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Macroscopic constitutive equations of thermo-poroelasticity derived using eigenstrain–eigenstress approaches

Alexander P. Suvorov and A.P.S. Selvadurai*

Department of Civil Engineering and Applied Mechanics, McGill University, QC, Canada H3A 2K6

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Macroscopic constitutive equations for thermoelastic processes in a fluid-saturated porous medium are re-derived using the notion of eigenstrain or, equivalently, eigenstress. The eigenstrain-stress approach is frequently used in micromechanics of solid multi-phase materials, such as composites. Simple derivations of the stress–strain constitutive relations and the void occupancy relationship are presented for both fully saturated and partially saturated porous media. Governing coupled equations for the displacement components and the fluid pressure are also obtained.

Keywords: fluid-saturated media; constitutive modeling; eigenstrain and eigenstress approaches; thermo-poroelastic models; unsaturated poroelastic media; homogenization techniques; effective poroelastic parameters

1. Introduction

The theory of poroelasticity can be regarded as an extension of the classical theory of elasticity for a continuum, to account for a continuous pore structure in the medium as well as the presence of fluids occupying the pore space. The classical theory in its complete form was proposed by Biot [1] and formed the basis for the theory of consolidation of a fluid-saturated soil with an elastic skeleton. Reviews of the subject and recent developments in the area are given by Scheidegger [2], Paria [3], Rice and Cleary [4], Selvadurai [5,6], Coussy [7], Verruijt [8] and Dormieux et al. [9], with applications in the areas of bone mechanics [10,11], soft tissue mechanics and indentation problems [12,13], energy resources extraction and recovery [14] and geoenvironmental applications related to geologic disposal of heat emitting nuclear fuel wastes [15–17].

In this paper, the equations governing the thermo-poroelastic response of a porous medium, with either a saturated or an unsaturated pore space, are derived by considering approaches used in mechanics of solid heterogeneous materials. In the application of the theory of heterogeneous materials to thermo-mechanics of poroelastic media, we consider two distinct scales: a microscale, over which the phases are alternated (or periodic), and the macroscale, which is comparable to the overall dimensions of the region of interest. The fields within the heterogeneous body,

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*Corresponding author. Email: patrick.selvadurai@mcgill.ca

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such as temperature or stress, usually exhibit rapid fluctuations over the microscale. Homogenization procedures enable the smoothing of these fluctuations but also allow for slow variations of the fields on a macroscale. The objective of the homogenization procedure is to determine the effective properties of the heterogeneous material, i.e. those properties that characterize the overall behavior or response of the material region to external influences, on a macroscale, and to formulate the relevant effective constitutive relations (i.e. the relationships between the macroscopic fields). There are numerous methods for determining the effective properties and examples are given in various articles and texts [18–23]. To an observer, a heterogeneous body will appear as a homogeneous body that has variations of the fields only on the macroscale. The homogenized problem can be solved more conveniently since it invariably involves a smaller number of degrees of freedom. The notion of effective properties of a heterogeneous porous medium also finds useful application in the modeling of permeability [24].

This paper specifically deals with heterogeneous materials that have a solid but porous deformable skeleton through which the liquid phase can move; rocks and soils are typical examples of such heterogeneous materials. Effective or macroscopic constitutive relations for soils were first formulated by Biot [1]. This theory was subsequently extended by several researchers to include the effects of compressibility of both the fluid and solid phases [4]. Other extensions of the theory include thermal effects, as given in several articles [16,25–27], where the constitutive relations for the thermo-poroelastic response of the medium, including the thermal expansion of both the solid grains and the liquid, are taken into consideration. Rutqvist et al. [28] presented a comparative study of four theories for partially saturated geomaterials containing both liquid and air.

The objective of this paper is to link micromechanics (used primarily in studies related to solid heterogeneous materials such as composites) with the macroscopic theory of poroelasticity and to re-derive the constitutive relations for the thermo-poroelastic behavior of a fluid-saturated medium. The approach we have chosen is different from those presented in classical studies [4,16,27,29,30]. It relies on a combination of the definitions of eigenstresses or eigenstrains, as defined by Mura [31] and Dvorak et al. [32], and the general principles of micromechanics. A similar approach was used by Chateau and Dormieux [33], who derived the equations of poroelasticity by treating the fluid pressure as a prestress applied to the skeleton and using Levin's formula [34] to evaluate the contribution of this prestress to the overall stress. This methodology can be extended or modified if there are contributions to the eigenstresses or eigenstrains resulting from processes other than thermal effects. Examples of these can include swelling, loss or gain of mass, eigenstrains associated with changes to the molecular structure of the phases or phase transformations and inelastic deformation.

The paper is organized as follows. Section 2 presents classical macroscopic constitutive equations for a fully-saturated thermo-poroelastic continuum. Section 3 gives, for completeness, a summary of the eigenstrain-stress approach. The derivation of the constitutive equations for a partially saturated porous medium based on the eigenstrain-stress approach is presented in Section 4. Section 5 contains an example of a one-dimensional column of a porous linear elastic material subjected to temperature change and surface loads at the boundary.

