Fluid pressure loading of a hyperelastic membrane

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A R T I C L E   I N F O

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A B S T R A C T

This paper examines the problem of the fluid pressure loading of a hyperelastic membrane made of a natural gum rubber, that is fixed along a circular boundary. The fluid pressure loading of hyperelastic membranes have several important technological applications particularly in the area of bio-medical engineering and the response of the membrane to fluid pressure loading can be useful in identifying the applicability of the various forms of strain energy functions that are proposed in the literature for describing hyperelastic behaviour. The results of the fluid pressure induced deflected profiles of the circular membrane together with computational modelling of the experiments are used to identify the range of applicability of several forms of strain energy functions for the hyperelastic materials.

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1. Introduction

The class of problems dealing with elastic membranes that experience large deformations has attracted attention over the past seven decades. The pioneering work in this area by R.S. Rivlin and co-workers in the 1940s has formed the basis for the scientific study and technological applications of the theory of hyperelastic materials. The studies by Rivlin and co-workers can be found in the collected works of R.S. Rivlin edited by Barenblatt and Joseph [1] and further developments related to mechanics of rubber-like hyperelastic materials are documented [2–8]. References to recent developments are also given by Lu’re [9], Gent [10,11], Dorfman and Muhr [12], Besdo et al. [13], Fu and Ogden [14], Selvadurai [15], Busfield and Muhr [16], Saccomandi and Ogden [17] and Goriely et al. [18,19]. Since the early applications of theories of hyperelasticity to rubber-like materials, their range of applications have been extended to include biological materials and synthetic soft tissues used in bio-medical engineering [20].

One important feature of hyperelastic materials relates to the choice of the strain energy function that can be used to describe its mechanical response. Numerous strain energy functions exist and the choice is based on the particular type of hyperelastic material including its molecular structure and cross-linking and rate-sensitivity [5,6,10,11,21–40]. Experiments dealing with the mechanics of membranes subjected to inflation pressures have been examined by Rivlin and Saunders [41] and more recent research in this area is due to Selvadurai [42], Gonçalves et al. [43], Sasso et al. [44]. This paper describes the experiments and the data analysis procedures used for parameter identification purposes of determining the appropriate form of the strain energy function for a wide range of strains. The validity of the strain energy function, however, needs to be established through a prediction of an alternative experimental configuration. The present paper will focus on the documentation of the test procedures and the techniques used to identify the most relevant form of the strain energy function that can characterize the hyperelastic behaviour of the natural gum rubber.

2. Basic equations

The mathematical modelling of the mechanical behaviour of rubber-like hyperelastic materials focuses to a large extent on the development of an appropriate form of a strain energy function applicable to the range of deformations of interest to practical applications. The early developments in the description of constitutive models for rubber-like hyperelastic materials focused on strain energy functions determined through tests conducted on solid rubber components [3,21,45–47] and rubber membranes [41,48–51]. Further references to studies in these areas are given by Selvadurai [15,42]. In general, the strain energy density of the strain energy function is assumed to be dependent on the three principal invariants $I_1$, $I_2$ and $I_3$ of the deformation tensor, which, for incompressible materials, can be defined in terms of the principal stretches as follows:

$$I_1 = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1 \lambda_2}; \quad I_2 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_1 \lambda_2}; \quad I_3 = \lambda_1 \lambda_2 \lambda_3 = 1$$

(1)

Considering the general form of the constitutive relationship for an isotropic hyperelastic material subjected to homogeneous stretches $\lambda_i (i=1,2,3)$ it can be shown (see e.g. [3,52]) that the
physical stress components of the Cauchy stress are given by

$$\sigma_{11} = 2\lambda_1^2 \frac{\partial W}{\partial \lambda_1} - 2\lambda_1^2 \lambda_2 \frac{\partial W}{\partial \lambda_2} + p$$

$$\sigma_{22} = 2\lambda_2^2 \frac{\partial W}{\partial \lambda_1} - 2\lambda_1^2 \lambda_2 \frac{\partial W}{\partial \lambda_2} + p$$

$$\sigma_{33} = 2\lambda_3^2 \frac{\partial W}{\partial \lambda_1} - 2\lambda_1^2 \lambda_2 \frac{\partial W}{\partial \lambda_2} + p$$

where $p$ is an undetermined isotropic stress and $W(l_1,l_2)$ is the strain energy function applicable to an isotropic incompressible elastic material.

