\xi. \tag{18}$$

Integral expressions similar to (17) can be developed for the stress components τ'^{ij} by making use of the result (16) and the expressions (4), (5) and (9). For example, the stress components τ'^{13} and τ'^{33} take the forms

$$\tau'^{13} = \frac{2T\alpha_4(1+k_2)}{\pi\theta\sqrt{k_2}}\int_0^\infty\left\{\cos\xi - \frac{\sin\xi}{\xi}\right\}\{\mathrm{e}^{-(\xi z/a\sqrt{k_1})} - \mathrm{e}^{-(\xi z/a\sqrt{k_2})}\}J_1(\xi r/a)\,\mathrm{d}\xi \tag{19a}$$

and

$$\tau'^{33} = \frac{2T\alpha_4}{\pi\theta}\int_0^\infty\left[\sqrt{\frac{k_1}{k_2}}\left(\frac{1+k_2}{1+k_1}\right)\left\{\frac{1}{k_1} + \frac{\alpha_1}{\alpha_4} - \frac{\tau^{11}}{\alpha_4} - \frac{\alpha_3}{\alpha_4} + 1\right\}\mathrm{e}^{-(\xi z/a\sqrt{k_1})} + \right.$$
$$\left. + \left\{1 + k_2\left(1 + \frac{\tau^{11}}{\alpha_4}\right)\right\}\mathrm{e}^{-(\xi z/a\sqrt{k_2})}\right]\left\{\cos\xi - \frac{\sin\xi}{\xi}\right\}J_0(\xi r/a)\,\mathrm{d}\xi \tag{19b}$$

respectively.

4. Condition for rupture

We now consider the problem relating to a finitely deformed incompressible elastic medium containing a penny-shaped crack, the surfaces of which are free from surface traction. The superposed incremental state of stress corresponds to

$$\tau'^{11} = r^2 \tau'^{22} = 0; \quad \tau'^{33} = T \tag{20}$$

Therefore, the combination of the solution derived in section 3 for the crack subjected to a uniform internal pressure with (20) yields an incremental state of stress τ'^{ij} which corresponds to the axial loading of a penny-shaped crack with traction free surfaces.

Using the result (17b) it can be shown that the normal component of the displacement of the crack surface ($z = 0^+$); $r \leq a$ is given by

$$w(r, 0) = \frac{2T}{\pi\theta}\left\{\frac{k_2(1+k_1) - \sqrt{k_1}(1+k_2)}{\sqrt{k_2}(1+k_1)}\right\}\sqrt{a^2 - r^2} \tag{21}$$

The presence of a circular penny-shaped crack lowers the potential energy of the medium by an amount

$$V = \frac{4}{3\theta}\left[\frac{k_2(1+k_1) - \sqrt{k_1}(1+k_2)}{\sqrt{k_2}(1+k_1)}\right] T^2 a^3 \tag{22}$$

The surface energy of the crack

$$U = 2\pi a^2 S \tag{23}$$

where S denotes the surface tension of the incompressible elastic material. The condition that the penny-shaped crack of radius a may just begin to spread requires that

$$\frac{\partial}{\partial a}\{V - U\} = 0 \tag{24}$$

From the condition (24) we observe that the crack will become unstable and propagate if T exceeds the critical value

$$T = \left[\frac{\pi\theta S\sqrt{k_2}(1+k_1)}{a\{k_2(1+k_1) - \sqrt{k_1}\,(1+k_2)\}}\right]^{1/2} \tag{25}$$

The above result extends the rupture condition for the classical penny-shaped crack located in an incompressible elastic medium, to include the effect of an initial finite deformation. This general result places no restriction on the form of the strain energy function $W(I_1, I_2)$.

In the ensuing we develop the result for the critical stress (25) applicable for an incompressible elastic material with a strain energy function of the Mooney-Rivlin type

$$W = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{26}$$

where C_1 and C_2 are constants.

In this particular case $k_1 = 1$ and $k_2 = 1/\mu^6$ and (25) gives

$$T = \left[\frac{2\pi(C_1 + \mu^2 C_2)\{\mu^9 + \mu^6 + 3\mu^3 - 1\}\, S}{a_0\mu^5(1+\mu^3)}\right]^{1/2} \tag{27}$$

Taking the limit of (27) as μ approaches unity we obtain

$$T = \left[\frac{2\pi GS}{a_0}\right]^{1/2} \tag{28}$$

where $G\{=2(C_1+C_2)\}$ is the linear elastic shear modulus of the incompressible elastic material. The result (28) is in agreement with the classical result for the critical rupture stress [8]. The influence of the initial finite deformation on the critical rupture stress can be examined by comparing $T(\mu)$ defined by (27) with the result $T(1)$ applicable for the unstressed incompressible elastic medium. We have

$$\frac{T(\mu)}{T(1)} = \left[\frac{(1+\mu^2\Gamma)(\mu^9+\mu^6+3\mu^3-1)}{2\mu^5(1+\Gamma)(1+\mu^3)}\right]^{1/2} \tag{29}$$

where $\Gamma = C_2/C_1$. The result $\Gamma = 0$, corresponds to the Neo-Hookean material.

