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ON THE DISPLACEMENT OF A PENNY-SHAPED RIGID INCLUSION EMBEDDED IN A TRANSVERSELY ISOTROPIC ELASTIC MEDIUM

A.P.S. Selvadurai
Department of Civil Engineering
Carleton University
Ottawa, Canada

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INTRODUCTION

This paper examines the axisymmetric problem related to the displacement of a penny-shaped rigid inclusion embedded in bonded contact with a transversely isotropic elastic medium of infinite extent. The solution of this problem is achieved by employing the classical integral transform formulation which reduces the problem to a system of dual integral equations. Explicit results are derived for the load-displacement relationship for the penny-shaped rigid inclusion.

ANALYSIS

The class of problem which examines the behaviour of disc-shaped inclusions embedded in elastic media has received considerable attention. The solution to the problem of a thin rigid circular disc embedded in an infinite isotropic elastic solid and subjected to a constant displacement normal to its plane was examined by Collins [1]. Keer [2] has considered a similar problem in which the bonded disc is displaced in its own plane. Kassir and Sih [3] subsequently extended these investigations to problems relating to elliptical disc inclusions. The class of problem in which separation or debonding phenomena occur at the elastic medium-circular inclusion interface is examined in the informative papers by Hunter and Gamblen [4] and Keer [5]. Also, solutions to problems associated with circular inclusions embedded in elastic media may be recovered as limiting cases of results developed for certain ellipsoidal and spheroidal rigid inclusion problems [6-8]. This paper examines the axisymmetric problem of the displacement of a penny-shaped rigid circular inclusion embedded in bonded contact with a transversely isotropic infinite elastic medium. The plane of the penny-shaped inclusion is assumed to coincide with the plane of transverse isotropy. In essence, this paper generalizes Collins' [1] result to include effects of transverse isotropy. In contrast to the complex potential function formulation adopted in [1], here we make use of the antisymmetry of the disc inclusion problem to reduce it to a mixed boundary value problem associated with a halfspace region. Using a Hankel transform development, this mixed boundary value problem is further reduced to a pair of dual integral equations, the solution of which is readily obtainable from the generalized results given by Titchmarsh [9] and Sneddon [10,11].

The load-displacement relationship for the bonded rigid disc inclusion is obtained in exact closed form. This result is of importance in connection with the analysis of the translational stiffness of bonded rubber mountings or in the geotechnical study of foundations embedded in soil and rock media.

FUNDAMENTAL FORMULAE

The methods of analysis of three-dimensional problems in transversely isotropic elastic materials make extensive use of the potential function techniques proposed by Elliott [12,13] and Lekhnitskii [14]. Complete accounts of these developments are given by Green and Zerna [15], Kassir and Sih [16] and Eubanks and Sternberg [17]. It can be shown that in the absence of body forces, the axisymmetric displacement and stress fields can be expressed in terms of two harmonic functions $\phi_1(r,z)$ and $\phi_2(r,z)$ which are solutions of

$$\left\{\nabla_1^2 + \frac{\partial^2}{\partial z_i^2}\right\} \phi_i(r,z) = o \quad ; \quad (i = 1,2) \tag{1}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad ; \quad z_i = \frac{z}{\sqrt{\nu_i}} \tag{2}$$

and ν_1 and ν_2 are roots of the equation

$$c_{11}c_{44}\,\nu^2 + \left[c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}\right]\nu + c_{33}c_{44} = o \tag{3}$$

The cylindrical polar coordinate system (r,θ,z) is chosen such that the z-axis is parallel to the material axis of symmetry. The roots ν_i may be real or complex depending upon the elastic constants c_{11}, c_{12}, c_{13}, c_{33} and c_{44}. The displacement and stress fields in the transversely isotropic elastic material can be represented in terms of the harmonic functions $\phi_i(r,z)$. For axial symmetry the displacement and stress components reduce to the forms

$$u_r = \frac{\partial}{\partial r}\{\phi_1 + \phi_2\} \quad ; \quad u_z = \frac{\partial}{\partial z}\{k_1\phi_1 + k_2\phi_2\} \tag{4}$$

and

$$\sigma_{rr} = \left[c_{11}\frac{\partial^2}{\partial r^2} + \frac{c_{12}}{r}\frac{\partial}{\partial r}\right](\phi_1+\phi_2) + c_{13}\frac{\partial^2}{\partial z^2}\{k_1\phi_1 + k_2\phi_2\},$$

$$\sigma_{\theta\theta} = \left[c_{12}\frac{\partial^2}{\partial r^2} + \frac{c_{11}}{r}\frac{\partial}{\partial r}\right](\phi_1+\phi_2) + c_{13}\frac{\partial^2}{\partial z^2}\{k_1\phi_1 + k_2\phi_2\} ,$$

$$\sigma_{zz} = \{k_1 c_{33} - \nu_1 c_{13}\} \frac{\partial^2 \phi_1}{\partial z^2} + \{k_2 c_{33} - \nu_2 c_{13}\} \frac{\partial^2 \phi_2}{\partial z^2} \ ,$$

$$\sigma_{rz} = c_{44} \left[(1 + k_1) \frac{\partial^2 \phi_1}{\partial r \partial z} + (1 + k_2) \frac{\partial^2 \phi_2}{\partial r \partial z} \right] \qquad (5)$$

respectively, where k_1 and k_2 are given by

$$k_i = \frac{c_{11}\nu_i - c_{44}}{c_{13} + c_{44}} \quad ; \qquad i = 1,2 \qquad (6)$$