The simplest form of the strain energy function $W(l_1,l_2)$ used in the study of the mechanics of rubber-like elastic materials is a neo-Hookean form of the strain energy function given by

$$W(l_1,l_2) = C_1(l_1 - 3)$$

where $C_1$ is a constant. This strain energy function has been applied by Treloar [5,6,22] for the study of inflated and stretched membranes of pure rubber. The modification to the neo-Hookean model is provided by the Mooney–Rivlin form of the strain energy function, given by

$$W(l_1,l_2) = C_1(l_1 - 3) + C_2(l_2 - 3)$$

where $C_2$ is a constant. It can be shown that $2(C_1 + C_2) = \mu$, where, $\mu$ is the linear elastic shear modulus. The model proposed by Yeoh [23] is based on the experimental data of Treloar [22]. In this model the strain energy function is given in the form

$$W(l_1,l_2) = \sum_{i=1}^{N} \frac{\mu_i}{\lambda_i^2} (\lambda_i^2 - 3)$$

where $N$, $\mu_i$ and $\lambda_i$ are material parameters and

$$\sum_{i=1}^{N} \mu_i = \mu$$

The result (7) reduces to the neo-Hookean form (5) when $N=1$, $\lambda_1=2$ and $\mu_1=\mu$. The linear elastic shear modulus can also be described in relation to a polymer network model, which gives

$$\mu = NkT$$

where $N$ is the number of chain molecules per unit volume of the rubber material, $k$ is Boltzmann’s constant and $T$ is the absolute temperature. The constitutive model for incompressible hyper-elastic rubber-like materials proposed by Yeoh [23] assumes that the strain energy function is independent of the second strain invariant and can be represented as a power series in terms of the variable $(l - 3)$ i.e.

$$W = \sum_{i=1}^{N} \tilde{C}_i (l - 3)^i$$

where $N$ is the number of terms in the series, $\tilde{C}_i$ are constants and, for a single term in the series, the Yeoh model reduces to the neo-Hookean form. We will restrict attention to the basic strain energy functions associated with the neo-Hookean, Mooney–Rivlin, Yeoh and Ogden forms of the strain energy functions, although many other strain energy functions can be found in the literature (see e.g. [11,39,40,42]).

3. Uniaxial testing

The rubber used in the experimental investigations was classified as a natural gum rubber. The mechanical properties of rubber were determined by performing uniaxial tensile tests on specimens of the rubber. The testing facility consisted of a servo-controlled MTS machine equipped with a load cell with a capacity of 150 kN. A set of Advantage™ Wedge Action Grips was used for the experiment since they have knurled clamping plates that provide good fixity at the ends of the specimen. The details of the experimental setup and the sample assembly are shown in Fig. 1. The tests involve the stretching a rubber specimen attached to the upper and lower set of grips. The lower set of grips is fixed during testing while the upper set moves either upwards or downwards in a displacement-controlled mode. The speed of movement of the cross-head during testing is controlled at a quasi-static strain rate of 20%/min. A special low capacity load cell (2224 N (500 lbs)) was used in the testing machine since the forces measured during uniaxial testing were very small compared to the peak load capacity of the MTS machine. Two special adaptors were added to the initial test arrangement in order to effectively attach the load cell to the MTS Testing frame. To prevent

