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Boundary element formulation of axisymmetric problems for an elastic halfspace

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ABSTRACT

Axisymmetric problems for an elastic halfspace are commonly analyzed by the boundary element (BE) method by employing the axisymmetric fundamental solution for the fullspace. In such cases, the discretization of the free surface is required, with its truncation at an appropriate location from the axis of symmetry. This paper presents the BE implementation of the axisymmetric fundamental solution for an elastic halfspace, given in terms of integrals of the Lipschitz–Hankel type, that satisfies in advance the boundary condition of zero traction on the free surface and the decay of displacements in the far field. Explicit equations for post-processing the results at internal points are provided, as well as adequate numerical schemes to evaluate the boundary integrals arising in the method. This formulation can be easily implemented in existing BE computational codes for axisymmetric fullspace problems, requiring only a few modifications. Numerical results are provided to validate the proposed formulation.

1. Introduction

The axisymmetric formulation in classical elasticity is useful for the analysis of problems in geomechanics [1,2], as well as contact problems for cylinders, spheres and circular plates [3–8]. Other applications involve the study of fracture mechanics phenomena and inclusions [5,9–11].

In particular, the BE method is advantageous for axisymmetric problems, since it reduces the analysis of the three-dimensional domain to a one-dimensional mesh discretization requiring only the evaluation of linear integrals. However, the fundamental solutions involved are more complex, requiring special considerations on their manipulation and integration to correctly evaluate the influence coefficients arising in the boundary integral equations. Extensive surveys on the existing axisymmetric fundamental solutions are given by Wang and Liao [12,13], Wang et al. [14] and Wideberg and Benitez [15].

The BE method for axisymmetric elasticity was first formulated by Cruse et al. [16], using the fullspace fundamental solution derived by Kermanidis [17]. Several contributions to the formulation of the axisymmetric problem may be cited, such as the expansion of non-symmetric boundary conditions by Fourier series suggested by Mayr [18] and Rizzo and Shippy [19,20], and the assessment of body forces by means of particular integrals incorporated by Park [21]. Also, axisymmetric formulations have been developed for transverse isotropy [22], thermoelasticity [23], elastoplasticity [24] and viscoplasticity [25]. In elastodynamics, the works by Wang and Banerjee [26,27], Tsinopoulos et al. [28] and Yang and Zhou [29] in the frequency domain should be mentioned. The method has also been successfully applied to contact problems [30] and fracture mechanics [31].

For axisymmetric halfspace problems, the BE formulation employed with the fullspace fundamental solution requires the discretization of the infinite free surface. In this case, the surface must be truncated at a reasonable distance from the axis of symmetry and the region of interest [32]. The disadvantage of such a scheme is that a large number of boundary elements is needed to model the remote boundary satisfactorily, so that relative displacements in particular can be accurately evaluated.

An alternative way to deal with this problem is to use infinite boundary elements, as suggested by Watson [33]. These infinite elements, which simulate the decay of the displacements and stresses in the far field, are mapped onto a finite region in terms of an intrinsic coordinate system to facilitate the integration. A variety of infinite elements can be found in the literature for three-dimensional elasticity, depending on the mapping scheme used and the application [34–36]. However, such elements are not available for problems with axisymmetry, probably because treating the integration of the singular kernels over the mapped

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infinite elements is not straightforward for the fullspace fundamental solution. Therefore, Kelvin's fundamental solution is usually employed together with the available three-dimensional surface infinite elements for axisymmetric applications in the halfspace [37–39], thus requiring the boundary surfaces to be discretized.

Another way to treat this problem is to implement the fundamental solution that satisfies in advance the traction free boundary condition on the free surface, which circumvents its numerical discretization. In elasticity, this approach was used by Telles and Brebbia [40] and Dumir and Mehta [41] to examine problems for an isotropic and orthotropic halfplane, respectively.

This work presents a BE formulation for axisymmetric elasticity problems for a halfspace [42] that makes use of the fundamental solutions due to radial and axial ring loads embedded in a halfspace derived by Hasegawa [43,44]. The resulting equations could be manipulated by expressing the fundamental solutions in terms of Lipschitz–Hankel integrals, as adopted by Selvadurai and Rajapakse [5] using extensions to the solutions developed by Mindlin [45] and Mindlin and Cheng [46]. Since the terms of the fullspace fundamental solution can be identified as constituents of the halfspace fundamental solution, the proposed formulation can be implemented by introducing only a few modifications in existing axisymmetric computational codes. Explicit equations are presented for expressing results at internal points as well as appropriate numerical schemes to accurately evaluate the integrals arising in the formulation. Problems related to torsional loads, not addressed in this work, involve simpler fundamental solutions and can be examined in a similar manner.

Section 2 of this paper introduces the axisymmetric fundamental solution for the elastic fullspace and an elastic halfspace. Section 3 presents the axisymmetric BE formulation, followed by Section 4 that deals with the numerical integration. Finally, Section 5 illustrates numerical examples that validate the proposed formulation.

2. Axisymmetric fundamental solution

The axisymmetric fundamental solution for elasticity consists of displacements $u^*_{ij}(\mathrm{P,Q})$ and stresses $\sigma^*_{ijk}(\mathrm{P,Q})$ due to ring loads in the i-direction applied at $\mathrm{P}(\xi,z')$ and centered in the z-axis. The continuum has shear modulus μ and Poisson's ratio ν. The solutions are given in the cylindrical coordinate system (r,z). The indices j and k stand for the displacement and stress components measured at $\mathrm{Q}(r,z)$.

For the fullspace, displacements due to ring loads were first derived by Kermanidis [17], by applying Betti's theorem to the Papkovich–Neuber solution [47] for an elastic medium of infinite extent. Subsequently, Cruse et al. [16] and Bakr and Fenner [23] solved Navier's equilibrium equations by expressing the displacements as Galerkin vectors [47] and considering ring loads as body forces. Also, Shippy et al. [48] integrated Kelvin's solution [47] for the three-dimensional infinite medium along a circular path centered on the axis of symmetry.

For the halfspace, Hasegawa [43,44] deduced displacements and stresses from stress functions [49] obtained by means of Fourier and Hankel transforms and considering ring loads as body forces. Later, Selvadurai and Rajapakse [5] imposed boundary conditions and continuity conditions to displacements and stresses expressed by Muki's solution [50,51] and arrived at the same solutions. These solutions were also obtained by Hanson and Wang [52] as a particular case of the problem for the medium with transverse isotropy.

Both axisymmetric fundamental solutions for fullspace and halfspace can be expressed by means of either integrals of the Lipschitz–Hankel type involving products of Bessel functions [53], or complete elliptic integrals of the first and second types [54], or Legendre functions [54]. In this work, the approach presented by Selvadurai and Rajapakse [5] is adopted. Expressions are written in terms of integrals of the Lipschitz–Hankel type [53]

$$I_{pq\lambda}(\xi,r;c)=\int_0^\infty J_p(\xi t)J_q(rt)e^{-ct}t^\lambda\,\mathrm{d}t \tag{1}$$

in which p, q and λ are integers, $J_p(\xi t)$ and $J_q(rt)$ are Bessel functions of the first kind of order p and q, respectively. The integrals arising in the axisymmetric fundamental solutions are convergent [53] and their closed form expressions in terms of complete elliptic integrals of the first, second and third kinds [54] are listed in Appendix A.

2.1. Ring loads in an elastic fullspace

The fundamental solution can be derived from Muki's solution [50,51] of the Navier equilibrium equations for an elastic isotropic medium,

$$(1-2\nu)\left(\nabla^2u_r-\frac{u_r}{r^2}\right)+\Delta_{,r}=0 \tag{2}$$

$$(1-2\nu)\nabla^2u_z+\Delta_{,z}=0 \tag{3}$$

where

$$\Delta=u_{r,r}+\frac{u_r}{r}+u_{z,z} \tag{4}$$

Muki represented displacements by means of harmonic and biharmonic functions and used Hankel transforms and their correspondence to generalized Fourier–Bessel transforms to arrive at a general asymmetric solution. This solution can be specialized for axisymmetry, leading to

$$u_r=\frac{1}{2}\int_0^\infty\left(\frac{\mathrm{d}G}{\mathrm{d}z}+2H\right)[J_1(rt)-J_{-1}(rt)]t^2\,\mathrm{d}t \tag{5}$$

$$u_z=\int_0^\infty\left[(1-2\nu)\frac{\mathrm{d}^2G}{\mathrm{d}z^2}-2(1-\nu)t^2G\right]J_0(rt)\,\mathrm{d}t \tag{6}$$

where

$$G(t,z)=(A+Bz)e^{zt}+(C+Dz)e^{-zt} \tag{7}$$

$$H(t,z)=Ee^{zt}+Fe^{-zt} \tag{8}$$

in which $A(t),B(t),\ldots,F(t)$ are unknown functions.

Consider a fullspace split into two parts, I and II, by a plane normal to z at $z=z'$ as shown in Fig. 1. Applying Eqs. (5) and (6) and the regularity conditions for the displacements and stresses as $z\to\pm\infty$,

$$u_i^{\mathrm{I,II}}(r,\pm\infty)=0,\quad \sigma_{ij}^{\mathrm{I,II}}(r,\pm\infty)=0 \tag{9}$$

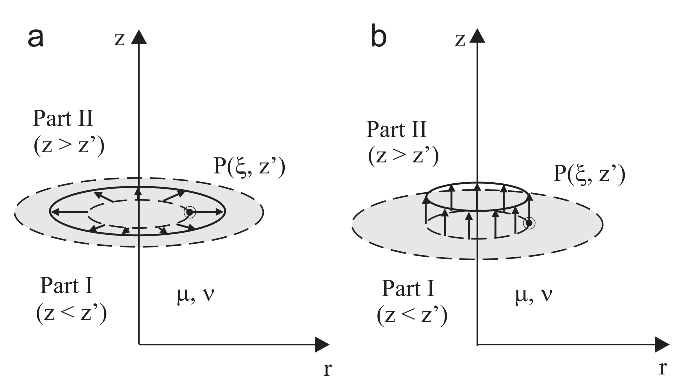

Fig. 1. Ring loads in the elastic fullspace: (a) radial direction; (b) axial direction.

the number of unknown functions can be reduced from 12 to 6. These functions can be determined using (i) the compatibility conditions of displacements at the interface

$$u_i^{\mathrm{I}}(r,z') = u_i^{\mathrm{II}}(r,z') \tag{10}$$

and (ii) the equilibrium conditions for radial and vertical unit ring loads applied at (ξ,z')

$$\sigma_{rr}^{\mathrm{I}}(r,z') - \sigma_{rr}^{\mathrm{II}}(r,z') = \frac{1}{2\pi\xi}\delta(r-\xi) = \frac{1}{2\pi\xi}\int_0^\infty J_1(\xi t)J_1(rt)\xi t\,\mathrm{d}t \tag{11}$$

$$\sigma_{rz}^{\mathrm{I}}(r,z') - \sigma_{rz}^{\mathrm{II}}(r,z') = 0 \tag{12}$$

and

$$\sigma_{zr}^{\mathrm{I}}(r,z') - \sigma_{zr}^{\mathrm{II}}(r,z') = 0 \tag{13}$$

$$\sigma_{zz}^{\mathrm{I}}(r,z') - \sigma_{zz}^{\mathrm{II}}(r,z') = \frac{1}{2\pi\xi}\delta(r-\xi) = \frac{1}{2\pi\xi}\int_0^\infty J_0(\xi t)J_0(rt)\xi t\,\mathrm{d}t \tag{14}$$

where δ is the Dirac delta function [55].

The final expressions for displacements $u_i^{\mathrm{I}}(r,z)$ and $u_i^{\mathrm{II}}(r,z)$ can be combined, leading to the following equations for displacements $u_{ij}^{*\mathrm{f}}(\mathrm{P,Q})$:

$$u_{rr}^{*\mathrm{f}} = \frac{1}{16\pi\mu(1-\nu)}\{(3-4\nu)\bar{I}_{110} - |\bar{z}|\,\bar{I}_{111}\} \tag{15}$$

$$u_{rz}^{*\mathrm{f}} = \frac{1}{16\pi\mu(1-\nu)}\bar{z}\bar{I}_{101} \tag{16}$$

$$u_{zr}^{*\mathrm{f}} = -\frac{1}{16\pi\mu(1-\nu)}\bar{z}\bar{I}_{011} \tag{17}$$

$$u_{zz}^{*\mathrm{f}} = \frac{1}{16\pi\mu(1-\nu)}\{(3-4\nu)\bar{I}_{000} + |\bar{z}|\bar{I}_{001}\} \tag{18}$$

where

$$\bar{z} = z'-z \quad \text{and} \quad \bar{I}_{pq\lambda} = I_{pq\lambda}(\xi,r;c=|\bar{z}|) \tag{19}$$

and the superscript f stands for the fullspace fundamental solution.