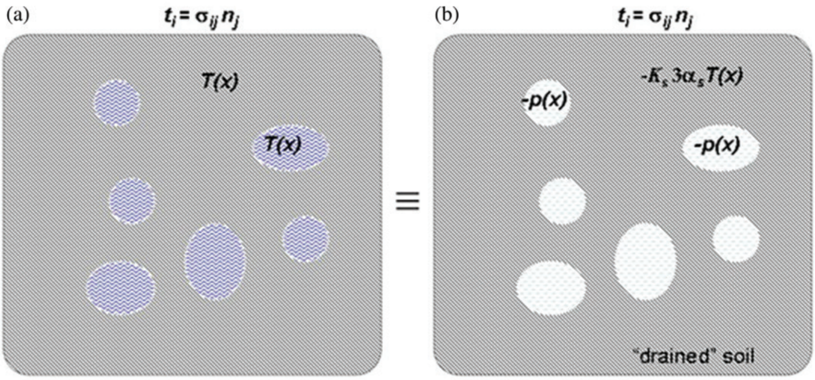

Figure 1. Fully saturated porous medium subjected to tractions t_i on the external boundary and temperature change T. By applying the eigen-fields to the drained soil (b), we can simulate the behavior of the porous medium under general, not necessarily drained conditions, as shown in (a).

2. Constitutive equations for thermo-poroelasticity

Consider a poroelastic body consisting of a porous fabric composed of an isotropic linear elastic solid and pores which contain a non-viscous liquid. The solid phase (i.e. the material composing the porous medium) and liquid are assumed to be compressible. We first consider the case where the continuous pore space of the medium is fully saturated with the liquid. The porosity of the medium (the ratio of the volume of the pores to the total volume) is denoted by n. Assume that this porous medium is subjected to heating (or cooling) and mechanical loads (Figure 1a). The constitutive relations that govern the behavior of such a porous medium at the macroscopic level will be presented in this section, without derivation.

Unlike composite materials, where the constitutive relation is written in terms of the elastic properties of the separate phases, the constitutive relationship for poroelastic materials is expressed in terms of the elastic constants of the drained fabric (i.e. in the absence of any interaction between the pore fluids and the porous skeleton). *Definition*: under drained conditions, the fluid is allowed to flow in and out of the soil, but the fluid pressure remains independent of time [4]. For example, fully drained conditions can prevail when the fluid pressure dissipates with time and eventually reaches zero. The effective mechanical properties and effective thermal expansion coefficient of such a "drained geomaterial" will be equivalent to those of the soil skeleton with empty pores (note, however, that even when pressure is zero, the liquid is present in the pore space since the soil is assumed to remain fully saturated).

For a fully saturated thermoelastic geomaterial (under general, not necessarily drained conditions), the macroscopic thermo-poroelastic relation is given by Rice and Cleary [4] and Savvidou and Booker [26]:

$$\sigma_{ij} = 2G_D\varepsilon_{ij} + \left(K_D - \frac{2G_D}{3}\right)\varepsilon_V\delta_{ij} - 3K_D\alpha_s T\delta_{ij} - \left(1 - \frac{K_D}{K_s}\right)p\delta_{ij}, \tag{1}$$

where σ_{ij} is the total stress tensor; ε_{ij} is the strain tensor; $\varepsilon_V = \varepsilon_{kk}$ is the volumetric strain; p is the fluid pressure; K_D, G_D are the bulk and shear moduli, respectively, of the geomaterial under drained conditions; K_s is the bulk modulus of the solid phase; α_s is the coefficient of linear thermal expansion of the solid phase and T is the temperature change. All field quantities (stresses, strains, pressure and temperature) in (1) are, in general, functions of coordinates $\mathbf{x}\,(= x_1, x_2, x_3)$ and time.

The last term in (1) represents the contribution of the fluid pressure to the total stress. The multiplier in front of the pressure is referred to as the Biot parameter:

$$\alpha = 1 - \frac{K_D}{K_s}. \tag{2}$$

The second constitutive relation is sometimes referred to as the void occupancy equation. The initial mass of fluid in the pore space is denoted by m_f, and the change in the mass of fluid is denoted by Δm_f. Then, the void occupancy equation can be expressed as [4]

$$n\frac{\Delta m_f}{m_f} - \alpha\left(\varepsilon_V + \frac{\alpha}{K_u - K_D}p\right) + n3\alpha_f T + (\alpha - n)3\alpha_s T = 0, \tag{3}$$

where K_u is the bulk modulus of the geomaterial under undrained conditions; α_s, α_f are coefficients of thermal expansion of solid grains and fluid, respectively; n is the porosity. Rice and Cleary [4] define K_u as follows:

$$K_u = K_D + \frac{\alpha^2 K_s K_f}{nK_s + (\alpha - n)K_f}, \tag{4}$$

where K_s, K_f denote the bulk modulus of the solid phase and liquid, respectively. *Definition*: during undrained deformations of the porous medium, the fluid is constrained from flowing in and out, giving the constraint $\Delta m_f = 0$, but the changes in the fluid pressure p can be induced. Substitution of (4) into (3) gives another form of the void occupancy equation:

$$-n\frac{\Delta m_f}{m_f} + \alpha\varepsilon_V + \frac{n}{K_f}p + \frac{\alpha - n}{K_s}p = n3\alpha_f T + (\alpha - n)3\alpha_s T. \tag{5}$$