![Fig. 1. The testing facility and details of the grips and the test specimen.](image-url)
any slippage between the specimen and the plates of the AdvantageM Wedge Action Grips, an additional layer of gum rubber was bonded to each end of the test specimen using a non-reactive instant adhesive (Krazy Glue®). This extra layer increased the thickness of the ends of the gum rubber specimen, which was then tightly clamped between the two knurled plates of the Wedge Action Grips. It was observed previously that slippage occurred without this additional layer [42,35]. Tests indicated that without the additional layer, slippage would be limited at small strains but become noticeable at moderate and large strains, primarily due to the progressive friction loss (associated with a Poisson-type contraction in the thickness direction) between the membrane and the clamping system during stretching. There is the possibility of chemical reaction between the instant adhesive and the rubber specimen if the testing takes place over a long period; however, since the tests on the rubber specimens were performed within one hour of application of the adhesive, the effects of any chemical reaction were disregarded. Rubber specimens of different thicknesses were tested. All samples were cut from the same sheet to minimise any batch-to-batch variations in the mechanical properties. The specimens used measured 150 mm × 30 mm in profile with cross sections of 30 mm × 0.794 mm and 30 mm × 1.588 mm in the undeformed configuration. The stress–strain data collected from the experiments are recorded in terms of engineering stress and engineering strain. In the experiments performed, the original length of the test specimen was taken as the distance between two edges of the specimen (L0 = 150 mm). During tensile testing, the specimen should ideally produce a homogeneous deformation. The homogeneity of the deformation is a requirement for the data analysis procedures that rely on the appropriate measures of stress and homogeneous strain. In reality, however, it is not possible to achieve perfect homogeneity of the specimen during straining over the entire grip length. Since the ends are gripped, the prevention of lateral contraction leads to non-homogeneity in the strain field in the vicinity of the grips. To examine the extent to which the end constraints influence the development of homogeneous straining, a comparison between the physical stretching of the specimen and the relative extension of the grips was made. The procedure used was identical to that given in Selvadurai [42]. A grid was drawn on the rubber specimen using a fine black marker with spacings of 10 and 30 mm between gridlines. As the sample stretched, the horizontal gridlines were used to calculate the real physical length in different sections on the rubber specimen. A 5 megapixels digital camera captured the deformed configuration of the specimen at different levels of strain. The distance between the horizontal gridlines was first measured in image pixels, and then calibrated against a known physical distance. The known physical distance was chosen as the distance between the two grips. The real distance at different extensions was then obtained via a conversion between the image pixels and the known physical distance. The test results indicate that the effects of the fixity constraints gives errors of 3.4%, 1.5% and 0.2% at average strains corresponding to ε0 = 27%, 54% and 81%, respectively. The strain was calculated as the percentage change of the original length. The results of the uniaxial tests conducted up to a strain of 100% are presented in this section. The experiments were conducted in the Materials Testing Laboratory where the room temperature was approximately 24 °C. The duration of each tensile test was approximately 35 min and during this period the temperature in the laboratory fluctuated by less than 1 °C. It can be assumed that the temperature fluctuations had virtually no effect on the change in the mechanical properties of the natural rubber.

The results of the uniaxial tests conducted on natural gum rubber of thicknesses 0.793 and 1.588 mm are shown in Fig. 2. The natural gum rubber undergoes moderate strains, i.e. up to 100% strain, during the tensile test. Due to the height limitation of the MTS machine, the material could not be tested up to failure; however, natural rubber is known to fail at a strain in the range 800–1000% [5]. The results show good repeatability between each set of experiments, and within the range of accuracy of the tests. From the loading–unloading stress–strain curves shown in Fig. 2, it can be observed that hysteresis in the sample of thickness 0.793 mm was negligible. The sample of thickness 1.588 mm displays some hysteresis. In a strict sense, the conventional theory of finite elasticity cannot be rigorously applied to describe hyperelasticity in the presence of the “Mullins Effect”. Despite this limitation, the theory can be applied to develop approximate representations of the monotonic loading process. For both membranes, the unloading behaviour of the gum rubber follows relatively closely the loading curve and the material does not display any significant irreversible deformation upon complete unloading. Other hyperelastic materials, such as PVC geosynthetics, exhibit creep and irreversible strains during uniaxial testing [33,34–37]. From an experimental point of view, natural rubber is relatively strain-rate independent. Results of uniaxial tests conducted at three different strain-rates, 20%/min, 40%/min and 60%/min, indicated that the strain rate has no significant influence on the stress–strain behaviour of the material.