Considering the constitutive equations in cylindrical coordinates

$$\sigma_{rr} = 2\mu\left(u_{r,r} + \frac{\nu}{1-2\nu}\Delta\right) \tag{20}$$

$$\sigma_{rz} = \mu(u_{r,z} + u_{z,r}) \tag{21}$$

$$\sigma_{zz} = 2\mu\left(u_{z,z} + \frac{\nu}{1-2\nu}\Delta\right) \tag{22}$$

the corresponding stresses $\sigma_{ijk}^{*\mathrm{f}}(\mathrm{P,Q})$ are obtained as

$$\sigma_{rrr}^{*\mathrm{f}} = \frac{1}{8r(1-\nu)}\{-(3-4\nu)\bar{I}_{110} + |\bar{z}|\bar{I}_{111} + (3-2\nu)r\bar{I}_{101} - r|\bar{z}|\bar{I}_{102}\} \tag{23}$$

$$\sigma_{rrz}^{*\mathrm{f}} = \frac{1}{8(1-\nu)}\{\mathrm{sign}(\bar{z})2(1-\nu)\bar{I}_{111} - \bar{z}\bar{I}_{112}\} \tag{24}$$

$$\sigma_{rzz}^{*\mathrm{f}} = \frac{1}{8(1-\nu)}\{-(1-2\nu)\bar{I}_{101} + |\bar{z}|\bar{I}_{102}\} \tag{25}$$

$$\sigma_{zrr}^{*\mathrm{f}} = \frac{1}{8r(1-\nu)}\{\bar{z}\bar{I}_{011} + r\,\mathrm{sign}(\bar{z})2\nu\bar{I}_{001} - r\bar{z}\bar{I}_{002}\} \tag{26}$$

$$\sigma_{zrz}^{*\mathrm{f}} = \frac{1}{8(1-\nu)}\{-(1-2\nu)\bar{I}_{011} - \bar{z}\bar{I}_{012}\} \tag{27}$$

$$\sigma_{zzz}^{*\mathrm{f}} = \frac{1}{8(1-\nu)}\{\mathrm{sign}(\bar{z})2(1-\nu)\bar{I}_{001} + \bar{z}\bar{I}_{002}\} \tag{28}$$

where, for a generic argument z,

$$\mathrm{sign}(z) = \begin{cases} 1 & \text{if } z \geq 0 \\ -1 & \text{if } z < 0 \end{cases} \tag{29}$$

If the ring load is applied at the axis of symmetry, i.e., for $\xi = 0$, the load in the radial direction is naturally void and, as a consequence, $u_{rj}^{*\mathrm{f}}|_{\xi=0} = 0$ and $\sigma_{rjk}^{*\mathrm{f}}|_{\xi=0} = 0$. In such a case, the fundamental solution for the vertical load simplifies to Kelvin's three-dimensional solution [47]. The expressions for $u_{zj}^{*\mathrm{f}}|_{\xi=0}$ and $\sigma_{zjk}^{*\mathrm{f}}|_{\xi=0}$ can be derived by taking the limit as $\xi \to 0$ in Eqs. (15)–(18) and Eqs. (23)–(28). Appendix A presents the limits of the integrals of the Lipschitz–Hankel type as $\xi \to 0$.

2.2. *Ring loads in an elastic halfspace*

An analogous procedure can be carried out for the axisymmetric halfspace. Consider a plane normal to z at $z = z'$ and split the halfspace defined for $z \leq 0$ into two parts, as depicted in Fig. 2.

Applying Eqs. (5) and (6) to each part of the halfspace leads to 12 unknown functions, as in the fullspace problem. These functions can be evaluated by applying regularity conditions for displacements and stresses at $z \to -\infty$ in part I,

$$u_i^{\mathrm{I}}(r,-\infty) = 0, \quad \sigma_{ij}^{\mathrm{I}}(r,-\infty) = 0 \tag{30}$$

traction free boundary condition at the surface of part II,

$$\sigma_{zj}^{\mathrm{II}}(r,0) = 0 \tag{31}$$

as well as displacement compatibility conditions and equilibrium conditions for the radial and vertical ring loads expressed in Eqs. (10)–(14).

The expressions of displacements for parts I and II can be combined and a similar procedure can also be applied to the complementary halfspace $z \geq 0$. The final expressions of displacements $u_{ij}^{*\mathrm{h}}(\mathrm{P,Q})$ and their corresponding stresses $\sigma_{ijk}^{*\mathrm{h}}(\mathrm{P,Q})$ are given by

$$u_{ij}^{*\mathrm{h}} = u_{ij}^{*\mathrm{f}} + u_{ij}^{*\mathrm{d}} \quad \text{and} \quad \sigma_{ijk}^{*\mathrm{h}} = \sigma_{ijk}^{*\mathrm{f}} + \sigma_{ijk}^{*\mathrm{d}} \tag{32}$$

in which $u_{ij}^{*\mathrm{f}}(\mathrm{P,Q})$ and $\sigma_{ijk}^{*\mathrm{f}}(\mathrm{P,Q})$ are the fullspace fundamental solutions given by Eqs. (15)–(18) and Eqs. (23)–(28). The index d in the remaining terms $u_{ij}^{*\mathrm{d}}(\mathrm{P,Q})$ and $\sigma_{ijk}^{*\mathrm{d}}(\mathrm{P,Q})$ refers to the difference between the halfspace and fullspace fundamental solutions and are given by

$$u_{rr}^{*\mathrm{d}} = \frac{1}{16\pi\mu(1-\nu)}\{(5-12\nu+8\nu^2)\hat{I}_{110} - (3-4\nu)|\hat{z}|\hat{I}_{111} + 2zz'\hat{I}_{112}\} \tag{33}$$

$$u_{rz}^{*\mathrm{d}} = \frac{1}{16\pi\mu(1-\nu)}\{-4(1-\nu)(1-2\nu)\mathrm{sign}(\hat{z})\hat{I}_{100} + (3-4\nu)\bar{z}\hat{I}_{101} + 2zz'\,\mathrm{sign}(\hat{z})\hat{I}_{102}\} \tag{34}$$

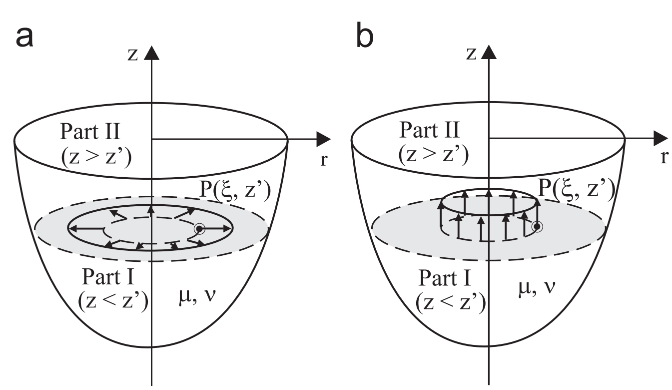

Fig. 2. Ring loads in the elastic halfspace: (a) radial direction; (b) axial direction.

$$u^{*d}_{zr} = \frac{1}{16\pi\mu(1-\nu)}\{-4(1-\nu)(1-2\nu)\mathrm{sign}(\hat{z})\hat{I}_{010} - (3-4\nu)\bar{z}\hat{I}_{011} + 2zz'\,\mathrm{sign}(\hat{z})\hat{I}_{012}\} \quad (35)$$

$$u^{*d}_{zz} = \frac{1}{16\pi\mu(1-\nu)}\{(5-12\nu+8\nu^2)\hat{I}_{000} + (3-4\nu)|\hat{z}|\hat{I}_{001} + 2zz'\hat{I}_{002}\} \quad (36)$$

and

$$\sigma^{*d}_{rrr} = \frac{1}{8r\pi(1-\nu)}\{-(5-12\nu+8\nu^2)\hat{I}_{110} + (3-4\nu)|\hat{z}|\hat{I}_{111} - 2zz'\hat{I}_{112} + r[(5-6\nu)\hat{I}_{101} - \mathrm{sign}(\hat{z})[(3-4\nu)z+3z']\hat{I}_{102} - 2zz'\hat{I}_{112} + 2zz'\hat{I}_{103}]\} \quad (37)$$

$$\sigma^{*d}_{rrz} = \frac{1}{8\pi(1-\nu)}\{-\mathrm{sign}(\hat{z})2(1-\nu)\hat{I}_{111} + [(3-4\nu)z+z']\hat{I}_{112} - \mathrm{sign}(\hat{z})2zz'\hat{I}_{113}\} \quad (38)$$

$$\sigma^{*d}_{rzz} = \frac{1}{8\pi(1-\nu)}\{(1-2\nu)\hat{I}_{101} + \mathrm{sign}(\hat{z})[(3-4\nu)z-z']\hat{I}_{102} - 2zz'\hat{I}_{103}\} \quad (39)$$

$$\sigma^{*d}_{zrr} = \frac{1}{8r\pi(1-\nu)}\{\mathrm{sign}(\hat{z})4(1-\nu)(1-2\nu)\hat{I}_{010} + (3-4\nu)\bar{z}\hat{I}_{011} - \mathrm{sign}(\hat{z})2zz'\hat{I}_{012} + r[-\mathrm{sign}(\hat{z})(2-3\nu)\hat{I}_{001} + [(3-4\nu)z-3z']\hat{I}_{002} + \mathrm{sign}(\hat{z})2zz'\hat{I}_{003}]\} \quad (40)$$

$$\sigma^{*d}_{zrz} = \frac{1}{8\pi(1-\nu)}\{(1-2\nu)\hat{I}_{011} - \mathrm{sign}(\hat{z})[(3-4\nu)z-z']\hat{I}_{012} - 2zz'\hat{I}_{013}\} \quad (41)$$

$$\sigma^{*d}_{zzz} = \frac{1}{8\pi(1-\nu)}\{-\mathrm{sign}(\hat{z})2(1-\nu)\hat{I}_{001} - [(3-4\nu)z+z']\hat{I}_{002} - \mathrm{sign}(\hat{z})2zz'\hat{I}_{003}\} \quad (42)$$

where

$$\hat{z} = z'+z \quad \text{and} \quad \hat{I}_{pq\lambda} = I_{pq\lambda}(\xi, r; c=|\hat{z}|) \quad (43)$$

The above equations are valid for the halfspace defined either for $z \le 0$ or $z \ge 0$.

If the ring load is applied at the axis of axisymmetry (i.e. $\xi = 0$), $u^{*h}_{rj}|_{\xi=0} = 0$ and $\sigma^{*h}_{rjk}|_{\xi=0} = 0$. In the case of a vertical load, $u^{*h}_{zj}|_{\xi=0}$ and $\sigma^{*h}_{zjk}|_{\xi=0}$ can be derived by taking the limit as $\xi \to 0$ in Eqs. (33)–(36) and Eqs. (37)–(42). The terms $u^{*d}_{ij}(\mathrm{P,Q})$ and $\sigma^{*d}_{ijk}(\mathrm{P,Q})$ are singular only at $z=0$. Notice that the implementation of the halfspace fundamental solution requires only a few changes to a code where the fullspace solution is already implemented.

3. Boundary element formulation

3.1. Boundary integral equation

In the absence of body forces, the displacements $u_i(\mathrm{P})$ in the domain Ω of a halfspace can be expressed in terms of displacements $u_i(\mathrm{Q})$ and traction forces $t_i(\mathrm{Q}) = \sigma_{ij}\eta_j$ along the boundary Γ by Somiglianas's identity for axisymmetric problems [16]

$$u_i(\mathrm{P}) = -2\pi\int_\Gamma t^{*h}_{ij}(\mathrm{P,Q})u_j(\mathrm{Q})r\,\mathrm{d}\Gamma + 2\pi\int_\Gamma u^{*h}_{ij}(\mathrm{P,Q})t_j(\mathrm{Q})r\,\mathrm{d}\Gamma \quad (44)$$

where $\Gamma(r,z) = \Gamma_i \cup \Gamma_s \cup \Gamma_0$ is the boundary of the meridian plane, shown in Fig. 3, and η_i is the outward unity normal to Γ. In this figure, Γ_i, Γ_s and Γ_0 represent the internal boundary, the loaded portion of the boundary at $z=0$ and the traction free extent of the boundary at $z=0$, respectively.

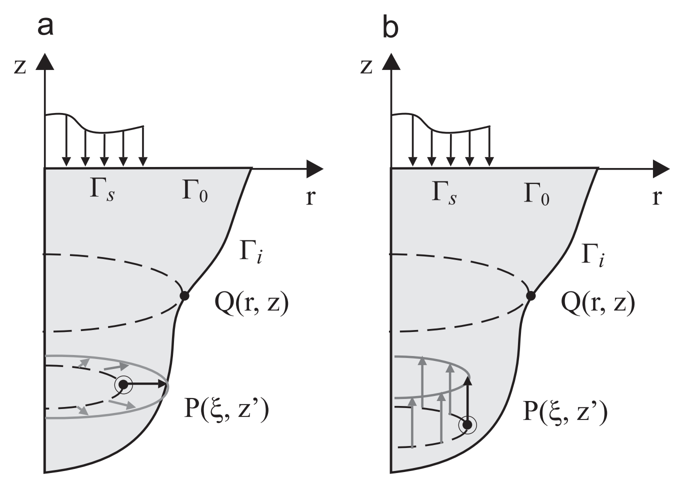

Fig. 3. Axisymmetric halfspace subjected to: (a) radial ring load; (b) vertical ring load.