The expression (29) for the critical rupture stress of a penny-shaped crack in a finitely deformed medium contains the term $(\mu^9+\mu^6+3\mu^3-1)$ in the numerator. Thus the critical stress becomes zero as μ approaches a value which is approximately 2/3. This result indicates that when the incompressible elastic medium is acted upon by a finite radial compression, the plane faces of the crack may experience instability. This critical value of μ is identical to that obtained by Green *et al.* [4] and others [10–12] in connection with the occurrence of instability in an incompressible elastic halfspace subjected to radial compression. The variation of $\{T(\mu)/T(1)\}$ with μ is graphically illustrated in Fig. 2. These results indicate that the application of a finite radial stretch increases the critical axial stress required for propagation of the penny-shaped crack.

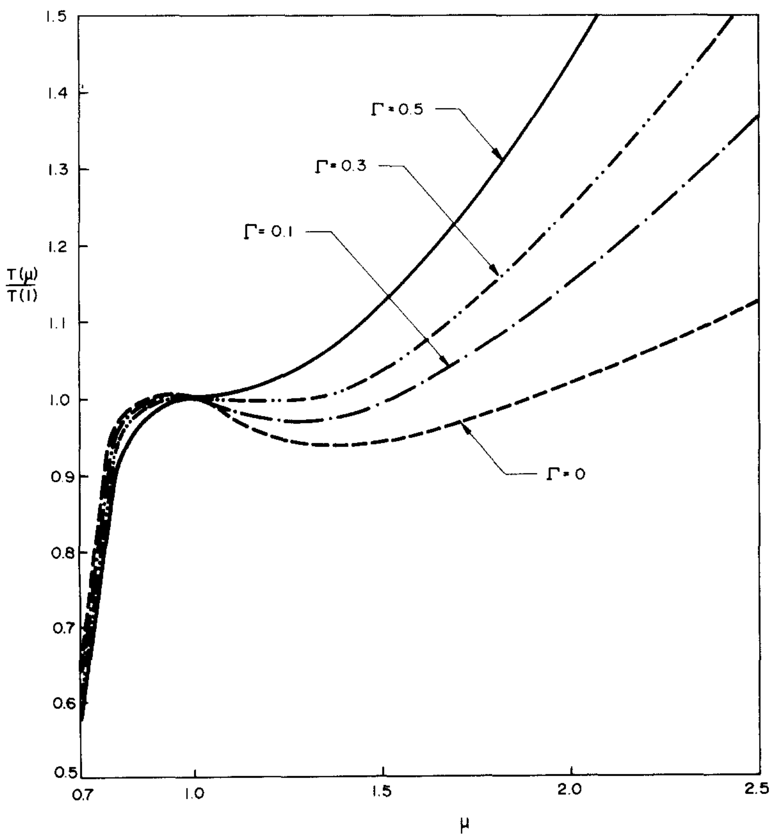

Fig. 2. The variation in the rupture stress of the penny-shaped crack with the initial finite deformation

In contrast, the application of an infinitesimal radial stress leaves the critical axial rupture stress unaltered [8].

Appendix 1

The constants $\alpha_i (i = 1, 2, 3, 4)$ are given by

$$\alpha_1 = \zeta_1 - \frac{2c}{\mu^4}; \quad \alpha_2 = \frac{2}{\mu^4}(\Phi + \mu^8\Psi) - \frac{4c}{\mu^4}; \quad \alpha_3 = \zeta_2 + \frac{2c}{\mu^4}; \alpha_4 = \frac{c}{\mu^4}$$

where

$$c = (\Phi + \mu^2\Psi)$$

$$\zeta_1 = 2A\left(\frac{1}{\mu^2} - \mu^4\right) + 2B\left(\frac{1}{\mu^4} - \mu^8\right) + 2F\left(1 + \frac{1}{\mu^6} - 2\mu^6\right)$$

$$\zeta_2 = 2\left(\frac{1}{\mu^2} - \mu^4\right)\left(\frac{A}{\mu^6} + \frac{2B}{\mu^2} + \frac{3F}{\mu^4}\right)$$

$$A = 2\frac{\partial^2 W}{\partial I_1^2}; \quad B = 2\frac{\partial^2 W}{\partial I_2^2}; \quad F = 2\frac{\partial^2 W}{\partial I_1 \partial I_2}.$$

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RÉSUMÉ

Dans le mémoire, on utilise la théorie des petites déformations superposées à de larges déformations pour examiner le problème axisymétrique d'une fissure en disque noyée dans un solide élastique infini incompressible soumis à un étirement uniforme fini radial. Les déformations axisymétriques de faible amplitude sont dues à une contrainte uniforme appliquée suivant la direction axiale. Des expressions intégrales formelles sont déduites des déplacements et des contraintes dans le solide élastique. Une expression exacte relative à la contrainte critique nécessaire pour la propagation d'une fissure en forme de disque est développée dans le cas d'un solide élastique déformé de manière finie.