The movement of the fluid phase within the pore space is described by Darcy's Law. For a porous medium with an isotropic pore structure [23], Darcy's Law can be written as

$$n v_i = -\frac{k}{\gamma}p_{,i}, \tag{6}$$

where v_i are components of the fluid velocity vector; k is the fluid conductivity and γ is the unit weight of the liquid. Both these properties are interpreted in an averaged sense and more sophisticated estimates can be derived for these parameters that take into consideration distribution of properties [24]. Equation (6) can be made more precise if the velocity of the fluid in relation to the velocity of the porous skeleton is used; however, in most practical applications, the velocity of the porous solid can be neglected in relation to the pore fluid velocity.

The process of heat transfer in the fluid-saturated porous medium is assumed to be dominated by heat conduction, governed by Fourier's law:

$$h_i = -k_c^* T_{,i}, \tag{7}$$

where $h_i = -k_c^* T_{,i}$ are components of the heat flux vector, and k_c^* is the effective thermal conductivity. Effective thermal properties, such as k_c^*, can be found for the porous medium filled with the liquid since the pore space is assumed to remain fluid-saturated, even under drained conditions.

3. Summary of the eigenstrain–eigenstress approach

The macroscopic constitutive relations (1) and the void occupancy equation (5) can be derived using the eigenstrain–eigenstress approach used quite extensively in the study of solid heterogeneous materials such as composites [18–20,31]. To better understand this derivation, and for completeness, a summary of the approach will be presented in this section. For simplicity, consider first a homogeneous linear elastic solid, where the strain (ε_{ij}) and stress (σ_{ij}) components are related by the constitutive equations:

$$\sigma_{ij}(x) = L_{ijkl}\varepsilon_{kl}(x) + \lambda_{ij}(x); \quad \varepsilon_{ij}(x) = M_{ijkl}\sigma_{kl}(x) + \mu_{ij}(x), \tag{8}$$

where L_{ijkl} is the elastic stiffness tensor; M_{ijkl} is the elastic compliance tensor; λ_{ij} is the eigenstress and μ_{ij} is the eigenstrain.

For an isotropic elastic body, the non-zero components of the elastic stiffness tensor can be expressed in terms of two elastic moduli: the bulk modulus K and the shear modulus G: i.e.

$$\begin{aligned} L_{1111} &= L_{2222} = L_{3333} = K + 4G/3 \\ L_{1122} &= L_{1133} = L_{2233} = K - 2G/3 \\ L_{1212} &= L_{2323} = L_{1313} = G. \end{aligned} \tag{9}$$

From energy conservation and the resulting elastic symmetries [35,36], it follows that $A_{ijkl} = A_{klij} = A_{jikl} = A_{ijlk}$, where A can be either M or L. For example, for the elastic stiffness tensor $L_{2211} = L_{1122}$ and $L_{1212} = L_{2112} = L_{1221} = L_{2121}$. The components of the stiffness and compliance tensors are related by the reciprocal relation:

$$M_{ijmn}L_{mnkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}). \tag{10}$$

It is evident from (8) that the eigenstress λ_{ij} can be defined as the stress that exists when all displacements and, thus, all the strains in the body are zero. Similarly, the eigenstrain μ_{ij} can be defined as the strain that exists when all stresses are zero. In a more likely situation where the stresses are non-zero, the eigenstrain is the difference, i.e. $(\varepsilon_{ij} - M_{ijkl}\sigma_{kl})$. Similarly, if the strains are not zero, the eigenstress is $(\sigma_{ij} - L_{ijkl}\varepsilon_{kl})$. From (8) and the reciprocal relation (10), it can be shown that the eigenstrain and eigenstress are related by the following:

$$\lambda_{ij} = -L_{ijkl}\mu_{kl}; \quad \mu_{ij} = -M_{ijkl}\lambda_{kl}. \tag{11}$$

The most easily recognisable example of an eigenstrain is thermal expansion (or contraction) $\alpha_{ij}T$, where α_{ij} are the linear thermal expansion coefficients, and T is the relative temperature change. If the stresses were zero within a body, the strains would be equal to the eigenstrain $\alpha_{ij}T$. Similarly, if the body were totally constrained from deformation, the strains would be zero within it, and the stress would be equal to the eigenstress $-L_{ijkl}\alpha_{kl}T$. Other eigen-fields can include actions due to swelling, removal or addition of mass and changes to the state of a body (e.g. freezing of water in the pore space).

From (8) it follows that if the deformability characteristics of the elastic body are small (i.e. elastic moduli $L_{ijkl} \to 0$), then the stress existing in the body is the eigenstress. Similarly, if the body is very rigid (i.e. $M_{ijkl} \to 0$), then the strain existing in the body is the eigenstrain.