The fundamental solutions u^{*h}_{ij} and $t^{*h}_{ij} = \sigma^{*h}_{ijk}n_k$ are displacements and traction forces in the halfspace that satisfy in advance the boundary conditions at $z=0$. Since, by definition, there are no tractions $t_i(\mathrm{Q})$ on Γ_0, Eq. (44) simplifies to

$$u_i(\mathrm{P}) = -2\pi\int_{\Gamma_i} t^{*h}_{ij}(\mathrm{P,Q})u_j(\mathrm{Q})r\,\mathrm{d}\Gamma + 2\pi\int_{\overline{\Gamma}} u^{*h}_{ij}(\mathrm{P,Q})t_j(\mathrm{Q})r\,\mathrm{d}\Gamma \quad (45)$$

where $\overline{\Gamma} = \Gamma_i \cup \Gamma_s$. Evaluating the above equation at the boundary leads to the following integral equation:

$$c_{ij}u_j(\mathrm{P}) = -2\pi \fint_{\Gamma_i} t^{*h}_{ij}(\mathrm{P,Q})u_j(\mathrm{Q})r\,\mathrm{d}\Gamma + 2\pi\int_{\overline{\Gamma}} u^{*h}_{ij}(\mathrm{P,Q})t_j(\mathrm{Q})r\,\mathrm{d}\Gamma \quad (46)$$

in which the first integral should be considered in the sense of Cauchy's principal value and

$$c_{ij} = \begin{cases} \delta_{ij} & \text{if } \mathrm{P}(\xi, z') \in \Gamma_s \\ \bar{c}_{ij} & \text{if } \mathrm{P}(\xi, z') \in \Gamma_i \end{cases} \quad (47)$$

The constants $\bar{c}_{ij}$ correspond to the discontinuous part of the first integral of Eq. (45) when $\mathrm{P}(\xi,z') \in \Gamma_i$ and their evaluation is presented in Section 3.2. Since the term t^{*d}_{ij} of the fundamental solution in Eq. (32) has no singularities, only t^{*f}_{ij} needs to be considered and the resulting constants c_{ij} are the same ones as for the implementation of the fullspace fundamental solution [16]. These constants can also be evaluated indirectly, by applying known analytical solutions, such as hydrostatic stress, plane stress and plane strain, to the final system of equations [56].

By approximating displacements and tractions along the boundary, which is discretized with n_n nodes, we obtain the conventional equation

$$H_{pq}u_q = G_{pq}t_q, \quad p,q = 1..2n_n \quad (48)$$

where H_{pq} and G_{pq} are the influence matrices, and u_q and t_q are nodal displacements and tractions. Solutions can be obtained by applying boundary conditions and rearranging the above equation. In this work, traction discontinuities were represented by duplicating the corresponding nodal degree of freedom [57].

3.2. Evaluation of the constants $\bar{c}_{ij}$

The constants $\bar{c}_{ij}$ correspond to the discontinuous part of the first integral of Eq. (45) and can be expressed as

$$\bar{c}_{ij} = \delta_{ij} + 2\pi\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} t^{*f}_{ij}(\mathrm{P,Q})r\,\mathrm{d}\Gamma = 2\pi\lim_{\epsilon\to 0}\int_{\overline{\Gamma}_\epsilon} t^{*f}_{ij}(\mathrm{P,Q})r\,\mathrm{d}\Gamma \quad (49)$$

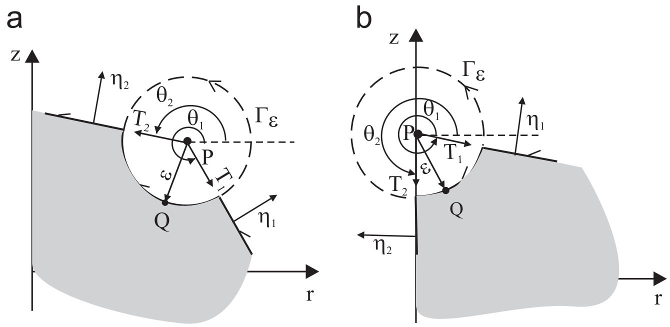

Fig. 4. Integration of constants c_{ij} for (a) $\xi \neq 0$ and (b) $\xi = 0$.

where Γ_ϵ and $\overline{\Gamma}_\epsilon$ are portions of the circumference of radius ϵ, as depicted in Fig. 4.

For the traction forces t_{ij}^{*f} in the fullspace, as the distance ρ between P(ξ,z') and Q(r,z) tends to zero, the modulus m of the complete elliptic integrals tends to unity, and accordingly $E(m)\to 1$ and $K(m)\to\infty$ in the integrals $\overline{I}_{pq\lambda}$. The integral in Eq. (49) can be simplified by expanding $K(m)$ as an infinite series for $m<1$ as [54]

$$K(m)=\frac{1}{2\pi}\left[1+\left(\frac{1}{2}\right)^2 m+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3+\cdots\right] \tag{50}$$

and using the following geometric relations:

$$r=\xi+\epsilon\cos\theta,\quad z=z'+\epsilon\sin\theta$$

$$n_r=-\cos\theta,\quad n_z=-\sin\theta,\quad d\overline{\Gamma}_\epsilon=-\epsilon\, d\theta \tag{51}$$

When m and n refer to the same node, the following expressions are obtained:

$$\overline{c}_{rr}=\frac{1}{4\pi(1-\nu)}\left[\frac{\sin 2\theta_1-\sin 2\theta_2}{2}+2(1-\nu)\Delta\theta\right]$$

$$\overline{c}_{rz}=\overline{c}_{zr}=\frac{1}{4\pi(1-\nu)}[\sin^2\theta_1-\sin^2\theta_2]$$

$$\overline{c}_{zz}=\frac{1}{4\pi(1-\nu)}\left[\frac{-\sin 2\theta_1-\sin 2\theta_2}{2}+2(1-\nu)\Delta\theta\right] \tag{52}$$

for $\xi\geq 0$, where $\theta_2=\theta_1-\Delta\theta$ and $\Delta\theta$ is the internal angle between θ_1 and θ_2, and

$$\overline{c}_{rr}=1$$

$$\overline{c}_{rz}=0$$

$$\overline{c}_{zr}=\frac{1}{4\pi(1-\nu)}[-\cos^3\theta_1+\cos^3\theta_2]$$

$$\overline{c}_{zz}=\frac{1}{4\pi(1-\nu)}[\sin\theta_1[2(1-\nu)-\cos^2\theta_1]-\sin\theta_2[2(1-\nu)-\cos^2\theta_2] \tag{53}$$

for $\xi=0$. On the other hand, when m and n do not refer to the same node, $\overline{c}_{mn}=0$.

The expressions given by Eq. (52) were obtained by Cruse et al. [16] and coincide with the constants for plane strain elasticity [58]. Correspondingly, Eq. (53) can be derived by integrating the constants for the three-dimensional elasticity [58] over the axis of symmetry. For $\xi=0$, only $\overline{c}_{zz}$ is required for a computational implementation since other constants are either multiplied by zero values of u_r or correspond to u_r values that a priori are known to be zero at the axis z, as remarked by Graciani et al. [59].

3.3. Displacements and stresses in the domain

From the solutions u_i(Q) and t_i(Q) along the boundary, displacements at a point P(ξ,z') in the domain can be obtained by Somigliana's identity, expressed in Eq. (45). Stresses in the domain can be evaluated by applying Somigliana's identity to the constitutive relations given by Eqs. (20)–(22), leading to

$$\sigma_{ij}(\mathrm{P})=2\pi\int_\Gamma \overline{t}_{ijk}^{*h}(\mathrm{P,Q})u_k(\mathrm{Q})r\, d\Gamma+2\pi\int_\Gamma \overline{u}_{ijk}^{*h}(\mathrm{P,Q})t_k(\mathrm{Q})r\, d\Gamma \tag{54}$$

Similar to the decomposition given by Eq. (32), functions $\overline{u}_{ijk}^{*h}$ and $\overline{t}_{ijk}^{*h}$ can also be expressed as

$$\overline{u}_{ijk}^{*h}=\overline{u}_{ijk}^{*f}+\overline{u}_{ijk}^{*d}\quad\text{and}\quad \overline{t}_{ijk}^{*h}=\overline{t}_{ijk}^{*f}+\overline{t}_{ijk}^{*d} \tag{55}$$

where $\overline{u}_{ijk}^{*f}$ and $\overline{t}_{ijk}^{*f}$ are derived from the fullspace fundamental solutions and were tabulated by Tan [60,61] in terms of complete elliptic integrals. The evaluation of $\overline{u}_{ijk}^{*h}$ and $\overline{t}_{ijkl}^{*h}$ is a cumbersome task since it involves the derivatives of the fundamental solutions u_{ij}^{*h} and t_{ij}^{*h}. However, they can be written in a more compact form in terms of integrals of the Lipschitz–Hankel type, as listed in Appendix B. This procedure is also valid for the evaluation of displacements and stresses on the non-discretized boundary Γ_0.

3.4. Displacements and stresses along the boundary

Stresses at a point P(ξ,z') of the boundary can be obtained by substituting for u_i in the constitutive relations for axisymmetry, Eqs. (20)–(22), with Eq. (45). As a result, the integral equation becomes hypersingular. This integral was first presented for fullspace problems by Lacerda and Wrobel [62], with contributions by Mukherjee [63] regarding its numerical integration.

Because of the complexity in evaluating these hypersingular integrals, this work adopts the approach of interpolating the nodal results along each boundary element in a local coordinate system [64].

4. Numerical integration

As only the meridian of the axisymmetric body needs to be discretized, the integrals can be evaluated along the boundary $\Gamma(r,z)$, for successive sub-boundaries Γ_t between two consecutive nodes of an element. Owing to the singularity of the fundamental solution, adequate numerical schemes must be adopted to evaluate the integrals

$$G_{i\alpha}^{t}=2\pi\int_{\Gamma_t} u_{ij}^{*h}(\mathrm{P,Q})N_\alpha(\mathrm{Q})r(\mathrm{Q})\, d\Gamma(r,z) \tag{56}$$

$$\widehat{H}_{i\alpha}^{t}=2\pi\int_{\Gamma_t} t_{ij}^{*h}(\mathrm{P,Q})N_\alpha(\mathrm{Q})r(\mathrm{Q})\, d\Gamma(r,z) \tag{57}$$

where the index t identifies the part of the boundary being integrated and N_α is the interpolation function for a given node α in the element. The various singularities occur only at the extremities of the integration intervals.

The singularities arising in the halfspace fundamental solutions u_{ij}^{*h}(P,Q) and t_{ij}^{*h}(P,Q) depend on the singularities of their individual terms, given in Eq. (32). Table 1 summarizes these singularities, where ρ is the distance between the points P(ξ,z') and Q(r,z). For the fullspace terms, the singularity type depends on the point P(ξ,z') at which the ring loads are applied. If the ring loads are placed on the axis of symmetry, i.e. $\xi=0$, the halfspace fundamental solutions coincide with the three-dimensional Kelvin's solutions [47] with their corresponding singularities. The remaining terms u_{ij}^{*d} and t_{ij}^{*d} present no singularities for

Table 1
Singularities of the halfspace fundamental solutions.

Ring load location	Singularity type		
	For all z		For $z=0$
	u_{ij}^{*f}	t_{ij}^{*f}	u_{ij}^{*d}
$\xi \neq 0$	$\ln(\rho)$	$\ln(\rho)$ and $1/\rho$	$\ln(\rho)$
$\xi = 0$	$1/\rho$	$1/\rho^2$	$1/\rho$

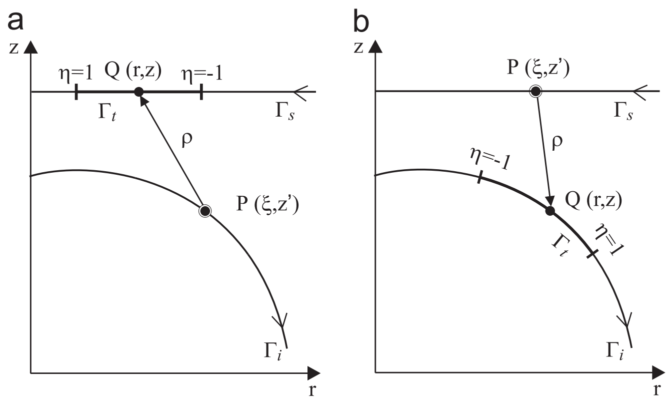

Fig. 5. Two illustrations of case 1: $P(\xi,z') \in \Gamma$ and $P(\xi,z') \notin \Gamma_t$ for: (a) $P(\xi,z') \in \Gamma_i$ and $Q(r,z) \in \Gamma_s$; (b) $P(\xi,z') \in \Gamma_s$ and $Q(r,z) \in \Gamma_i$.

$z \neq 0$. On the other hand, for $z=0$ the traction term t_{ij}^{*d} vanishes and only u_{ij}^{*d} presents singularity.

In this section, the integration cases identified for $G_{i\alpha}^{t}$ and $\widehat{H}_{i\alpha}^{t}$ are grouped according to the position of $P(\xi,z')$ in relation to the part of the boundary along which the integration is carried out as well as to the axis of symmetry. The numerical schemes employed in this work to evaluate regular integrals, weakly singular integrals of logarithmic terms and the finite part of singular integrals of order $1/\rho$ are briefly presented in Appendix C.

4.1. *Case 1:* $P(\xi,z') \notin \Gamma_t$

If the point $P(\xi,z')$ does not belong to the portion of the boundary being integrated, as illustrated in Fig. 5, then $\rho > 0$ and accordingly both $G_{i\alpha}^{t}$ and $\widehat{H}_{i\alpha}^{t}$ expressed in Eqs. (56) and (57) are regular and can be evaluated by the Gauss–Legendre quadrature rule [65]. For each portion of the boundary, these integrals can be rewritten in terms of the natural coordinate $\eta \in [-1,1]$ as

$$G_{i\alpha}^{t} = 2\pi \int_{-1}^{1} u_{ij}^{*h} N_\alpha(\eta) r(\eta) J(\eta)\, d\eta \tag{58}$$

$$\widehat{H}_{i\alpha}^{t} = 2\pi \int_{-1}^{1} t_{ij}^{*h} N_\alpha(\eta) r(\eta) J(\eta)\, d\eta \tag{59}$$

where $J(\eta)$ is the Jacobian of the coordinate transformation. In comparison to the fullspace implementation, it suffices to add to the existing integration scheme the terms represented by u_{ij}^{*d} and t_{ij}^{*d}.