4. Parameter identification

If we consider uniaxial stressing of the test specimen in the 1-direction, we have

$$\sigma_0 = \begin{bmatrix} \sigma_{11}^0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the undetermined isotropic stress p can be evaluated. The single non-zero axial stress σ11(=σ0) can be expressed in the following form [52]:

$$\sigma_0 = 2 \left( \frac{L^2 - 1}{L} \right) \left( \frac{\sigma W}{2L_1} + \frac{1.2W}{2L_2} \right)$$

where L is the stretch in the direction of testing. The objective of the research therefore is to determine the form of the strain energy function that can best match the results derived from uniaxial stretching tests. In order to accomplish this objective, it is necessary to postulate a form of the strain energy function. There are as many forms of the strain energy function as there are hyperelastic materials and it is prudent to select some forms that have already been successfully applied to describe the mechanical behaviour of...
rubber-like materials. The experimental data derived from the uniaxial tests can now be used to identify the parameters characterizing the three strain energy functions. Considering the result (12) for the physical stress $\sigma_0$, we can write

$$\sigma_0 = \frac{2\lambda}{(\lambda-\lambda^2)} \left( \frac{\partial W}{\partial \lambda} + \frac{1}{\lambda} \frac{\partial W}{\partial \mu} \right)$$

(13)

The parameter identification for the neo-Hookean and the Mooney–Rivlin materials is relatively straightforward if the results for $\sigma_0/(2(\lambda-\lambda^2))$ are plotted against $1/\lambda$. For the Ogden model, the three parameters $(N, n, \mu)$ can be altered to obtain a match to the experiments results. Ideally a non-linear optimisation algorithm should be used to determine the best match. In this instance, the parameter $N$ is set to unity and the values of $n$ and $\mu$ varied over a range to achieve the best match. The Mooney-plot used to determine the parameters $C_1$ and $C_2$ are shown in Fig. 3. Mooney–Rivlin constants obtained from tests conducted on the two thicknesses of natural gum rubber are shown in Table 1. The values of $G$ for each thickness should theoretically be the same since both samples are from the same type of rubber. It was found from the uniaxial test results that the Mooney–Rivlin constants can vary; a small range of Mooney–Rivlin constants can exist for the material, as long as Eq. (12) is satisfied (Fig. 4).

![Fig. 3. Mooney plot from uniaxial stretching of a rubber membrane with $C_1=0.153$ MPa and $C_2=0.216$ MPa.](image)

![Fig. 4. Results for a range of Mooney–Rivlin constants.](image)

Table 1

<table>
<thead>
<tr>
<th>Specimen thickness (mm)</th>
<th>$C_1$ (MPa)</th>
<th>$C_2$ (MPa)</th>
<th>$G$ (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.793</td>
<td>0.281 ±</td>
<td>0.075 ±</td>
<td>0.712 ±</td>
</tr>
<tr>
<td>1.588</td>
<td>0.153 ±</td>
<td>0.216 ±</td>
<td>0.738 ±</td>
</tr>
</tbody>
</table>

large library of hyperelastic models designed for rubber materials, namely: polynomial, Mooney–Rivlin, reduced polynomial, neo-Hookean, Yeoh, Arruda-Boyce, Ogden and Van der Waals. The implementation of other versatile strain energy functions similar to the one proposed by Gent [10] is yet to be implemented. The Mooney–Rivlin and reduced polynomial models are particular forms of the polynomial model. The neo-Hookean and Yeoh models are special cases of the reduced polynomial model. The ABAQUS software allows the evaluation of the hyperelastic material behaviour by automatically creating response curves using selected strain energy functions. ABAQUS allows the input of either uniaxial tension data, or equi-biaxial extension data, or planar tension data, or any combination of these loading states to be used for parameter/model identification. Although the ABAQUS manuals suggest that the input of data obtained from multi-axial stress states will optimise the accuracy of hyperelastic model predictions, only uniaxial tension test data were used in this study to determine the material parameters characterizing the various forms of the strain energy functions. Each parameter was determined using the least squares method. Fig. 5 shows a comparison of the modelling of the tensile behaviour of the natural rubber material obtained using different forms of strain energy functions. Fig. 5 shows a comparison of the modelling of the tensile behaviour of the natural rubber material obtained using different forms of strain energy functions. Fig. 5 shows a comparison of the modelling of the tensile behaviour of the natural rubber material obtained using different forms of strain energy functions. Fig. 5 shows a comparison of the modelling of the tensile behaviour of the natural rubber material obtained using different forms of strain energy functions.
5. Fluid pressure loading of a circular membrane