Consider now a heterogeneous elastic body consisting of two homogeneous phases. Assume that the scale over which the phases are alternated is much smaller than the overall dimensions of the body. The stiffness and compliances of these phases are denoted by $L^{(1)}_{ijkl}$, $L^{(2)}_{ijkl}$ and $M^{(1)}_{ijkl}$, $M^{(2)}_{ijkl}$, respectively. Assume that the volume fraction of the first phase is n, giving the volume fraction of the second phase as $(1-n)$. A porous medium with either empty pores or with pores filled with liquid is a typical example of such a heterogeneous body. On a microscopic level, the constitutive equations at point $\mathbf{x}$ can be written as

$$\begin{aligned}\sigma_{ij}(\mathbf{x}) &= L_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}) + \lambda_{ij}(\mathbf{x}); \quad L_{ijkl}(\mathbf{x}) = L^{(1)}_{ijkl} \quad \text{or } L^{(2)}_{ijkl}\\ \varepsilon_{ij}(\mathbf{x}) &= M_{ijkl}(\mathbf{x})\sigma_{kl}(\mathbf{x}) + \mu_{ij}(\mathbf{x}); \quad M_{ijkl}(\mathbf{x}) = M^{(1)}_{ijkl} \quad \text{or } M^{(2)}_{ijkl},\end{aligned} \tag{12}$$

where, for example, the stiffness $L_{ijkl}(\mathbf{x})$ is either $L^{(1)}_{ijkl}$ or $L^{(2)}_{ijkl}$ depending on whether the point $\mathbf{x}$ lies within the first phase or the second. The stress and strain fields, as well as the eigen-fields in (12) can have rapid fluctuations on the scale of the microstructure and much slower variations on a relatively larger length scale. It is of interest to formulate the constitutive equation on a macroscopic level, in which all rapid fluctuations can be neglected. To accomplish this objective, consider a *cube* centered at point $\mathbf{x}$, of a size that is large in comparison to the microstructure but much smaller than the size of the entire body (see [22] for further definitions of the meso-sized cubic window). We now define the averaged values of the stress and strain of the heterogeneous medium at point $\mathbf{x}$ as

$$\sigma^*_{ij}(\mathbf{x}) = n\bar{\sigma}^{(1)}_{ij}(\mathbf{x}) + (1-n)\bar{\sigma}^{(2)}_{ij}(\mathbf{x}); \quad \varepsilon^*_{ij}(\mathbf{x}) = n\bar{\varepsilon}^{(1)}_{ij}(\mathbf{x}) + (1-n)\bar{\varepsilon}^{(2)}_{ij}(\mathbf{x}), \tag{13}$$

where all the barred quantities are *local* averages of the fields obtained by integration over the volume of either phase one or phase two within the cube. Using this procedure, the rapid oscillations of the averaged fields in (13), observed on the length of the microstructure, will be smoothed out. On the other hand, the averaged fields will depend, in general, on the location of the meso-sized cube, centered at $\mathbf{x}$. Thus, they are still a function of position $\mathbf{x}$ and exhibit slow variations at the macroscopic level. The averaged stress and strain of the heterogeneous medium are related through the effective constitutive equations:

$$\sigma^*_{ij}(\mathbf{x}) = L^*_{ijkl}\varepsilon^*_{kl}(\mathbf{x}) + \lambda^*_{ij}(\mathbf{x}); \quad \varepsilon^*_{ij}(\mathbf{x}) = M^*_{ijkl}\sigma^*_{kl}(\mathbf{x}) + \mu^*_{ij}(\mathbf{x}), \tag{14}$$

where L^*_{ijkl} and M^*_{ijkl} are effective stiffness and compliance tensors, $\lambda^*_{ij}(\mathbf{x})$ and $\mu^*_{ij}(\mathbf{x})$ are, respectively, the effective eigenstress and eigenstrain. Note that components of L^*_{ijkl} and M^*_{ijkl} are no longer functions of position since, on the macroscopic level, the heterogeneous body behaves like a homogeneous medium. It may be noted that either the effective stiffness L^*_{ijkl} or the effective compliance M^*_{ijkl} is not simply the local average of $L_{ijkl}(\mathbf{x})$ or $M_{ijkl}(\mathbf{x})$ appearing in (12); instead the effective tensors will exhibit a *nonlinear dependence* on the local stiffness or compliance tensors and depend on the microstructure. A comprehensive review of approaches for determining the effective properties of heterogeneous bodies is given by Milton [22]. Similarly, it can be verified, from the definitions (13) and connections (14), that the effective eigen-fields are not equal to the local averages of eigen-fields in the separate phases. In other words, the results (13) are not valid when written in terms of eigenstresses or eigenstrains.

To determine the effective stiffness (or compliance) tensor and effective eigenstress (or eigenstrain) we introduce the notion of a mechanical *concentration factor* [18]. Consider the problem for a given heterogeneous body, subjected only to mechanical loads, where all eigenstresses and eigenstrains are prescribed as zero. We define the mechanical strain concentration factors for the two phases as

$$\bar{\varepsilon}^{(1)}_{ij}(\mathbf{x}) = A^{(1)}_{ijkl}\,\varepsilon^*_{kl}(\mathbf{x}); \quad \bar{\varepsilon}^{(2)}_{ij}(\mathbf{x}) = A^{(2)}_{ijkl}\,\varepsilon^*_{kl}(\mathbf{x}), \tag{15}$$

which connect the averaged strain of the heterogeneous body $\varepsilon^*_{ij}(\mathbf{x})$ to the averages of the strains in either phase. Again, the averaged strains may have variations on a macroscopic level, but the concentration factors, defined in (15), depend only on the phase number.