4.2. *Case 2:* $P(\xi,z') \in \Gamma_t$, $\Gamma_t \subset \Gamma_i$

If $Q(r,z) \in \Gamma_i$ and $P(\xi,z')$ belongs to the portion of the boundary along which the integration is carried out, the singularities in the fundamental solutions are entirely due to the fullspace terms u_{ij}^{*f} and t_{ij}^{*f}, for either case $\xi > 0$ or $\xi = 0$. The integration of $G_{i\alpha}^{t}$ and $\widehat{H}_{i\alpha}^{t}$ for these singular terms is discussed below in Cases 2.1 and 2.2.

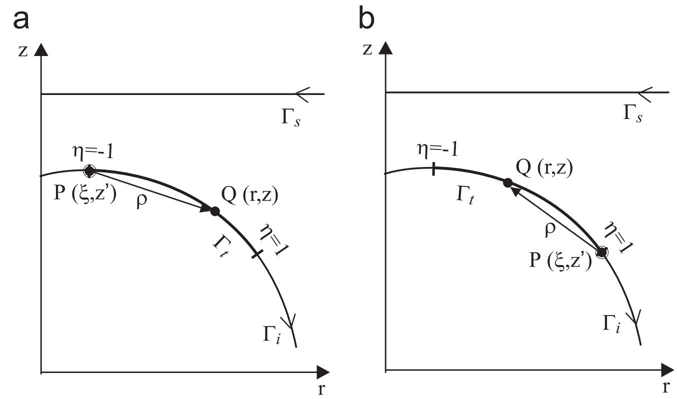

Fig. 6. Case 2.1: $P(\xi,z') \in \Gamma_i$ and $P(\xi,z') \in \Gamma_t$ for $\xi \neq 0$: (a) $\rho(\eta)|_{\eta=-1}=0$; (b) $\rho(\eta)|_{\eta=1}=0$.

The terms u_{ij}^{*d} and t_{ij}^{*d} present no singularities for $z \neq 0$ and therefore, it is sufficient to add to existing fullspace codes the integration of these terms in $G_{i\alpha}^{t}$ and $\widehat{H}_{i\alpha}^{t}$ by using the Gauss–Legendre quadrature scheme.

Case 2.1: $\xi \neq 0$. If the point $P(\xi,z')$ coincides with one of the nodes of the portion of the element being integrated and $\xi \neq 0$, then $\rho(\eta)|_{\eta=\eta'}=0$ for either $\eta'=-1$ or $\eta'=1$, as depicted in Fig. 6. In this case, because of the complete elliptic integrals, both u_{ij}^{*f} and t_{ij}^{*f} present singularity of order $\ln(\rho)$. Moreover, t_{ij}^{*f} also presents a singularity of order $1/\rho$. As a consequence, $G_{i\alpha}^{t}$ is weakly singular and $\widehat{H}_{i\alpha}^{t}$ presents both weak and strong singularities, the latter ones to be evaluated in terms of finite part.

The singular terms in $G_{i\alpha}^{t}$ and $\widehat{H}_{i\alpha}^{t}$ can be isolated by decomposing displacements and traction forces as

$$u_{ij}^{*f}(P,Q) = u_{K_{ij}}^{*f} K(m) + u_{E_{ij}}^{*f} E(m) \tag{60}$$

$$t_{ij}^{*f}(P,Q) = t_{K_{ij}}^{*f} K(m) + t_{E_{ij}}^{*f} E(m) \tag{61}$$

where $u_{K_{ij}}^{*f}$, $u_{E_{ij}}^{*f}$ and $t_{K_{ij}}^{*f}$ are regular functions and $t_{E_{ij}}^{*f}$ has singularity of order $1/\rho$. Also, the complete elliptic integrals $K(m)$ and $E(m)$ may be approximated as [54]

$$K(m) = K_1(\overline{m}) - K_2(\overline{m}) \ln \overline{m} \tag{62}$$

$$E(m) = E_1(\overline{m}) - E_2(\overline{m}) \ln \overline{m} \tag{63}$$

where

$$m = \frac{4\xi r}{(\xi+r)^2+\overline{z}^2}, \quad \overline{m} = \frac{\rho^2}{(\xi+r)^2+\overline{z}^2} \tag{64}$$

are the modulus and the complementary modulus of the complete elliptic integrals of the fundamental solutions. The polynomials $K_1(\overline{m})$, $K_2(\overline{m})$, $E_1(\overline{m})$ and $E_2(\overline{m})$ are listed in Appendix C.

Using Eqs. (60)–(63) in Eq. (56), $G_{i\alpha}^{t}$ can be evaluated by the numerical scheme suggested by Bialecki et al. [66], summarized in Appendix C. This approach splits the original integral into a regular integral and a weakly singular logarithmic integral, yielding

$$\begin{aligned} G_{i\alpha}^{t} = 2\pi \int_{-1}^{1} \Bigg\{ & u_{K_{ij}}^{*f} \left[K_1 + K_2 \ln \frac{(1-\eta'\eta)^2}{4\overline{m}} \right] \\ & + u_{E_{ij}}^{*f} \left[E_1 + E_2 \ln \frac{(1-\eta'\eta)^2}{4\overline{m}} \right] \Bigg\} N_\alpha(\eta) r(\eta) J(\eta)\, d\eta \\ & - 8\pi \int_0^1 [u_{K_{ij}}^{*f} K_2 + u_{E_{ij}}^{*f} E_2] \ln \tilde{\eta}\; N_\alpha(\tilde{\eta}) r(\tilde{\eta}) J(\tilde{\eta})\, d\tilde{\eta} \end{aligned} \tag{65}$$

where $\eta \in [-1,1]$ and $\tilde{\eta} \in [0,1]$ are the natural coordinate variables. The first integral is regular and can be evaluated by the

Gauss–Legendre quadrature rule while the second integral has logarithmic singularity and should be evaluated by the weighted logarithmic Gauss quadrature rule [65]. Although the complete elliptic integral $E(m)$ presents no singularity, its nonsingular approximation also includes logarithmic terms which are also isolated to enhance numerical convergence.

Owing to Eqs. (60)–(63), $\widehat{H}^{t}_{i\alpha}$ as given by Eq. (57) also includes logarithmic terms in $t^{*f}_{K_{ij}}K(m)$ and $t^{*f}_{E_{ij}}E_2(m)\ln \overline{m}$. Thus, their integration can be carried out by means of a regular integral and a weakly logarithmic singular integral, similar to the procedure proposed for $G^{t}_{i\alpha}$. On the other hand, the integral of $t^{*f}_{E_{ij}}E_1(m)$ exists only in terms of the finite part, to be numerically evaluated by the scheme proposed by Dumont and Souza [67] for singular integrals in terms of order $1/\rho$ over a curved boundary. This procedure employs the Gauss–Legendre quadrature and an additional correction term, as summarized in Appendix C. Hence, $\widehat{H}^{t}_{i\alpha}$ can be given by

$$
\begin{aligned}
\widehat{H}^{t}_{i\alpha} = 2\pi \int_{-1}^{1} \Bigg\{ & t^{*f}_{K_{ij}}\left[K_1 + K_2 \ln\frac{(1-\eta'\eta)^2}{4\overline{m}}\right] \\
& + t^{*f}_{E_{ij}}\left[E_1 + E_2 \ln\frac{(1-\eta'\eta)^2}{4\overline{m}}\right]\Bigg\} N_\alpha(\eta) r(\eta) J(\eta)\, d\eta \\
& -8\pi \int_{0}^{1} [t^{*f}_{K_{ij}}K_2 + t^{*f}_{E_{ij}}E_2] \ln\tilde{\eta}\; N_\alpha(\tilde{\eta}) r(\tilde{\eta}) J(\tilde{\eta})\, d\tilde{\eta} \\
& -2\pi\eta'[t^{*f}_{E_{ij}}\rho N_\alpha(\eta) r(\eta)]_{\eta=\eta'} \left\{ \ln|2J|_{\eta=\eta'} - \sum_{m=1}^{n_g} \frac{w^{g}_{m}}{1+\eta^{g}_{m}} \right\}
\end{aligned} \tag{66}
$$

where η' is the value of η at the singularity point (which is either -1 or 1) and η^{g}_{m} and w^{g}_{m} are the abscissae and weights of the Gauss–Legendre quadrature scheme for n_g points. In the above equation, the limit of $t^{*f}_{E_{ij}}\rho N_\alpha(\eta) r(\eta)$ for $\eta \to n'$ is difficult to obtain analytically and has been evaluated numerically by extrapolation.

Alternatively, the elements of the matrix $\widehat{H}^{t}_{i\alpha}$ that require the evaluation of singular integrals can be obtained indirectly by applying to Eq. (48) known analytical solutions for simple stress states, as suggested by Bakr and Fenner [56].

Case 2.2: $\xi = 0$. If $\xi = 0$, as depicted in Fig. 7, despite the $1/\rho$ singularity of the displacement fundamental solutions, $G^{t}_{i\alpha}$ becomes regular due to the presence of r multiplying its integrand and can be integrated as presented in Eq. (58) for Case 1.

Similarly, the presence of r in the integrand of $\widehat{H}^{t}_{i\alpha}$ makes its singularity of order $1/\rho$, even though the singularity of t^{*f}_{ij} is actually of order $1/\rho^2$. Thus, the integration scheme suggested by Dumont and Souza [67] can be applied as well, leading to

$$
\widehat{H}^{t}_{i\alpha} = 2\pi \int_{-1}^{1} t^{*f}_{ij} N_\alpha(\eta) r(\eta) J(\eta)\, d\eta
$$

$$
-2\pi\eta'[t^{*f}_{ij}\rho N_\alpha(\eta) r(\eta)]_{\eta=\eta'} \left\{ \ln|2J|_{\eta=\eta'} - \sum_{m=1}^{n_g} \frac{w^{g}_{m}}{1-\eta^{g}_{m}} \right\} \tag{67}
$$

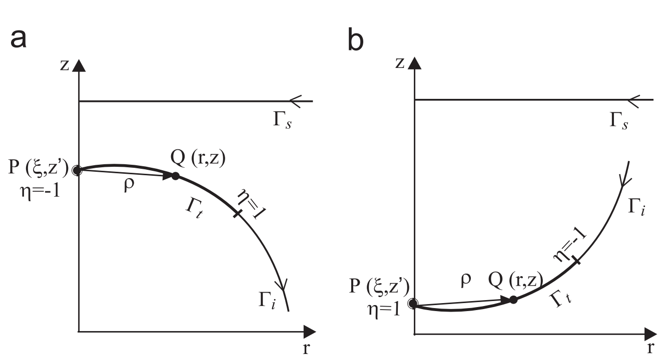

Fig. 7. Case 2.2: $P(\xi,z') \in \Gamma_i$ and $P(\xi,z') \in \Gamma_t$ for $\xi = 0$: (a) $\rho(\eta)|_{\eta=-1} = 0$; (b) $\rho(\eta)|_{\eta=1} = 0$.

4.3. *Case 3:* $P(\xi,z') \in \Gamma_t$, $\Gamma_t \subset \Gamma_s$

If $Q(r,z) \in \Gamma_s$, i.e. $z=0$, and $P(\xi,z')$ belongs to the portion of the boundary along which the integration is carried out, singularities arise in the fundamental solutions u^{*h}_{ij} for either case $\xi > 0$ or $\xi = 0$. In this case, it is preferable to manipulate and simplify the entire expressions rather than treat the singularities in the fullspace and the remaining terms separately. The integral $\widehat{H}^{t}_{i\alpha}$ vanishes since $t^{*h}_{ij} = 0$ satisfies the traction free boundary condition. The integration of $G^{t}_{i\alpha}$ for these singular terms are discussed below in Cases 3.1 and 3.2.

*Case*3.1: $\xi \neq 0$. If $Q(r,z) \in \Gamma_s$, $\xi \neq 0$ and $P(\xi,z')$ belongs to the portion of the element along which the integration is carried out, as depicted in Fig. 8, the term u^{*d}_{ij} presents logarithmic singularity and the evaluation of $G^{t}_{i\alpha}$ involves an improper integral.

For $z=0$ and $\xi \neq 0$, the fundamental solution u^{*h}_{ij} may be decomposed as

$$
u^{*h}_{ij}(P,Q) = u^{*h}_{K_{ij}}K(m) + u^{*h}_{E_{ij}}E(m) + u^{*h}_{L_{ij}} \tag{68}
$$

The remaining functions are given by

$$
u^{*h}_{K_{rr}} = \frac{(1-\nu)(\xi^2+r^2)}{2\pi^2\mu(\xi+r)\xi r}, \quad u^{*h}_{K_{rz}} = 0
$$

$$
u^{*h}_{K_{zr}} = 0, \quad u^{*h}_{K_{zz}} = \frac{(1-\nu)}{\pi^2\mu(\xi+r)} \tag{69}
$$

$$
u^{*h}_{E_{rr}} = -\frac{(1-\nu)(\xi+r)}{2\pi^2\mu\xi r}, \quad u^{*h}_{E_{rz}} = 0, \quad u^{*h}_{E_{zr}} = 0, \quad u^{*h}_{E_{zz}} = 0 \tag{70}
$$

$$
u^{*h}_{L_{rr}} = 0, \quad u^{*h}_{L_{rz}} = \begin{cases} \dfrac{-(1-2\nu)hsign}{4\pi\mu\xi} & \text{if } r < \xi \\ \dfrac{-(1-2\nu)hsign}{8\pi\mu\xi} & \text{if} r = \xi \\ 0 & \text{if } r > \xi \end{cases}
$$

$$
u^{*h}_{L_{zr}} = \begin{cases} 0 & \text{if } r < \xi, \\ \dfrac{-(1-2\nu)hsign}{8\pi\mu r} & \text{if} r = \xi, \\ \dfrac{-(1-2\nu)hsign}{4\pi\mu r} & \text{if } r > \xi, \end{cases} \quad u^{*h}_{L_{zz}} = 0 \tag{71}
$$

where *hsign* is either 1 or -1 for the halfspace defined for $z \leq 0$ or $z \geq 0$, respectively.