To validate the results of the constitutive model development, a further experiment was conducted, where the rubber membrane was subjected to a multi-axial state of stress. The specific problem chosen for the experimental study is that of a natural rubber membrane that was fixed along a circular boundary and subjected to a fluid pressure applied in the transverse direction. The membrane loading experimental technique would induce strains in approximately the same range as those applied to determine the constitutive properties of the rubber.

5.1. The test facility

Photographic and schematic views of the experimental facility used to conduct fluid loading of the rubber membrane are shown in Fig. 6. The apparatus consists of a series of precision manufactured glass cylinders of internal diameter 15 cm, length 61 cm and wall thickness of 0.50 cm that are connected to form a one-dimensional column of height 260 cm. This test facility is described in detail by Selvadurai and Dong [61] who used the apparatus to examine fluid flow in porous media. The fluid used
in this experiment was tap water at approximately 24 °C. The membrane testing facility was adapted to apply fluid loading to a membrane that was fixed along a circular boundary. The rubber membranes used were 146 mm in diameter with thicknesses of 0.794 and 1.588 mm (Fig. 7a). The fixed boundary condition was achieved by clamping the membrane between Plexiglas and aluminum plates. The boundaries of the Plexiglas and aluminum plates were shaped to a circular cross section to minimize stress concentration along the clamped edge. Careful placement of the membrane was necessary to prevent both leakage and slippage. To prevent leakage between the glass column and the membrane, two rubber gaskets sandwiched the membrane and a thin aluminum plate was placed at the bottom; the membrane–plate assembly was secured with 8–32 screws. Furthermore, to prevent slippage, an additional layer of hard neoprene rubber was bonded to one side of the membrane (Fig. 7a), using a non-reactive adhesive. The schematic view of the clamped assembly and the fabricated rubber membrane specimen ready for assembly are shown in Fig. 7b. Observations indicated that this additional layer eliminated the slippage problem and prevented tearing of the rubber membrane at the clamped edge. Furthermore, since the preparation of the specimen and the duration of the experiment are relatively short, the possibility of any long-term chemical reaction between the adhesive and the rubber membrane were neglected.

5.2. Experimental results

The measurement of the deflected shape of the rubber membrane for different pressures was the main focus of this experimental research, and results would then be validated using a finite element simulation. The deflected profile was examined by applying a fluid load in an incremental manner until the rubber membrane reached a maximum strain of approximately 100%. The deflected profile was measured using an optical technique. Using a 5 Mega pixel digital camera, a photographic record of the deflected profile was captured for a specific pressure. The distance of the camera to the test specimen does not require a fixed location since in the data extraction procedure of the visual images are related to a distance in image pixels rather than an actual physical distance. The central optical axis of the camera, however, was aligned and normal to the datum of the object. In the experiments, the datum was taken as the midpoint of the thickness of the rubber membrane. The camera was positioned to capture a representative focused image through an azimuthal plane through the axis of symmetry of the membrane. To obtain the deformation at the central deflection, the image pixels were calibrated against a known physical distance, which in this case was the diameter of the aluminum plate. The physical deflection
of the membrane can then be determined by converting between the image pixels and the known distance. This method was found to minimise any parallax or barrel distortion. The image resolution is an important factor for the accuracy of the data. An image resolution of 2304 x 1728 pixels was used and gave accurate results. During the experiment, the images of the deflected profile were recorded for different water levels. Three experiments were performed for each rubber thickness. Water was added to the column in an incremental manner until a central maximum displacement of $D_{\text{max}} = 63.9$ mm was obtained for the 0.793 mm thick membrane and a displacement of $D_{\text{max}} = 51.5$ mm was obtained for the 1.588 mm thick rubber membrane. The fluid pressure vs. displacement at the centre of the membrane obtained for a series of tests conducted on the membranes of two thickness are shown in Fig. 8. The strain in the membrane at different pressures was measured using the same optical technique described earlier. A grid was drawn on the rubber membrane using a fine black marker with spacings of 10 mm between gridlines. As the membrane stretched, the position of the gridlines was used to calculate the strain at different pressures. It was found that the central maximum displacement corresponds approximately to an overall average strain of 113% in the radial direction for the membrane of thickness 0.793 mm and an overall average strain of 70% in the radial direction for the membrane of thickness 1.588 mm. The results show a good repeatability between the sets of experiments conducted on both thin and thick membranes.