It can be shown [34,37] that the effective stiffness tensor L^*_{ijkl} and the effective eigenstress $\lambda^*_{ij}(\mathbf{x})$ can be expressed in terms of the strain concentration factors as follows:

$$L^*_{ijkl} = nL^{(1)}_{ijmn}\,A^{(1)}_{mnkl} + (1-n)L^{(2)}_{ijmn}\,A^{(2)}_{mnkl} \tag{16}$$

$$\lambda^*_{ij}(\mathbf{x}) = n\bar{\lambda}^{(1)}_{kl}(\mathbf{x})A^{(1)}_{klij} + (1-n)\bar{\lambda}^{(2)}_{kl}(\mathbf{x})A^{(2)}_{klij}, \tag{17}$$

where, again, quantities with bars are the averages of the respective fields over the volume of the separate phases in the meso-sized cube centered at $\mathbf{x}$, and n is the volume fraction of phase one.

Similarly, we can define the mechanical stress concentration factors as

$$\bar{\sigma}^{(1)}_{ij}(\mathbf{x}) = B^{(1)}_{ijkl}\,\sigma^*_{kl}(\mathbf{x}); \quad \bar{\sigma}^{(2)}_{ij}(\mathbf{x}) = B^{(2)}_{ijkl}\,\sigma^*_{kl}(\mathbf{x}). \tag{18}$$

Then the effective compliance M^*_{ijkl} and the effective eigenstrain $\mu^*_{ij}(\mathbf{x})$ can be obtained as

$$M^*_{ijkl} = nM^{(1)}_{ijmn}\,B^{(1)}_{mnkl} + (1-n)M^{(2)}_{ijmn}\,B^{(2)}_{mnkl} \tag{19}$$

$$\mu^*_{ij}(\mathbf{x}) = n\bar{\mu}^{(1)}_{kl}(\mathbf{x})\,B^{(1)}_{klij} + (1-n)\bar{\mu}^{(2)}_{kl}(\mathbf{x})\,B^{(2)}_{klij}. \tag{20}$$

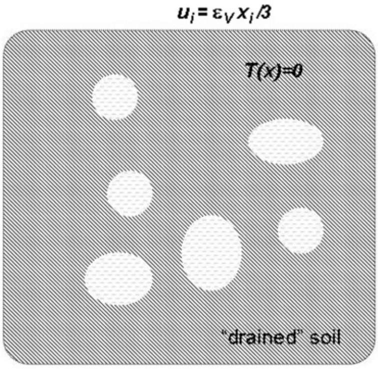

Figure 2. Porous media under fully drained conditions subjected to displacements derived from a uniform volumetric strain field ε_V.

It is clear that the *concentration factors* depend on the effective properties of the heterogeneous body L^*_{ijkl} or M^*_{ijkl}. From (16) or (19), it is evident that the concentration factors fully characterize the overall or the effective behavior of the heterogeneous body similar to the effective stiffness or compliance tensors.

It can be shown that the concentration factors satisfy the following relationships:

$$nA^{(1)}_{ijkl} + (1-n)A^{(2)}_{ijkl} = \delta_{ik}\delta_{jl} \tag{21}$$

$$nB^{(1)}_{ijkl} + (1-n)B^{(2)}_{ijkl} = \delta_{ik}\delta_{jl}. \tag{22}$$

Consider an example of a geomaterial specimen of porosity n subjected to the displacements $u_i = \varepsilon_V x_i/3$ on the boundary (Figure 2). Here, ε_V is a given constant, equal to the prescribed volumetric strain. It can be shown that this strain is equal to the averaged strain $\varepsilon_V = \varepsilon^*_{ii}$. The objective is to derive volumetric concentration factors for the solid phase and for the pores. Assume that the deformation of the geomaterial takes place under fully drained conditions. Also, assume that the effective bulk modulus of the geomaterial under drained conditions is known and equal to K_D. The bulk modulus of the solid phase is denoted by K_s.

From (15), the volumetric concentration factor relates the averaged volumetric strain ε_V to the averages of the volumetric strains of the solid and void phases:

$$\varepsilon_{Vp} = A_p \varepsilon_V; \quad \varepsilon_{Vs} = A_s \varepsilon_V, \tag{23}$$

where A_p, A_s are the (volumetric) concentration factors for the pores and the solid phases, respectively, and ε_{Vp}, ε_{Vs} are the averages of the volumetric strains in the pores and the solid phases, respectively (Here we often omit bars and stars for the averaged quantities). Since the prescribed averaged strain ε_V is uniform, the local averaged strain fields are also uniform, i.e. not functions of the macroscopic coordinate **x** (which makes this problem convenient for obtaining the concentration factors). From (21) the relationship between concentration factors is

$$nA_p + (1-n)A_s = 1. \tag{24}$$

If we denote $\sigma_I = \sigma_{kk}/3$ as the isotropic part of a stress tensor σ_{ij}, then, we can write