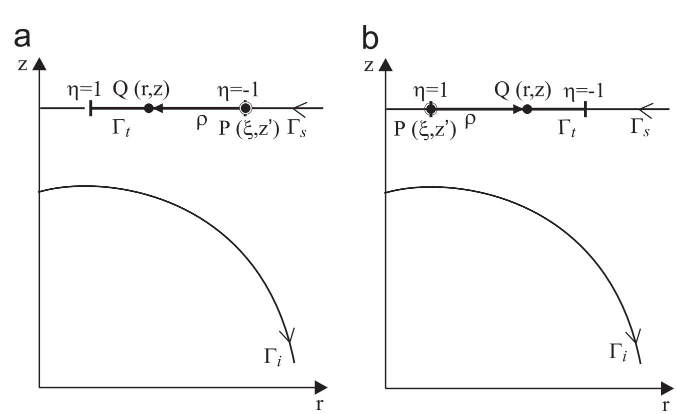

Fig. 8. Case 3.1: $P(\xi,z') \in \Gamma_s$, $P(\xi,z') \in \Gamma_t$ and $\xi \neq 0$ for: (a) $\rho(\eta)|_{\eta=-1} = 0$; (b) $\rho(\eta)|_{\eta=1} = 0$.

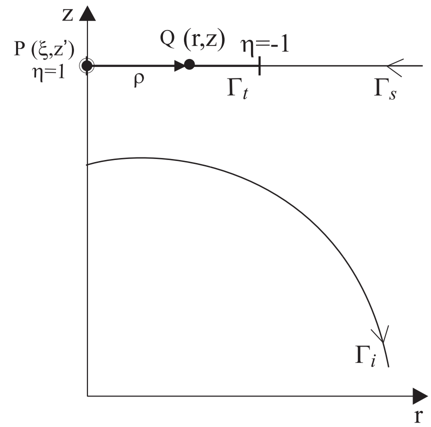

Fig. 9. Case 3.2: $P(\xi,z') \in \Gamma_s$, $P(\xi,z') \notin \Gamma_t$, $\xi=0$ and $\rho(\eta)|_{\eta=1}=0$.

The complete elliptic integrals can be approximated as in Eqs. (62) and (63). Similar to the procedure presented for Case 2.1, one arrives at

$$G^t_{i\alpha} = 2\pi \int_{-1}^{1} \left\{ u^{*h}_{K_{ij}} \left[K_1 + K_2 \ln \frac{(1-\eta'\eta)^2}{4\overline{m}} \right] + u^{*h}_{E_{ij}} \left[E_1 + E_2 \ln \frac{(1-\eta'\eta)^2}{4\overline{m}} \right] + u^{*h}_{L_{ij}} \right\} N_\alpha(\eta) r(\eta) J(\eta)\, d\eta - 8\pi \int_0^1 [u^{*h}_{K_{ij}} K_2 + u^{*h}_{E_{ij}} E_2] \ln \tilde{\eta} N_\alpha(\tilde{\eta}) r(\tilde{\eta}) J_{\tilde{\eta}}(\tilde{\eta})\, d\tilde{\eta} \quad (72)$$

in which the polynomials K_1, K_2, E_1 and E_2 are listed in Appendix C.

Case 3.2: $\xi=0$. If $Q(r,z) \in \Gamma_s$, $\xi=0$ and $P(\xi,z')$ belongs to the portion of the element along which the integration is carried out, as depicted in Fig. 9, the term u^{*d}_{ij} also has singularity of order $1/\rho$. However, similar to Case 2.2, the presence of r multiplying the integrand of $G^t_{i\alpha}$ makes the integral regular and the Gauss–Legendre quadrature rule can be applied.

Only a few modifications are needed in the integration scheme when comparing the fullspace and halfspace computational codes. In the boundary element implementation, only the case of $G^t_{i\alpha}$ for the integration carried out along part of Γ_s should be coded separately. In all other cases, the additional terms in the halfspace fundamental solution can be handled using the Gauss–Legendre quadrature rule.

5. Numerical examples

In the following, some numerical examples are presented in order to validate the proposed formulation. The BE method for axisymmetric problems was programmed in FORTRAN 90/95, for linear and quadratic elements. Six and eight integration points were used in the Gauss–Legendre and the logarithmic-weighted Gauss quadratures, respectively.

5.1. Example 1: circular load on a halfspace

Fig. 10 illustrates the halfspace $z \le 0$ with shear modulus μ and Poisson's ratio $\nu=0.25$, subjected to a uniform compressive normal stress p over a circular region of radius R=5 m. The analytical expressions for vertical displacement and axial stress can be found in Selvadurai [68] and Milovic [69] as

$$u_z = \frac{pR}{2\mu}[2(1-\nu)I_{10-1}(R,r;z) + zI_{100}(R,r;z)] \quad (73)$$

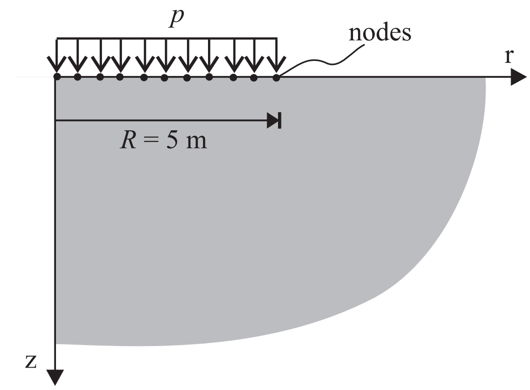

Fig. 10. Boundary element model of a halfspace subjected to a uniform pressure p on a circular surface of radius R=5 m.

$$\sigma_{zz} = p\left\{ A - \frac{n}{\pi\sqrt{n^1+(1+t)^2}} \left[\frac{n^2+t^2-1}{n^2+(1-t)^2} E(k) + \frac{1-t}{1+t} \Pi_0(k,q) \right] \right\} \quad (74)$$

in which

$$n = \frac{z}{R}, \quad t = \frac{r}{R}, \quad k^2 = \frac{4t}{n^2+(t+1)^2}, \quad q = \frac{4t}{(t+1)^2} \quad (75)$$

and

$$A = \begin{cases} 1 & \text{if } r<R \\ 1/2 & \text{if } r=R \\ 0 & \text{if } r>R \end{cases} \quad (76)$$

and the z-axis is positive in the direction shown in the figure.

This problem was modeled with 11 nodes and five quadratic elements, as depicted in Fig. 10. Results at internal points were evaluated using a square grid of 25 m spaced by 0.5 m. Fig. 11 shows good agreement between analytical and numerical results for vertical displacement u_z and axial stress σ_{zz}. Displacements and stresses on the surface z=0 for $r>R$ are evaluated as results at internal points, because of the halfspace formulation. In this case, as the free surface Γ_0 does not require any discretization, the evaluation of results in its nearby region is exempted from the treatment of the quasi-singular integrals arising in the fullspace formulation due to the boundary effect.

5.2. Example 2: halfspace medium with an irregularly shaped cavity subjected to a stress field

Consider a localized ring load $\mathbf{p}^*=(1,1)$ MN applied to the coordinates $P=(3\text{ m},-5\text{ m})$ of an elastic halfspace with shear modulus $\mu=10$ MPa and Poisson's ratio $\nu=0.3$. As proposed, this ring load has unit projections in both radial (p^*_r) and axial (p^*_z) directions, according to Fig. 2. The displacement and the stress fields produced at any point $Q=(r,z)$ can be evaluated by directly applying the halfspace fundamental solution of Eq. (32)

$$u_i = u^{*h}_{rj} p^*_r + u^{*h}_{zj} p^*_z \quad (77)$$

$$\sigma_{ij} = \sigma^{*h}_{rjk} p^*_r + u^{*h}_{zjk} p^*_z \quad (78)$$

Let an irregular, axisymmetric patch, given by the segment ABCDEF shown in Fig. 12, be drawn in this elastic medium, generating a cavity by removal of the enclosed material. Since the ring load is inside the cavity, as shown in the figure, the corresponding displacement and stress fields given by the above equations remain unaltered in the halfspace domain of interest if the traction forces corresponding to Eq. (78) are consequently

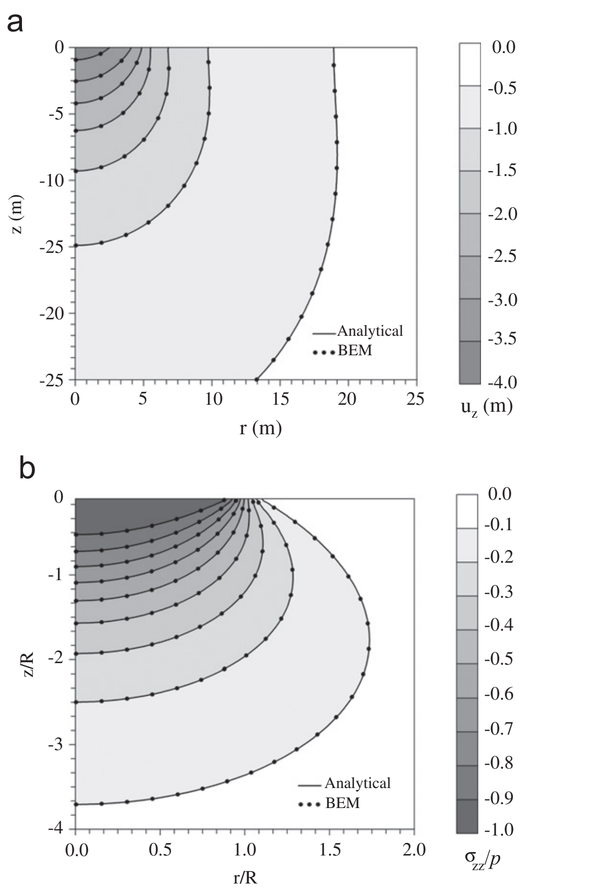

Fig. 11. Results of a vertical circular load p on a halfspace: (a) vertical displacements $u_z(r,z)$ (m); (b) vertical stresses $\sigma_{zz}(r,z)/p$.

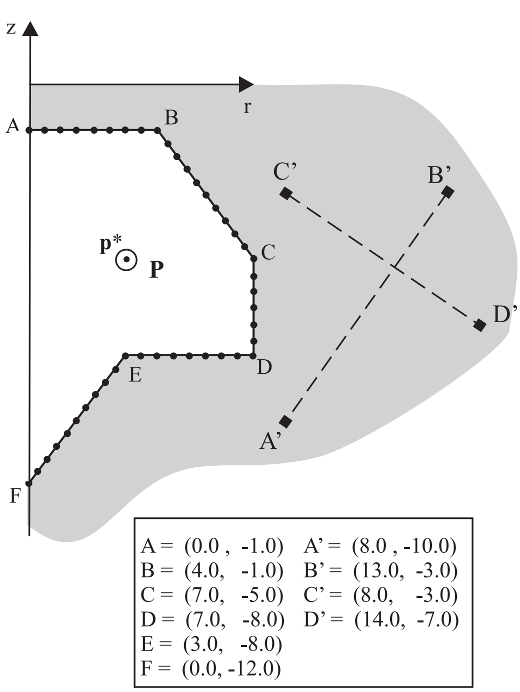

Fig. 12. Boundary element model of a halfspace medium with an irregularly shaped cavity subjected to a stress field (dimensions in meters).

applied to the cavity's surface. Observe that no stress singularities actually occur in the vicinity of the reentrant corners given by nodes B, C, E and F, for the applied ring load, although a high stress gradient is expected in the vicinity of the whole cavity. A variety of numerical problems with mixed boundary conditions can be formulated for the proposed example, as displacements or boundary tractions caused by the ring load can be complementarily prescribed along different parts of the cavity's boundary, according to Eqs. (77) and (78).