6. Computational modelling of the fluid loading

A finite element approach was used to analyse the hyperelastic behaviour of the natural gum rubber membrane subjected to fluid pressure. The modelling used the finite element software ABAQUS. This commercial finite element code contains many features related to the modelling of large strain phenomenon, including the possibility of implementing a chosen constitutive model into the computational algorithm. This programme has been extensively used.
to study engineering applications related to hyperelastic materials. A comprehensive study is also given by Selvadurai [42]. The objective here is to model a rubber membrane that is subjected to different pressures, observe its central deflection and to compare the computational predictions with experimental observations. The circular rubber membrane was modelled as a three-dimensional domain. Since the thickness of the specimen is small compared to its radius and since the fluid pressure loading transforms the flat membrane into a nearly spherical shell, a three-dimensional deformable shell element was used to create the model. Although due to the symmetry of the structure, a two-dimensional analysis could be conducted, a three-dimensional model gives a better overall visual representation of the deformed shape at different pressures. Also, since both the two-dimensional and the three-dimensional model gave the same results for the deflected shape and since the mesh generation and computing time were only marginally increased, the three-dimensional modelling was used.

Membrane elements were chosen to model the circular rubber membrane. The use of membrane elements is appropriate for non-linear analyses; these elements represent thin surfaces that transmit forces in its plane but have no bending stiffness. ABAQUS offers a number of membrane elements in its element library, each of which is placed in three different categories: general membrane elements, which include both triangular and quadrilateral types elements, cylindrical membrane elements and axisymmetric membrane elements. General membrane elements are used in most three-dimensional models in which the deformations of the region can evolve in three dimensions. Axisymmetric membrane elements allow torsional loading and general material anisotropy. The membrane model used in this study consisted of M3D8R elements—quadrilateral, 8-noded with reduced integration. Each node has three displacement and three rotational degrees of freedom. Either triangular or quadrilateral elements could be used for the analysis of the membrane. The choice depends upon the complexity of the structure, the accuracy of the results required as well as minimising computing time. Since the membrane has a very simple geometry, it was found that the computing time was virtually independent of the choice of the element. Therefore, quadrilateral elements were chosen over triangular elements since they have better convergence properties and are relatively insensitive to mesh orientation. The second-order form of the quadrilateral elements was

![Diagram of the rubber membrane](image)

**Fig. 9.** Computational results for the deflection of the rubber membrane. (a) Fluid loading on rubber membrane. (b) Mesh configuration and boundary conditions (total number of elements: 840 and 2287, for membranes of thickness 0.793 mm and 1.588 mm, respectively). (c) Deformed shape during maximum loading ($\Delta_{\text{max}}=63.9$ mm and $\Delta_{\text{max}}=51.5$ mm for the 0.793 and 1.588 mm thick rubber membranes, respectively).
selected because it provides higher accuracy for problems that do not involve complex contact conditions or severe element distortions in the analysis. Second-order elements have extra mid-side nodes in each element making computation of both small and finite deformations more effective. The reduced-integration option is preferred for quadratic elements because it uses a lower-order integration to produce the element stiffness and decreases the computing time of an analysis, especially in three dimensions. Also, since the accuracy of the analysis is of prime importance, second-order reduced integration elements were chosen because this typically yields more accurate results than the first order elements. Solid (continuum) elements could also be used to model the rubber membrane. Simulations indicated a less than 0.3% difference in the results for the maximum deflection of the rubber membrane if either the solid or membrane element type was chosen. Solid elements have a hybrid incompressible formulation (ABAQUS [59]) since they are intended primarily for use with incompressible and almost incompressible materials. For nearly incompressible cases (where Poisson’s ratio, ν, is greater than 0.4999999), the material will produce extremely large changes in pressure for a very small change in displacement. Hybrid elements treat the isotropic pressure $p$ (Eq. (2)) as an independently interpolated basic solution variable and couples it to the displacement. ABAQUS recommends the use of hybrid solid elements for hyperelastic materials. A convergence study was conducted to determine if the mesh size of the final model provided accurate results and whether or not there should be a mesh refinement or coarser meshes should be used to reduce the computing time during analysis. Since the default mesh size in ABAQUS was five, it was expected that no further refinement was needed. The maximum deflection at the central point of the membrane was computed for different mesh sizes. The number of elements used for the membrane ranged from 42 to 20,266. Based on the results, the model which consisted of 840 and 2287 elements for the membranes of thickness 0.793 and 1.588 mm, respectively, provided satisfactory accuracy.