$$\varepsilon_V = K_D^{-1}\sigma_I^* = K_D^{-1}(-np + (1-n)\sigma_{Is}) = K_D^{-1}(1-n)\sigma_{Is} = K_D^{-1}(1-n)K_s\varepsilon_{Vs}, \qquad (25)$$

where σ_{Is} is the average of the isotropic stress in the solid phase and p is the pressure (equal to zero under fully drained conditions). The first relationship follows from the effective constitutive relation for an isotropic solid (14), while the second is due to the definition of the averaged stress σ_{ij}^* in (13). In the last relationship in (25), the local constitutive relation for the solid phase is used. Thus, from (25) and (23), one obtains the strain concentration factor for the solid phase:

$$A_s = \frac{1}{1-n}\frac{K_D}{K_s}. \qquad (26)$$

The strain concentration factor for the pore space follows from (24):

$$A_p = \frac{1}{n}\left(1 - \frac{K_D}{K_s}\right). \qquad (27)$$

We can now derive the stress concentration factors: the isotropic stress concentration factors are defined as

$$\sigma_{Ip} = B_p\,\sigma_I^*; \quad \sigma_{Is} = B_s\,\sigma_I^*. \qquad (28)$$

Note that if the stress in the pores (or pore fluid pressure) is zero, then from (28) the concentration factor for the pores B_p is also zero. Thus, from the relationship (22)

$$B_s = \frac{1}{1-n} \qquad (29)$$

the volumetric concentration factors derived here will be used in the subsequent section.

4. Constitutive relationships for a partially saturated poroelastic medium

In this section, we apply the eigenstrain-stress approach for a heterogeneous medium to re-derive the constitutive relationships (1), the void occupancy Equation (5) and the governing equations for a partially saturated porous medium.

The saturation is defined as

$$s = \frac{V_f}{V_p}; \quad s \in (0,1), \qquad (30)$$

where V_f is the volume of the fluid and V_p is the pore volume.

If $s = 1$, the porous medium is fully saturated. Note that

$$1 - s = \frac{V_a}{V_p}, \qquad (31)$$

where V_a is the volume of air in the pore space.

The material of the solid phase is considered to be linear elastic and compressible. The fluid is assumed to be non-viscous and compressible. The derivation allows for thermal deformation of both the solid grains and the fluid to account for heating (cooling) of the porous medium. The derivation can be extended or modified if other forms of eigenstresses or eigenstrains are present in the porous medium (for example, swelling, change of molecular structure of the phases, inelastic strains).

4.1. *Effective stress–strain relations*

As mentioned previously, the constitutive relations are expressed in terms of the elastic properties of the geomaterial under drained conditions. Thus, all eigenstresses or eigenstrains must be applied to the porous medium under drained conditions. By applying these eigen-fields, we should be able to simulate the behavior of the porous medium under general, not necessarily drained conditions (Figure 1). Following the general form of the effective stress-strain relation (14) in the presence of eigen-fields, the total stress in the isotropic porous medium can be expressed as

$$\sigma_{ij} = 2G_D\varepsilon_{ij} + \left(K_D - \frac{2G_D}{3}\right)\varepsilon_V\delta_{ij} + \lambda_T\delta_{ij} + \lambda_P\delta_{ij} + \text{other eigenstresses,} \tag{32}$$

where ε_{ij} are the strain components; K_D, G_D are the bulk and shear moduli, respectively, of the soil under drained conditions; λ_T is the effective eigenstress due to temperature change, and λ_P is the effective eigenstress due to the existence of fluid pressure. In the following, we omit all bars or superscripts-stars for both the *averaged fields* and the *effective properties*. Note that all fields in (32) are, in general, functions of position $\mathbf{x}$ and time.

We now obtain λ_T and λ_P. When the temperature change T is applied to the solid phase, the thermal (volumetric) eigenstrain in the solid phase is $\mu_s = 3\alpha_s T$. It follows from the relationship between eigenstresses and eigenstrains (11) that the thermal eigenstress in the solid phase is $\lambda_s = -K_s 3\alpha_s T$, where K_s is the bulk modulus of the solid phase (Figure 1b). To convert this local eigenstress to the effective eigenstress, applied to the drained soil, we use the result (17) [34]:

$$\lambda_T = (1-n)A_s\lambda_s, \tag{33}$$

where A_s is the (volumetric) strain concentration factor for the solid grains and n is the porosity. The concentration factor A_s was derived in Section 3, and it is given by the formula (26); thus, the effective eigenstress due to a temperature change in the solid phase is

$$\lambda_T = (1-n)A_s[-K_s 3\alpha_s T] = -K_D 3\alpha_s T. \tag{34}$$

Next, the total stress applied to the pore space is $-sp_f - (1-s)p_a = -sp$ if the pressure in the air phase p_a and the surface tension at the air-fluid interface are neglected. Thus, $-sp$ is the local eigenstress applied to the pores of the drained material (see Figure 1b for the case $s = 1$). The capillary pressure, defined as the difference between the pressure in the air and the fluid, is now equal to the fluid pressure multiplied by -1, i.e. $p_c = p_a - p_f = -p$.