The following numerical results are obtained for traction forces given by Eq. (78) applied to the cavity – Neumann boundary conditions – for a BE model with 43 nodes and 21 quadratic elements, as depicted in Fig. 12. The nodal displacement results along the boundary ABCDEF are shown in Fig. 14, as compared with the analytical solution given by Eq. (77). Displacements and stresses evaluated at 20 internal points along the segments A′B′ and C′D′ of Fig. 12 are presented in Fig. 13. Table 2 presents the global relative errors for the displacements and stresses evaluated along the boundary and in the domain using the norm

$$\text{Error} = \sqrt{\frac{\sum (x_{a_i} - x_{n_i})^2}{\sum x_{a_i}^2}} \tag{79}$$

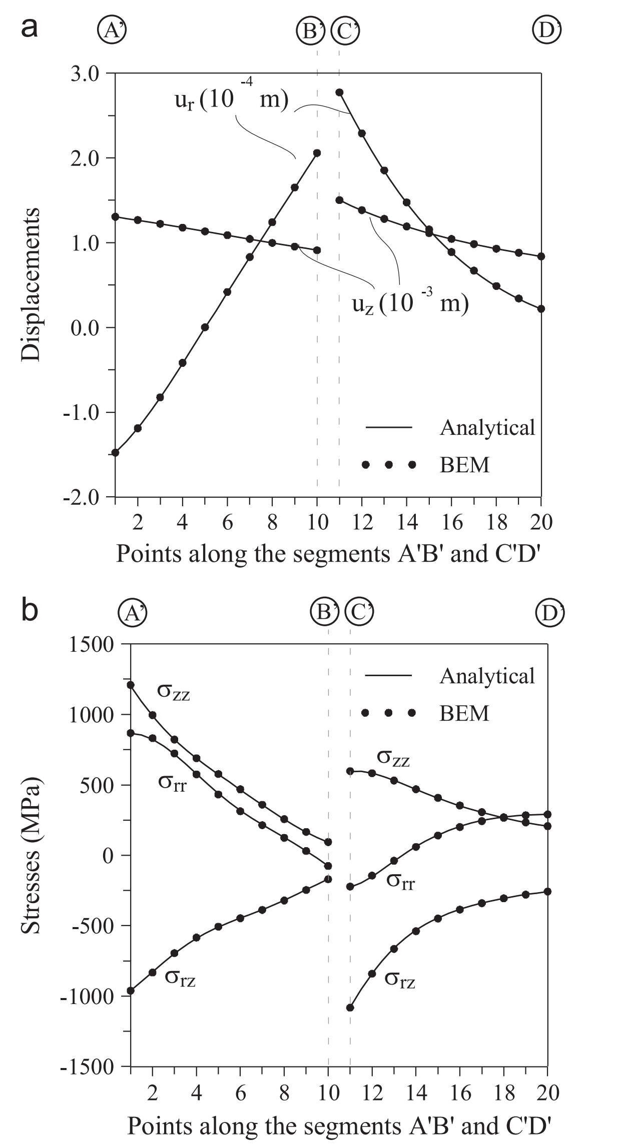

Fig. 13. Results along the segments A′B′ and C′D: (a) radial and vertical displacements; (b) radial, vertical and shear stress components.

Table 2
Global errors of displacements and stresses along the boundary and in the domain.

Error (%)						
Along the boundary		In the domain				
u_r	u_z	u_r	u_z	σ_{rr}	σ_{rz}	σ_{zz}
1.17	0.35	0.44	0.06	1.06	0.45	0.48

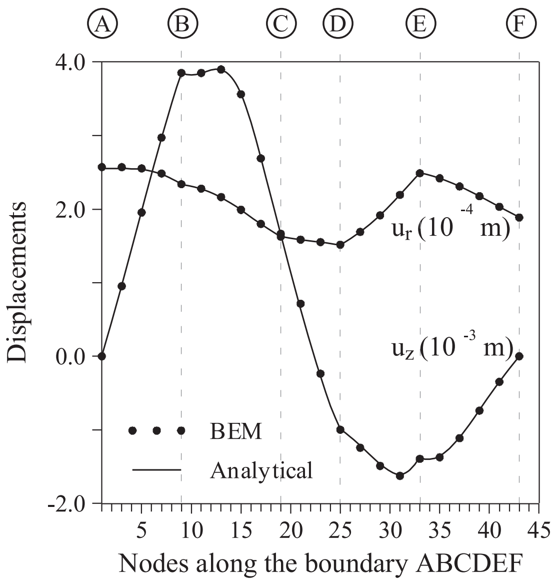

Fig. 14. Radial and vertical displacements along the boundary ABCDEF.

where x_a are the analytical results given by Eqs. (77) and (78), x_n are the numerical results and the summation index i refers to the number of results evaluated. All numerical results present good agreement with the analytical results. This example has also been analyzed for mixed boundary conditions, with displacements given by Eq. (77) considered as prescribed at some of the nodal points: the numerical results turned out to be visually indistinguishable from the ones of Figs. 13 and 14.

6. Concluding remarks

The expression of the fundamental solutions by means of integrals of the Lipschitz–Hankel type with products of Bessel functions was shown advantageous for the case of axisymmetric problems. For both fullspace and halfspace problems, each term of the fundamental solutions was investigated, allowing the order of the singularities to be identified and isolated. Moreover, all expressions that appear in the boundary element formulation could be written in a more compact manner than given in the technical literature, providing more concise equations to be implemented computationally and making it easier to find their limiting expressions in the cases of ring loads applied on the axis of symmetry. This is evident in the expressions for evaluating displacements and stresses at domain points in terms of Somigliana's identity.

Finally, this more compact representation has made explicit that the halfspace fundamental solution incorporates the fullspace fundamental solution. As the difference terms between these two fundamental solutions present singularities only on the surface of the halfspace, the implementation of the halfspace formulation turned out to require only a few modifications to existing codes for examining the fullspace problem.

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Appendix A. Integrals of the Lipschitz–Hankel type involving products of Bessel functions

The integrals of the Lipschitz–Hankel type involving products of Bessel functions can be represented by

$$I_{pq\lambda}(\xi,r;c) = \int_0^\infty J_p(\xi t) J_q(rt) e^{-ct} t^\lambda \, dt \tag{A.1}$$

where p, q and λ are integers; and $J_p(\xi t)$ and $J_q(rt)$ are Bessel functions of the first kind of order p and q, respectively. The convergent integrals of this type were tabulated by Eason et al. [53] and the expressions used in this formulation are

$$I_{000} = \frac{2k}{\pi A_1} K(m) \tag{A.2}$$

$$I_{110} = -\frac{2(k^2-2)}{\pi k A_1} K(m) - \frac{4}{\pi k A_1} E(m) \tag{A.3}$$

$$I_{100} = \begin{cases} -\dfrac{kc}{2\pi\xi\sqrt{\xi r}} K(m) - \dfrac{\Lambda_0(n,m)}{2\xi} + \dfrac{1}{\xi} & \text{if } \xi > r \\ -\dfrac{kc}{2\pi\xi^2} K(m) + \dfrac{1}{2\xi} & \text{if } \xi = r \\ -\dfrac{kc}{2\pi\xi\sqrt{\xi r}} K(m) + \dfrac{\Lambda_0(n,m)}{2\xi} & \text{if } \xi < r \end{cases} \tag{A.4}$$

$$I_{001} = \frac{2ck^3}{\pi \bar{k}^2 A_1^3} E(m) \tag{A.5}$$

$$I_{111} = -\frac{4ck}{\pi A_1^3} K(m) - \frac{2ck(k^2-2)}{\pi \bar{k}^2 A_1^3} E(m) \tag{A.6}$$

$$I_{101} = \frac{k}{\pi \xi A_1} K(m) + \frac{k^3 A_2}{\pi \xi \bar{k}^2 A_1^3} E(m) \tag{A.7}$$

$$I_{002} = -\frac{2c^2k^5}{\pi \bar{k}^2 A_1^5} K(m) - \frac{2k^3}{\pi \bar{k}^2 A_1^3} \left[1 + \frac{2c^2k^2(k^2-2)}{\bar{k}^2 A_1^2} \right] E(m) \tag{A.8}$$

$$I_{112} = \frac{2k}{\pi A_1^3} \left[2 + \frac{c^2k^2(k^2-2)}{\bar{k}^2 A_1^2} \right] K(m) + \frac{2k}{\pi \bar{k}^2 A_1^3} \left[k^2 - 2 + \frac{2c^2k^2(k^4+\bar{k}^2)}{\bar{k}^2 A_1^2} \right] E(m) \tag{A.9}$$

$$I_{102} = -\frac{ck^5 A_2}{\pi \xi \bar{k}^2 A_1^5} K(m) + \frac{ck^3}{\pi \xi \bar{k}^2 A_1^3} \left[3 - \frac{2k^2 A_2 (k^2-2)}{\bar{k}^2 A_1^2} \right] E(m) \tag{A.10}$$

$$I_{003} = \frac{2ck^5}{\pi \bar{k}^2 A_1^5} \left[3 + \frac{4c^2k^2(k^2-2)}{\bar{k}^2 A_1^2} \right] K(m)$$

$$-\frac{2ck^5}{\pi\bar{k}^4A_1^5}\left[-6(k^2-2)-\frac{c^2k^2}{A_1^2}\left(\frac{8k^4}{\bar{k}^2}+23\right)\right]E(m) \tag{A.11}$$

$$I_{113}=-\frac{2ck^3}{\pi\bar{k}^2A_1^5}\left[3(k^2-2)+\frac{2c^2k^2}{A_1^2}\left(\frac{2k^4}{\bar{k}^2}+3\right)\right]K(m) -\frac{2ck^3}{\pi\bar{k}^4A_1^5}\left[6(k^4+\bar{k}^2)+\frac{c^2k^2(k^2-2)(3\bar{k}^2+8k^4)}{\bar{k}^2A_1^2}\right]E(m) \tag{A.12}$$

$$I_{103}=\frac{k^5}{\pi\xi\bar{k}^2A_1^5}\left[A_2-5c^2+\frac{4k^2(k^2-2)c^2A_2}{\bar{k}^2A_1^2}\right]K(m) +\frac{k^3}{\pi\xi\bar{k}^2A_1^3}\left[-3+\frac{2k^2(k^2-2)(A_2-5c^2)}{\bar{k}^2A_1^2}+\frac{c^2k^4A_2}{\bar{k}^2A_1^4}\left(\frac{8k^4}{\bar{k}^2}+23\right)\right]E(m) \tag{A.13}$$

$$I_{004}=-\frac{2k^5}{\pi\bar{k}^2A_1^5}\left[3+\frac{24c^2k^2(k^2-2)}{\bar{k}^2A_1^2}+\frac{c^4k^4(24k^4+41\bar{k}^2)}{\bar{k}^4A_1^4}\right]K(m) -\frac{4k^5}{\pi\bar{k}^4A_1^5}\left[3(k^2-2)+\frac{3c^2k^2(8k^4+23\bar{k}^2)}{\bar{k}^2A_1^2}+\frac{4c^4k^4(k^2-2)(6k^4+11\bar{k}^2)}{\bar{k}^4A_1^4}\right]E(m) \tag{A.14}$$

$$I_{114}=\frac{6k^3}{\pi\bar{k}^2A_1^5}\left[k^2-2+\frac{4c^2k^2(2k^4+3\bar{k}^2)}{\bar{k}^2A_1^2}+\frac{c^4k^4(k^2-2)(8k^4+5\bar{k}^2)}{\bar{k}^4A_1^4}\right]K(m) +\frac{12k^3}{\pi\bar{k}^4A_1^5}\left[k^2+\bar{k}^4+\frac{c^2k^2(k^2-2)(3\bar{k}^2+8k^4)}{\bar{k}^2A_1^2} +\frac{c^4k^4(8-4\bar{k}^2-3\bar{k}^4-4\bar{k}^6+8\bar{k}^8)}{\bar{k}^4A_1^4}\right]E(m) \tag{A.15}$$

$$I_{104}=-\frac{ck^5}{\pi\xi\bar{k}^2A_1^5}\left[-15+\frac{4k^2(k^2-2)(3A_2-7c^2)}{\bar{k}^2A_1^2}+\frac{c^2k^4A_2(24k^4+71\bar{k}^2)}{\bar{k}^4A_1^4}\right]K(m) -\frac{ck^5}{\pi\xi\bar{k}^4A_1^5}\left[-30(k^2-2)+\frac{k^2(8k^4+23\bar{k}^2)(3A_2-7c^2)}{\bar{k}^2A_1^2} +\frac{8c^3k^4A_2(k^2-2)(6k^4+11\bar{k}^2)}{\bar{k}^4A_1^4}\right]E(m) \tag{A.16}$$

in which $I_{pq\lambda}(\xi,r;c)=I_{qp\lambda}(r,\xi;c)$ and

$$A_1=2\sqrt{\xi r},\quad A_2=\xi^2-r^2-c^2,\quad A_3=-\xi^2+r^2-c^2 \tag{A.17}$$

In the above expressions, $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kinds, respectively,

$$K(m)=\int_0^{\pi/2}(1-m\sin\theta^2)^{-1/2}\,d\theta \tag{A.18}$$

$$E(m)=\int_0^{\pi/2}(1-m\sin\theta^2)^{1/2}\,d\theta \tag{A.19}$$

The modulus k, the complementary modulus $\bar{k}$ and the parameter m are given by

$$k=\frac{2\sqrt{\xi r}}{\sqrt{(\xi+r)^2+c^2}},\quad \bar{k}=\sqrt{1-k^2}\quad\text{and}\quad m=k^2 \tag{A.20}$$

In Eq. (A.4), $\Lambda_0(n,m)$ is the Heuman complete elliptic integral expressed as

$$\Lambda_0(n,m)=\frac{2}{\pi}\left[\sqrt{1-n}\sqrt{1-\frac{m}{n}}\Pi(n,m)\right] \tag{A.21}$$

where $\Pi(n,m)$ is the complete elliptic integral of the third kind defined as

$$\Pi(n,m)=\int_0^{\pi/2}(1-n\sin\theta^2)^{-1}(1-m\sin\theta^2)^{-1/2}\,d\theta \tag{A.22}$$

and n is the characteristic number

$$n=\frac{A_1^2}{(\xi+r)^2} \tag{A.23}$$

Note that all Lipschitz–Hankel integrals $I_{pq\lambda}(\xi,r;c)$ listed above are written in terms of $K(m)$, $E(m)$ and $\Pi(m)$, which can be numerically evaluated by duplication as proposed by Carlson [70,71].