6.1. Computational results

The schematic view of the rubber membrane problem under fluid loading is illustrated in Fig. 9a. The boundary conditions and finite element discretization used in the computational modelling are shown in Fig. 9b. The boundary condition of the membrane is

![Fig. 10. Variation in the central deflection of the membrane with fluid pressure—results for a natural gum rubber membrane with a strain energy function of the Mooney–Rivlin form.](image)

![Fig. 11. Variation in the central deflection of the membrane with fluid pressure—results for a natural gum rubber membrane with a strain energy function of the neo-Hookean form.](image)

![Fig. 12. Variation in the central deflection of the membrane with fluid pressure—results for a natural gum rubber membrane with a strain energy function of the Ogden form.](image)

![Fig. 13. Variation in the central deflection of the membrane with fluid pressure—results for a natural gum rubber membrane with a strain energy function of the Yeoh form.](image)
fixed; i.e. neither a displacement nor a rotation is allowed. Uniform pressure is applied on the membrane surface for pressures up to 13.7 and 24.5 kPa for the membranes of thickness 0.793 and 1.588 mm, respectively. Fig. 10 illustrates a comparison between the fluid pressure vs. central displacement of the membrane, obtained for the Mooney–Rivlin material, the material parameters of which are described in Fig. 5a. A constant pressure was assumed to act on the membrane. It was found that the influence of the weight of the fluid in the deformed region can essentially be neglected; the difference between the real and the assumed pressure corresponds to a discrepancy of approximately 2%, which falls within the overall range of accuracy of the tests. Analogous results that show comparisons between the experimental results and the computational predictions made by choosing respectively the neo-Hookean, Ogden and Yeoh forms of the strain energy functions are given in Figs. 11–13. A summary of the results for the variation in the central deflection with fluid pressure is shown in Fig. 14. Identical computations and comparisons were performed with results derived from experiments conducted on the membrane of thickness 1.588 mm. These results are presented in Fig. 15–19. It is evident that due to the presence of nominal hysteresis in the membranes, the error between the computational and experimental results is increased for the unloading portion. Overall, the computational results provide a comparable trend and a satisfactory prediction of the deflection of the membrane determined from the experiments.
form the strain energy function can adequately model the mechanical behaviour of the membrane at moderately large strains (i.e. $\varepsilon_0 = 65\%$), and that computational simulations indicate that the Mooney–Rivlin form of the strain energy function can adequately predict the experimental response of membranes that experience strains of up to 65% during fluid loading. The strain energy function proposed by the Ogden model is more versatile and able to match the observed behaviour over a wider range of strains. In this regard, other recent developments in this area including the strain energy function proposed by Gent merits further investigation. Constitutive model development is an important aspect of the characterisation of rubber-like materials that exhibit hyperelasticity. As has been observed, the ability to duplicate the one-dimensional experiment is a necessary condition for the acceptability of a form of a strain energy function. It is shown that in the moderately small range of strains, most strain energy functions can provide a reasonable match to the experimental data on membrane deflections. A conclusive proof of the applicability of a particular form of a strain energy function also requires the model to be able to predict the performance of a completely different experimental configuration that preferably involves inhomogeneous strain fields.

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References