THE PENNY-SHAPED INCLUSION PROBLEM

We consider the axisymmetric problem related to a penny-shaped rigid inclusion embedded in bonded contact with a transversely isotropic elastic medium of infinite extent. The disc (radius a) is subjected to a total load P which acts in the z-direction and the resulting rigid body displacement of the disc is denoted by δ (Figure 1). It is evident that the inclusion problem thus formulated is antisymmetric in normal stress σ_{zz} and radial displacement u_r, about the plane $z = o$. We may therefore restrict the analysis to a halfspace region of the transversely isotropic elastic infinite medium, in which the plane $z = o$ is subjected to the mixed boundary conditions

$$u_r(r,0) = 0 \quad ; \qquad r \geq 0 \qquad (7a)$$

$$u_z(r,0) = \delta \quad ; \qquad 0 \leq r < a \qquad (7b)$$

$$\sigma_{zz}(r,0) = 0 \quad ; \qquad a \leq r \leq \infty \qquad (7c)$$

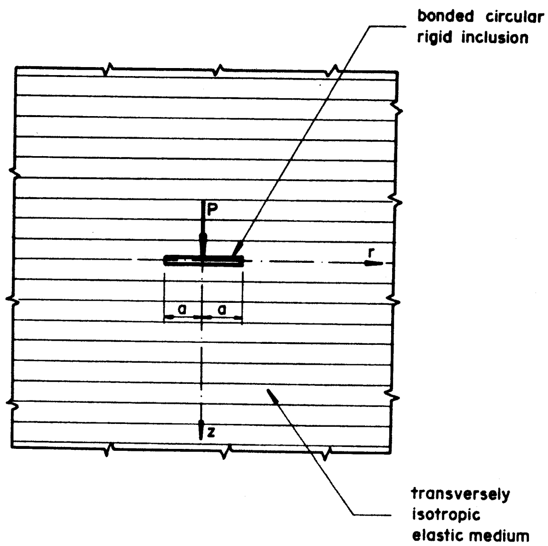

Figure 1

We shall restrict our attention to the halfspace region $z \geq o$. Following Sneddon [11], we introduce the zero order Hankel transform of $\phi_i(r,z)$ as follows:

$$\bar{\phi}_i^o\,(\xi,z) = H_o\{\xi;\ \phi_i(r,z)\} = \int_o^\infty r\ \phi_i(r,z)\ J_o(\xi r/a)\ dr\ . \tag{8a}$$

The appropriate Hankel inversion theorem is

$$\phi_i(r,z) = H_o^{-1}\ \{r;\ \bar{\phi}_i^o\ (\xi,z)\} = \frac{1}{a^2}\int_o^\infty \xi\ \bar{\phi}_i^o(\xi,z) J_o(\xi r/a) d\xi . \tag{8b}$$

Operating on (1) with the zero order Hankel transform we obtain a second-order ordinary differential equation for $\bar{\phi}_i^o(\xi,z)$; choosing the solution appropriate for the region $z \geq o$ it can be shown that $\phi_i(r,z)$ take the form

$$\phi_i(r,z) = \frac{1}{a^2}\int_0^\infty \xi\, A_i(\xi)\, e^{-\lambda_i z}\, J_o(\xi r/a)\,d\xi \;;\; (i = 1,2) \;, \tag{9}$$

where $A_i(\xi)$ are arbitrary functions and $\lambda_i = \xi/a\sqrt{\nu_i}$. From (4) and (9) it is evident that, in order to satisfy the boundary condition (7a), we require

$$A_1(\xi) = -\,A_2(\xi) \;\; (= A(\xi)) \quad . \tag{10}$$

By making use of the above result in the general expression (9) and with aid of the expressions for $u_z(r,z)$ and $\sigma_{zz}(r,z)$ given by (4) and (5) it can be shown that the boundary conditions (7b) and (7c) are equivalent to

$$-\frac{\{k_1\sqrt{\nu_2}-k_2\sqrt{\nu_1}\}}{\sqrt{\nu_1\nu_2}\;\;a^3}\int_0^\infty \xi^2 A(\xi) J_o(\xi r/a)\,d\xi = \delta \;;\quad \text{for } r \le a \tag{11a}$$

$$\int_0^\infty \xi^3\, A(\xi) J_o(\xi r/a)\,d\xi = o \;;\; \text{for } r \ge a \quad . \tag{11b}$$