In the following, some useful limits are given, for $\rho_0=\sqrt{r^2+c^2}$:

$$\lim_{\xi\to0}I_{000}=\frac{1}{\rho_0}\qquad \lim_{\xi\to0}\frac{I_{102}}{\xi}=-\frac{3c(3r^2-2c^2)}{2\rho_0^7}$$
$$\lim_{\xi\to0}\frac{I_{110}}{\xi}=\frac{r}{2\rho^3}\qquad \lim_{\xi\to0}I_{012}=\frac{3rc}{\rho_0^5}$$
$$\lim_{\xi\to0}\frac{I_{100}}{\xi}=\frac{c}{2\rho_0^3}\qquad \lim_{\xi\to0}I_{003}=-\frac{3c(3r^2-2c^2)}{\rho_0^7}$$
$$\lim_{\xi\to0}I_{010}=\frac{c-\rho_0}{r\rho_0}\qquad \lim_{\xi\to0}\frac{I_{113}}{\xi}=-\frac{15rc(3r^2-4c^2)}{2\rho_0^9}$$
$$\lim_{\xi\to0}I_{001}=\frac{c}{\rho_0^3}\qquad \lim_{\xi\to0}\frac{I_{103}}{\xi}=\frac{3(3r^4-24r^2c^2+8c^4)}{2\rho_0^9}$$
$$\lim_{\xi\to0}\frac{I_{111}}{\xi}=\frac{3rc}{2\rho_0^5}\qquad \lim_{\xi\to0}I_{013}=-\frac{3r(r^2-4c^2)}{\rho_0^7}$$
$$\lim_{\xi\to0}\frac{I_{111}}{\xi}=\frac{3rc}{2\rho_0^5}\qquad \lim_{\xi\to0}I_{013}=-\frac{3r(r^2-4c^2)}{\rho_0^7}$$
$$\lim_{\xi\to0}\frac{I_{101}}{\xi}=-\frac{r^2-2c^2}{2\rho_0^5}\qquad \lim_{\xi\to0}I_{004}=\frac{3(3r^4-24r^2c^2+8c^4)}{\rho_0^9}$$
$$\lim_{\xi\to0}I_{011}=\frac{r}{\rho_0^3}\qquad \lim_{\xi\to0}\frac{I_{114}}{\xi}=\frac{45r(r^4-12r^2c^2+8c^4)}{2\rho_0^{11}}$$
$$\lim_{\xi\to0}I_{002}=-\frac{r^2-2c^2}{\rho_0^5}\qquad \lim_{\xi\to0}\frac{I_{104}}{\xi}=\frac{15c(15r^4-40r^2c^2+8c^4)}{2\rho_0^{11}}$$
$$\lim_{\xi\to0}\frac{I_{112}}{\xi}=-\frac{3r(r^2-4c^2)}{2\rho_0^7}\qquad \lim_{\xi\to0}I_{014}=-\frac{15cr(3r^2-4c^2)}{\rho_0^9}$$

Appendix B. Expressions for evaluating stresses in the domain

As presented in Section 3.3, stresses in the domain can be recovered by integrating the terms $\bar{u}^{*h}_{ijk}(\mathrm{P,Q})=\bar{u}^{*f}_{ijk}+\bar{u}^{*d}_{ijk}$ and $\bar{t}^{*h}_{ijk}(\mathrm{P,Q})=\bar{t}^{*f}_{ijk}+\bar{t}^{*d}_{ijk}$ along the boundary, where

$$\bar{t}^{*()}_{ijk}=\bar{\sigma}^{*()}_{ijkl}\eta_l=\bar{\sigma}^{*()}_{ijlk}\eta_l \tag{B.1}$$

For $\xi\neq0$, the terms above can be written in terms of the Lipschitz–Hankel integrals as

$$\bar{u}^{*f}_{rrr}=\frac{1}{8\pi(1-\nu)}\left\{\frac{1}{\xi}[-P_5\bar{I}_{110}+|\bar{z}|\bar{I}_{111}]+P_4\bar{I}_{001}-|\bar{z}|\bar{I}_{012}\right\} \tag{B.2}$$

$$\bar{u}^{*f}_{rrz}=\frac{1}{8\pi(1-\nu)}\left\{-\frac{\bar{z}\bar{I}_{101}}{\xi}-\mathrm{sign}(\bar{z})2\nu\bar{I}_{001}+\bar{z}\bar{I}_{002}\right\} \tag{B.3}$$

$$\bar{u}^{*f}_{rzr}=\frac{1}{8\pi(1-\nu)}\{-\mathrm{sign}(\bar{z})2P_1\bar{I}_{111}+\bar{z}\bar{I}_{112}\} \tag{B.4}$$

$$\bar{u}^{*f}_{rzz}=\frac{1}{8\pi(1-\nu)}\{-P_2\bar{I}_{101}-|\bar{z}|\bar{I}_{102}\} \tag{B.5}$$

$$\overline{u}^{*f}_{zzr} = \frac{1}{8\pi(1-\nu)}\{-P_2\bar{I}_{011}+|\bar{z}|\bar{I}_{012}\} \tag{B.6}$$

$$\overline{u}^{*f}_{zzz} = \frac{1}{8\pi(1-\nu)}\{-\text{sign}(\bar{z})2P_1\bar{I}_{001}-\bar{z}\bar{I}_{002}\} \tag{B.7}$$

$$\overline{u}^{*d}_{rrr} = \frac{1}{8\pi(1-\nu)}\left\{\frac{1}{\xi}[-P_7\hat{I}_{110}+P_5|\hat{z}|\hat{I}_{111}-2zz'\hat{I}_{112}]+P_6\hat{I}_{011} -\text{sign}(\hat{z})(P_5z'+3z)\hat{I}_{012}+2zz'\hat{I}_{013}\right\} \tag{B.8}$$

$$\overline{u}^{*d}_{rrz} = \frac{1}{8\pi(1-\nu)}\left\{\frac{1}{\xi}[\text{sign}(\hat{z})4P_1P_2\hat{I}_{100}-P_5\bar{z}\hat{I}_{101} -\text{sign}(\hat{z})2zz'\hat{I}_{102}]-\text{sign}(\hat{z})2P_3\hat{I}_{001}+(P_5z'-3z)\hat{I}_{002} +\text{sign}(\hat{z})2zz'\hat{I}_{003}\right\} \tag{B.9}$$

$$\overline{u}^{*d}_{rzr} = \frac{1}{8\pi(1-\nu)}\{-\text{sign}(\hat{z})2P_1\hat{I}_{111}+(P_5z'+z)\hat{I}_{112}-\text{sign}(\hat{z})2zz'\hat{I}_{113}\} \tag{B.10}$$

$$\overline{u}^{*d}_{rzz} = \frac{1}{8\pi(1-\nu)}\{P_2\hat{I}_{101}-\text{sign}(\hat{z})(P_5z'-z)\hat{I}_{102}-2zz'\hat{I}_{103}\} \tag{B.11}$$

$$\overline{u}^{*d}_{zzr} = \frac{1}{8\pi(1-\nu)}\{P_2\hat{I}_{011}+\text{sign}(\hat{z})(P_5z'-z)\hat{I}_{012}-2zz'\hat{I}_{013}\} \tag{B.12}$$

$$\overline{u}^{*d}_{zzz} = \frac{1}{8\pi(1-\nu)}\{-\text{sign}(\hat{z})2P_1\hat{I}_{001}-(P_5z'+z)\hat{I}_{002} -2\text{sign}(\hat{z})zz'\hat{I}_{003}\} \tag{B.13}$$

$$\overline{\sigma}^{*f}_{rrrr} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}[P_4\bar{I}_{101}-|\bar{z}|\bar{I}_{102}]+\frac{1}{\xi r}[-P_5\bar{I}_{110}+|\bar{z}|\bar{I}_{111}] +\frac{1}{r}[P_4\bar{I}_{011}-|\bar{z}|\bar{I}_{012}]-3\bar{I}_{002}-|\bar{z}|\bar{I}_{003}\right\} \tag{B.14}$$

$$\overline{\sigma}^{*f}_{rrrz} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}[\text{sign}(\bar{z})2P_1\bar{I}_{111}-\bar{z}\bar{I}_{112}]-\text{sign}(\bar{z})2\bar{I}_{012}+\bar{z}\bar{I}_{013}\right\} \tag{B.15}$$

$$\overline{\sigma}^{*f}_{rrzz} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}[-P_2\bar{I}_{101}+|\bar{z}|\bar{I}_{102}]+\bar{I}_{002}-|\bar{z}|\bar{I}_{003}\right\} \tag{B.16}$$

$$\overline{\sigma}^{*f}_{rzrr} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{r}[-\text{sign}(\bar{z})2P_1\bar{I}_{111}+\bar{z}\bar{I}_{112}]+\text{sign}(\bar{z})2\bar{I}_{102}-\bar{z}\bar{I}_{103}\right\} \tag{B.17}$$

$$\overline{\sigma}^{*f}_{rzrz} = \frac{\mu}{4\pi(1-\nu)}\{\bar{I}_{112}-|\bar{z}|\bar{I}_{113}\} \tag{B.18}$$

$$\overline{\sigma}^{*f}_{rzzz} = \frac{\mu\bar{z}\bar{I}_{103}}{4\pi(1-\nu)} \tag{B.19}$$

$$\overline{\sigma}^{*f}_{zzrr} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{r}[-P_2\bar{I}_{011}+|\bar{z}|\bar{I}_{012}]+\bar{I}_{002}-|\bar{z}|\bar{I}_{003}\right\} \tag{B.20}$$

$$\overline{\sigma}^{*f}_{zzrz} = -\frac{\mu\bar{z}\bar{I}_{013}}{4\pi(1-\nu)} \tag{B.21}$$

$$\overline{\sigma}^{*f}_{zzzz} = \frac{\mu}{4\pi(1-\nu)}\{\bar{I}_{002}+|\bar{z}|\bar{I}_{003}\} \tag{B.22}$$

$$\overline{\sigma}^{*d}_{rrrr} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}[P_6\hat{I}_{101}-\text{sign}(\hat{z})(P_5z+3z')\hat{I}_{102} +2zz'\hat{I}_{103}]+\frac{1}{\xi r}[-P_7\hat{I}_{110}-2zz'\hat{I}_{112}+P_5|\hat{z}|\hat{I}_{111}] +\frac{1}{r}[P_6\hat{I}_{011}-\text{sign}(\hat{z})(P_5z'+3z)\hat{I}_{012}+2zz'\hat{I}_{013}] -5\hat{I}_{002}+3|\hat{z}|\hat{I}_{003}-2zz'\hat{I}_{004}\right\} \tag{B.23}$$

$$\overline{\sigma}^{*d}_{rrrz} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}[-\text{sign}(\hat{z})2P_1\hat{I}_{111}+(P_5z+z')\hat{I}_{112} -\text{sign}(\hat{z})2zz'\hat{I}_{113}]+\text{sign}(\hat{z})2\hat{I}_{012}-(3z+z')\hat{I}_{013} -\text{sign}(\hat{z})2zz'\hat{I}_{014}\right\} \tag{B.24}$$

$$\overline{\sigma}^{*d}_{rrzz} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{\xi}[P_2\hat{I}_{101}+\text{sign}(\hat{z})(P_5z-z')\hat{I}_{102}-2zz'\hat{I}_{103}] -\hat{I}_{002}+\text{sign}(\hat{z})(-3z+z')\hat{I}_{003}+2zz'\hat{I}_{004}\right\} \tag{B.25}$$

$$\overline{\sigma}^{*d}_{rzrr} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{r}[-\text{sign}(\hat{z})2P_1\hat{I}_{111}+(P_3z'+z)\hat{I}_{112} -\text{sign}(\hat{z})2zz'\hat{I}_{113}]+\text{sign}(\hat{z})2\hat{I}_{102}-(z+3z')\hat{I}_{103} +\text{sign}(\hat{z})2zz'\hat{I}_{104}\right\} \tag{B.26}$$

$$\overline{\sigma}^{*d}_{rzrz} = \frac{\mu}{4\pi(1-\nu)}\{-\hat{I}_{112}+|\hat{z}|\hat{I}_{113}-2zz'\hat{I}_{114}\} \tag{B.27}$$

$$\overline{\sigma}^{*d}_{rzzz} = \frac{\mu}{4\pi(1-\nu)}\{-\bar{z}\hat{I}_{103}-\text{sign}(\hat{z})2zz'\hat{I}_{104}\} \tag{B.28}$$

$$\overline{\sigma}^{*d}_{zzrr} = \frac{\mu}{4\pi(1-\nu)}\left\{\frac{1}{r}[P_2\hat{I}_{011}+\text{sign}(\hat{z})(P_5z'-z)\hat{I}_{012}-2zz'\hat{I}_{013}] -\hat{I}_{002}+\text{sign}(\hat{z})(z-3z')\hat{I}_{003}+2zz'\hat{I}_{004}\right\} \tag{B.29}$$

$$\overline{\sigma}^{*d}_{zzrz} = \frac{\mu}{4\pi(1-\nu)}\{\bar{z}\hat{I}_{013}-\text{sign}(\hat{z})2zz'\hat{I}_{014}\} \tag{B.30}$$

$$\overline{\sigma}^{*d}_{zzzz} = \frac{\mu}{4\pi(1-\nu)}\{-\hat{I}_{002}-|\hat{z}|\hat{I}_{003}-2zz'\hat{I}_{004}\} \tag{B.31}$$

where

$$P_1 = 1-\nu, \quad P_2 = 1-2\nu, \quad P_3 = 2-3\nu$$

$$P_4 = 3-2\nu, \quad P_5 = 3-4\nu, \quad P_6 = 5-6\nu, \quad P_7 = 5-12\nu+8\nu^2 \tag{B.32}$$

For $\xi = 0$, these functions can be obtained in terms of limits only. The required expressions for $\lim_{\xi\to 0}\bar{I}_{pq\lambda}$ and $\lim_{\xi\to 0}\hat{I}_{pq\lambda}$ can be found in Appendix A. The terms $\overline{u}^{*f}_{ijk}$ and $\overline{t}^{*f}_{ijk}$ were also presented by Tan [60,61] in terms of elliptic integrals.