To convert this local eigenstress to the effective eigenstress, applied to the drained skeleton, we make use of the result (17), which gives

$$\lambda_P = -nA_f sp, \tag{35}$$

where A_f is the (volumetric) strain concentration factor for the pore. Note that the negative sign in (35) implies that the pressure is positive when the fluid is in compression. From (27):

$$A_f = A_p = \frac{1}{n}\left(1 - \frac{K_D}{K_s}\right)$$

and, hence, the effective eigenstress due to the existence of pressure in the liquid is

$$\lambda_P = -\left(1 - \frac{K_D}{K_s}\right)sp. \tag{36}$$

Substituting (34) and (36) into (32) leads to the final form of the thermo-poroelastic constitutive relation for a partially-saturated porous medium as

$$\sigma_{ij} = 2G_D\varepsilon_{ij} + \left(K_D - \frac{2G_D}{3}\right)\varepsilon_V\delta_{ij} - 3K_D\alpha_s T\delta_{ij} - \left(1 - \frac{K_D}{K_s}\right)sp. \tag{37}$$

Usually, if the solid phase is locally incompressible, $K_D \ll K_s$ and the Biot parameter $\alpha = 1 - K_D/K_s \approx 1$. However, this assumption may not be valid if the porosity n is small, even if the solid phase is incompressible; it can be shown that $n \le \alpha \le 1$ [38].

To obtain the first governing equation, the standard equations of equilibrium and the strain-displacement relations are used. In the absence of body forces, the equilibrium equations are

$$\sigma_{ij,j} = 0. \tag{38}$$

The strain components ε_{ij} are related to the displacement components u_i by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{39}$$

with $\varepsilon_V = \varepsilon_{ii} = u_{i,i}$. Using (37) and (39), the equilibrium Equation (38) can be written as

$$\left(K_D + \frac{G_D}{3}\right)u_{k,ki} + G_D u_{i,kk} - K_D 3\alpha_s T_{,i} - \left(1 - \frac{K_D}{K_s}\right)(sp)_{,i} = 0. \tag{40}$$

The dependent variables are the displacement components u_i, the pressure p and the temperature T.

4.2. *Void occupancy equation*

To derive the void occupancy equation, we use the local constitutive equations (12) for the solid phase and the liquid phase under general conditions. The notion of the geomaterial under drained conditions is not used here. The normal isotropic stress in the liquid phase is equal to $-p$ and thus $-p/K_f$ is the volumetric strain in the liquid phase if all eigenstrains are zero. In the presence of the thermal eigenstrain and the change in the liquid mass, the volumetric strain in the fluid becomes

$$\varepsilon_{Vf} = -\frac{p}{K_f} + 3\alpha_f T + \frac{\Delta m_f}{m_f}, \tag{41}$$

where $\Delta m_f / m_f$ is the eigenstrain associated with a loss or gain of liquid mass Δm_f.

The strain in the fluid is *not* the only strain in the pores. One must also take into account the volumetric strain of air:

$$\varepsilon_{Va} = \frac{\Delta V_a}{V_a}, \tag{42}$$

where V_a is the initial volume of air in the pores space. Let us represent this strain in a different form:

$$\begin{aligned}\varepsilon_{Va} &= \frac{\Delta V_a}{V_a} = \frac{\Delta V_a}{(1-s)V_p} = \frac{\Delta(V_p - V_f)}{(1-s)V_p} = \frac{\Delta(V_p - sV_p)}{(1-s)V_p} = \frac{\Delta(V_p(1-s))}{(1-s)V_p} \\ &= \frac{\Delta V_p}{V_p} + \frac{1}{1-s}(-\Delta s) = \varepsilon_{Vp} + \frac{1}{1-s}(-\Delta s),\end{aligned} \tag{43}$$

where $V_p = V_a + V_f$ is the volume of pores, ε_{Vp} is the volumetric strain of the entire pore space, and Δs is the change in saturation.

The volumetric strain in the solid phase is

$$\varepsilon_{Vs} = \frac{\sigma_{Is}}{K_s} + 3\alpha_s T, \tag{44}$$

where σ_{Is} is the average of the isotropic stress in the solid phase, or phase number two, i.e. $\sigma_{Is} = (\sigma_{11}^{(2)} + \sigma_{22}^{(2)} + \sigma_{33}^{(2)})/3$.

From (13), the averaged strain of the entire porous medium is equal to the sum of averages of the strains in the fluid, air and solid phases, i.e.

$$\varepsilon_V = ns\varepsilon_{Vf} + n(1-s)\varepsilon_{Va} + (1-n)\varepsilon_{Vs}. \tag{45}$$

On the other hand, ε_V can be expressed in terms of the averaged strain of the entire pore space ε_{Vp} and the averaged strain of the solid phase:

$$\varepsilon_V = n\varepsilon_{Vp} + (1-n)\varepsilon_{Vs}. \tag{46}$$

Now, using (43) and (46), we obtain from (45) the following equation:

$$s\varepsilon_V = (1-n)s\varepsilon_{Vs} + ns\varepsilon_{Vf} - n\Delta s. \tag{47}$$