Introducing the substitutions

$$w_o = -\frac{\delta a^3\sqrt{\nu_1\nu_2}}{\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}} \;;\; \rho = \frac{r}{a} \;;\; \xi^2 A(\xi) = C(\xi) \;, \tag{12}$$

the equations (11) can be reduced to the following pair of dual integral equations:

$$\int_o^\infty C(\xi)\; J_o(\xi\rho)\; d\xi = w_o \quad ; \quad o \le \rho \le 1$$

$$\int_o^\infty \xi\; C(\xi)\; J_o(\xi\rho)\; d\xi = o \quad ; \quad \rho \ge 1 \quad . \tag{13}$$

The solution of this system of dual integral equations is given by Sneddon [10,11] and the details of the method of solution will not be pursued here. It can be shown that the general solution of (13) is

$$C(\xi) = \frac{2w_o \sin \xi}{\pi \xi} = \xi^2 A(\xi) \quad . \tag{14}$$

Formal integral expressions for the displacement and stress fields in the halfspace region $z \geq o$ can be determined by making use of equations (4), (5), (9) and (14).

LOAD-DISPLACEMENT RELATIONSHIP FOR THE PENNY-SHAPED INCLUSION

In this note we are primarily interested in establishing the load-displacement relationship for the disc inclusion embedded in bonded contact with the transversely isotropic elastic medium. Using the results derived in the previous section, it can be shown that the normal stress σ_{zz} acting on the plane $z = o$ is given by

$$\sigma_{zz}(r,o) = -\frac{2c_{33}\{k_1\nu_2 - k_2\nu_1\}\delta}{\pi a\sqrt{\nu_1\nu_2}\,\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}} \int_o^\infty \sin \xi \, J_o(\xi r/a)\,d\xi \tag{15}$$

From Erdelyi *et al*, [18] the value of $\sigma_{zz}(r,o)$ given above reduces to the following:

$$\sigma_{zz}(r,o) = \begin{cases} \dfrac{2c_{33}\{k_1\nu_2 - k_2\nu_1\}\delta}{\pi\sqrt{\nu_1\nu_2}\,\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}\,\sqrt{a^2 - r^2}} & ;\ r \leq a \\[2ex] o & ;\ r > a \quad . \end{cases} \tag{16}$$

By considering the behaviour of the halfspace region $z \leq o$ a result similar to (16) can be obtained for the normal contact stress distribution at the plane surface of the circular region $z = 0-$. The normal stress component on the penny-shaped inclusion is given by

$$[\sigma_{zz}]_{z = 0^{\pm}} = \pm \frac{2c_{33}\{k_2\nu_1 - k_1\nu_2\}\delta}{\pi\sqrt{\nu_1\nu_2}\,\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}\,\sqrt{a^2 - r^2}} \quad , \tag{17}$$

where the upper and lower signs refer to the plane faces $z = 0^+$ and $z = 0^-$ of the inclusion. The force exerted by the transversely isotropic medium on the penny-shaped inclusion is given by

$$P = 2\pi \int_0^a r\left[(\sigma_{zz})_{0+} - (\sigma_{zz})_{0-}\right] dr \quad . \tag{18}$$

Evaluating (18) we obtain the load-displacement relationship for the penny-shaped rigid inclusion as

$$P = \frac{8c_{33}\,\delta a\,\{k_2\nu_1 - k_1\nu_2\}}{\sqrt{\nu_1\nu_2}\,\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}} \quad . \tag{19}$$

In the limiting case when $\nu_1, \nu_2 \to 1$, we recover from (19) the solution to the problem of a penny-shaped inclusion embedded in an isotropic elastic medium. We note that as $\nu_1, \nu_2 \to 1$,

$$\frac{k_2\nu_1 - k_1\nu_2}{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}} = -\frac{2c_{44}}{c_{11} + c_{44}} \quad , \tag{20a}$$

where

$$c_{11} = c_{33} = (\lambda + 2\mu) \quad ; \quad c_{44} = \mu \tag{20b}$$

and λ, μ are Lame's constants for the isotropic elastic material. Making use of these results in (19) we obtain the following result for the load-displacement relationship for a rigid penny-shaped inclusion embedded in bonded contact with an isotropic elastic infinite medium:

$$P = -\frac{32\delta\ \mu a\ (1 - \nu)}{(3 - 4\nu)} \tag{21}$$

This relationship is in agreement with the results obtained by Collins [1], Kanwal and Sharma [7] and Selvadurai [8] for the penny-shaped inclusion problem, by making use of complex potential function techniques, singularity methods and direct spheroidal harmonic function techniques, respectively.

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