Appendix C. Numerical integration schemes

This appendix presents the numerical schemes used to evaluate the integrals arising in the boundary element formulations for axisymmetric problems. As only the meridian of the axisymmetric boundary needs to be discretized, these integrals are evaluated along the boundary $\Gamma(r,z)$, for each portion between consecutive nodes of an element.

C.1. Regular integral

Let $f(r,z)$ be a regular function on Γ, in the sense that it can be approximated by a polynomial of a not too high degree in the domain of interest. Then, its integral can be expressed in a natural coordinate system η in the interval $[-1,1]$ and approximated by the Gauss–Legendre quadrature rule [65], arriving at

$$\int_\Gamma f(r,z)\,\mathrm{d}\Gamma = \int_{-1}^{1} f(\eta)J(\eta)\,\mathrm{d}\eta \cong \sum_{m=1}^{n_g}[f(\eta)J(\eta)]_{\eta=\eta_m^g}w_m^g \tag{C.1}$$

where

$$J(\eta)=\sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\eta}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}\eta}\right)^2} \tag{C.2}$$

is the Jacobian transformation between the global and natural coordinate systems. The coefficients η_m^g and w_m^g are the abscissas and weights of the Gauss–Legendre quadrature rule for n_g points within the interval $(-1,1)$, which suffice to exactly evaluate the integral of a polynomial of order $2n_g-1$.

C.2. Weakly singular integral of logarithmic terms

Let $f(r,z)$ be a regular function and $\rho(r,z)$ the distance between the points $P(\xi,z')$ and $Q(r,z)$ on the boundary $\Gamma(r,z)$. It is necessary to evaluate the following weakly singular integral:

$$\int_\Gamma f(r,z)\ln\rho(r,z)\,\mathrm{d}\Gamma=\int_{-1}^{1} f(\eta)\ln\rho(\eta)J(\eta)\,\mathrm{d}\eta \tag{C.3}$$

for the case $\rho(-1)=0$ or $\rho(1)=0$. A unified treatment of both cases may be obtained by expressing

$$\rho(\eta)=\overline{\rho}(\eta,\eta')(1-\eta'\eta) \tag{C.4}$$

where η' is equal to either -1 or 1 and $\overline{\rho}(\eta,\eta')$ is the non-vanishing part of $\rho(\eta)$ for $\eta\in(-1,1)$. Then, the integral of Eq. (C.3) may be decomposed as [72]

$$\int_\Gamma f(r,z)\ln\rho(r,z)\,\mathrm{d}\Gamma=\int_{-1}^{1} f(\eta)\ln[2\overline{\rho}(\eta)]J(\eta)\,\mathrm{d}\eta+2\int_0^1 f(\tilde\eta)\ln\tilde\eta J(\tilde\eta)\,\mathrm{d}\tilde\eta \tag{C.5}$$

in which the transformation to the natural coordinate system $\tilde\eta\in[0,1]$ is given by

$$\tilde\eta=\tfrac{1}{2}(1-\eta'\eta) \tag{C.6}$$

The resulting integrals can be approximated by the Gauss–Legendre and logarithmic weighted Gauss quadratures rules [65], leading to

$$\int_\Gamma f(r,z)\ln\rho(r,z)\,\mathrm{d}\Gamma\cong\sum_{m=1}^{n_g}[f(\eta)\ln[2\overline{\rho}(\eta)]J(\eta)]_{\eta=\eta_m^g}w_m^g+\sum_{m=1}^{n_l}[f(\tilde\eta)\ln\tilde\eta J(\tilde\eta)]_{\tilde\eta=\eta_m^l}w_m^l \tag{C.7}$$

The coefficients η_m^l and w_m^l are the abscissas and weights of the logarithmic weighted Gauss quadrature rule for n_l points within the interval $(0,1)$, which suffice to exactly evaluate the integral of a polynomial of order $2n_l-1$.

The above integration scheme is obtained from a transformation of variables and the use of Gauss–Legendre and logarithmic weighted Gauss quadrature rules. Other approaches can also be employed [67,72].

C.2.1. Weakly singular integral of terms with the complete elliptic integral of the first order

Let $f(r,z)$ be a regular function and $K(m)$ the complete elliptic integral of the first order with modulus

$$m=\frac{4\xi r}{(\xi+r)^2+(z'-z)^2} \tag{C.8}$$

given in terms of the coordinates of points $P(\xi,z')$ and $Q(r,z)$ on the boundary $\Gamma(r,z)$. The following weakly singular integral

$$\int_\Gamma f(r,z)K(m)\,\mathrm{d}\Gamma=\int_{-1}^{1} f(\eta)K(m)J(\eta)\,\mathrm{d}\eta \tag{C.9}$$

needs to be evaluated, which actually encompasses two singularities in the case of $K(m)\to\infty$ since $m=1$ for either $\eta=-1$ or $\eta=1$. The integration scheme presented was proposed by Bialecki et al. [66].

The complete elliptic integral $K(m)$ can be approximated, for $0\le m<1$ and within an error $\epsilon<2\times10^{-8}$, by the expression [54]

$$K(m)=K_1(\overline{m})-K_2(\overline{m})\ln\overline{m} \tag{C.10}$$

where

$$\overline{m}=\frac{\rho^2}{(\xi+r)^2+(z'-z)^2} \tag{C.11}$$

is the complementary modulus of the complete elliptic integral and

$$K_1(\overline{m})=a_0+a_1\overline{m}+\cdots+a_4\overline{m}^4$$

$$K_2(\overline{m})=b_0+b_1\overline{m}+\cdots+b_4\overline{m}^4 \tag{C.12}$$

are polynomials whose coefficients are given by

$$a_0=1.38629436112,\quad b_0=0.5$$

$$a_1=0.09666344259,\quad b_1=0.12498593597$$

$$a_2=0.03590092383,\quad b_2=0.06880248576$$

$$a_3=0.03742563713,\quad b_3=0.03328355346$$

$$a_4=0.01451196212,\quad b_4=0.00441787012 \tag{C.13}$$

Substituting the approximation given by Eq. (C.10) in the weakly singular integral in Eq. (C.9), the singular term can be isolated to obtain

$$\int_\Gamma f(r,z)K(m)\,\mathrm{d}\Gamma=\int_\Gamma f(r,z)\left[K_1(\overline{m})+K_2(\overline{m})\ln\frac{\rho(r,z)^2}{\overline{m}}\right]\mathrm{d}\Gamma-2\int_\Gamma f(r,z)K_2(\overline{m})\ln\rho(r,z)\,\mathrm{d}\Gamma \tag{C.14}$$

Applying the schema for regular and weakly singular integrals, presented in the previous sections, to the first and second integrals of the above equation, respectively, leads to

$$\int_\Gamma f(r,z)K(m)\,\mathrm{d}\Gamma\cong\sum_{m=1}^{n_g}\left\{f(\eta)\left[K_1(\overline{m})+2K_2(\overline{m})\ln\frac{1-\eta'\eta}{2\sqrt{\overline{m}}}\right]J(\eta)\right\}_{\eta=\eta_m^g}w_m^g \tag{C.15}$$

$$-4\sum_{m=1}^{n_l}[f(\tilde\eta)K_2(\overline{m})\ln\tilde\eta J(\tilde\eta)]_{\tilde\eta=\eta_m^l}w_m^l \tag{C.16}$$

where $\tilde\eta$ is given in Eq. (C.6).

C.2.2. Weakly singular integral of terms with the complete elliptic integral of the second order

Let $f(r,z)$ be a regular function and $E(m)$ the complete elliptic integral of the second order with modulus m, given in terms of the coordinates of points $P(\xi,z')$ and $Q(r,z)$ on the boundary $\Gamma(r,z)$. The following weakly singular integral

$$\int_\Gamma f(r,z)E(m)\,\mathrm{d}\Gamma=\int_{-1}^{1} f(\eta)E(m)J(\eta)\,\mathrm{d}\eta \tag{C.17}$$

needs to be evaluated for the case of $m=1$ for either $\eta=-1$ or $\eta=1$. Although $E(m)\neq\infty$ for this case, the quasi-singular terms can be isolated to enhance the convergence of the numerical integration.

The complete elliptic integral $E(m)$ can be approximated, for $0\le m<1$ and within an error $\epsilon<2\times10^{-8}$, by the expression [54]

$$E(m)=E_1(\overline{m})-E_2(\overline{m})\ln\overline{m} \tag{C.18}$$

where $\overline{m}$ is given by Eq. (C.8) and

$$E_1(\overline{m}) = 1 + a_1\overline{m} + \cdots + a_4\overline{m}^4$$

$$E_2(\overline{m}) = b_1\overline{m} + \cdots + b_4\overline{m}^4 \tag{C.19}$$

are the polynomials whose coefficients are given by

$$a_1 = 0.44325141463, \quad b_1 = 0.24998368310$$

$$a_2 = 0.06260601220, \quad b_2 = 0.09200180037$$

$$a_3 = 0.04757383546, \quad b_3 = 0.04069697526$$

$$a_4 = 0.01736506451, \quad b_4 = 0.00526449639 \tag{C.20}$$

The polynomial approximation of $E(m)$ presents no singularity, since $E_2(\overline{m})$ has no free coefficients, according to Eq. (C.18). However, the presence of $\ln\overline{m}$ causes the integrand of Eq. (C.17) to be non-analytical, which requires a special numerical treatment.

In a manner similar to that used in the previous section, the following expression can be obtained for the numerical evaluation of the weakly singular integral given by Eq. (C.17)

$$\int_\Gamma f(r,z)E(m)\,\mathrm{d}\Gamma \cong \sum_{m=1}^{n_g}\left\{f(\eta)\left[E_1(\overline{m})+2E_2(\overline{m})\ln\frac{1-\eta'\eta}{2\sqrt{\overline{m}}}\right]J(\eta)\right\}_{\eta=\eta_m^g} w_m^g \tag{C.21}$$

$$-4\sum_{m=1}^{n_l}\left[f(\tilde{\eta})E_2(\overline{m})\ln\tilde{\eta}J(\tilde{\eta})\right]_{\tilde{\eta}=\eta_m^l} w_m^l \tag{C.22}$$

for $\tilde{\eta}$ given by Eq. (C.6).

C.3. Cauchy principal value of the singular integral of order $1/\rho$

Let $f(r,z)$ be a regular function and $\rho(r,z)$ the distance between the points $P(\xi,z')$ and $Q(r,z)$ on the boundary $\Gamma(r,z)$. The strongly singular integral

$$\int_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma \tag{C.23}$$

has to be evaluated for the case $\rho(-1)=0$ or $\rho(1)=0$. This integral may be obtained as a sum of a Cauchy principal value and a discontinuous term as

$$\int_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma = \mathrm{PV}\int_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma + c \tag{C.24}$$

The evaluation of the discontinuous term c of the strongly singular integrals appearing in the boundary element formulations is addressed in Section 3.2.

The Cauchy principal value is best evaluated in terms of two finite-part integrals, denoted by $⨍$, for the boundary segments adjacent to the singularity point $\rho(r,z)=0$.

In what follows, the integration scheme proposed by Dumont and Souza [67] is used. Using the notation of Eq. (C.6), the regular function can be expanded as a Taylor series to obtain the following normalized integral of Eq. (C.23) over the curved boundary Γ

$$⨍_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma = -\eta'[f(\eta)\ln\overline{|\rho|}]_{\eta=\eta'} + ⨍_{-1}^{1}\frac{f(\eta)}{\rho(\eta)}J(\eta)\,\mathrm{d}\eta \tag{C.25}$$

The resulting quadrature rule for evaluating Cauchy's principal value of the strongly singular integral of (C.23) is given by

$$⨍_\Gamma \frac{f(r,z)}{\rho(r,z)}\,\mathrm{d}\Gamma \cong \sum_{m=1}^{n_g}\left[\frac{f(\eta)}{\rho(\eta)}J(\eta)\right]_{\eta=\eta_m^g} w_m^g$$

$$-\eta'[f(\eta)]_{\eta=\eta'}\left\{[\ln|2\overline{\rho}|]_{\eta=\eta'} - \sum_{m=1}^{n_g}\frac{w_m^g}{1-\eta_m^g}\right\} \tag{C.26}$$

where

$$[\overline{\rho}(\eta)]_{\eta=\eta'} = [J(\eta)]_{\eta=\eta'} \tag{C.27}$$

The above scheme, that employs the Gauss–Legendre quadrature rule and an additional correction term, evaluates exactly this integral for a polynomial function of order $2n_g$. Other numerical integration schemes for the strongly singular integral can be used [72,73].

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