This is the void occupancy equation, in a concise form, for the case of a partially saturated porous medium. Using the expressions for the volumetric strain of the solid grains (44) and the volumetric strain in the liquid (41), we can reduce (47) to the result:

$$s\varepsilon_V = (1-n)s\left(\frac{\sigma_{Is}}{K_s} + 3\alpha_s T\right) + ns\left(-\frac{p}{K_f} + 3\alpha_f T + \frac{\Delta m_f}{m_f}\right) - n\Delta s. \tag{48}$$

We will attempt to eliminate the isotropic stress of the solid phase σ_{Is} from (48). On the one hand, from (13) the total (isotropic) stress of the porous medium σ_I is

$$\sigma_I = -nsp + (1-n)\sigma_{Is}. \tag{49}$$

This is true because the stress in the air phase is zero. On the other hand, from the effective stress–strain relationship (37) it follows that this stress is

$$\sigma_I = K_D\varepsilon_V - 3K_D\alpha_s T - \left(1-\frac{K_D}{K_s}\right)sp. \tag{50}$$

By equating (49) and (50), the stress of the solid phase σ_{Is} can be found. Substitution of this stress into (48) leads to, after simplification:

$$\begin{aligned} &-ns\frac{\Delta m_f}{m_f} + \left(1-\frac{K_D}{K_s}\right)s\varepsilon_V + ns\frac{p}{K_f} + \left(1-\frac{K_D}{K_s}-n\right)\frac{s^2 p}{K_s} + n\Delta s \\ &= ns3\alpha_f T + \left(1-\frac{K_D}{K_s}-n\right)s3\alpha_s T. \end{aligned} \tag{51}$$

To verify Equation (51), consider a special case of a fully saturated porous medium with very low permeability. The pores are filled with, in general, a compressible liquid. Under such conditions, the liquid mass remains constant and, thus, $\Delta m_f = 0$. Assume that the temperature change is uniform and that the thermal expansion coefficients of the solid phase and the liquid are equal, giving $\alpha_f T = \alpha_s T$. Also, the external boundary of the porous body is free of any tractions or stresses. Under these conditions, the stresses in the solid phase and the pressure p in the liquid are zero. From Equation (51), one can then obtain $(1-\frac{K_D}{K_s})\varepsilon_V = (1-\frac{K_D}{K_s})3\alpha T$ or $\varepsilon_V = 3\alpha T$. This means that the volumetric strain of the porous medium is equal to the applied thermal eigenstrain, which is the expected result.

It can be shown that the eigenstrain due to the change of fluid mass is

$$\frac{\Delta m_f}{m_f} = -\int_0^t v_{l,l}\,\mathrm{d}t, \tag{52}$$

where $v_{l,l}$ is the divergence of the fluid velocity vector $\mathbf{v}$. Note that the divergence of the velocity in (52) cannot be obtained by taking a time derivative of the fluid strain (41). The velocity in (52) is the velocity of the fluid entering and exiting the control volume, which can be taken as the meso-sized cube centered at point $\mathbf{x}$.

Substitution of (52) into the void occupancy Equation (51) and subsequent differentiation with respect to time gives

$$\begin{aligned} &nsv_{l,l} + \left(1-\frac{K_D}{K_s}\right)s\frac{\mathrm{d}\varepsilon_V}{\mathrm{d}t} + \frac{ns}{K_f}\frac{\mathrm{d}p}{\mathrm{d}t} + \left(1-\frac{K_D}{K_s}-n\right)\frac{s^2}{K_s}\frac{\mathrm{d}p}{\mathrm{d}t} + n\frac{\mathrm{d}s}{\mathrm{d}t} \\ &= ns3\alpha_f\frac{\mathrm{d}T}{\mathrm{d}t} + \left(1-\frac{K_D}{K_s}-n\right)s3\alpha_s\frac{\mathrm{d}T}{\mathrm{d}t}. \end{aligned} \tag{53}$$

In deriving (53), we used the fact that

$$\frac{\mathrm{d}\Delta s}{\mathrm{d}t} = n\frac{\mathrm{d}s}{\mathrm{d}t}. \tag{54}$$

Also, we took into account the fact that

$$\frac{\mathrm{d}n}{\mathrm{d}t} \times \text{any strain} \ll n \times \frac{\mathrm{d(strain)}}{\mathrm{d}t}$$

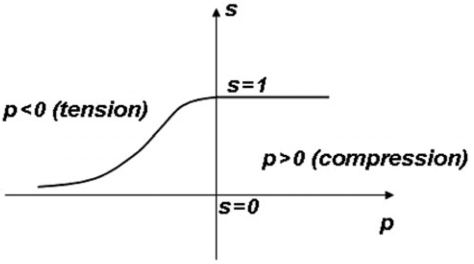

Figure 3. Dependence of saturation on fluid pressure.

and

$$\frac{\mathrm{d}s}{\mathrm{d}t} \times \text{any strain} \ll s \times \frac{\mathrm{d(strain)}}{\mathrm{d}t}; \quad \frac{\mathrm{d}n}{\mathrm{d}t} \times \Delta s \ll n\frac{\mathrm{d}\Delta s}{\mathrm{d}t};$$

usually, the derivative of saturation with respect to time is represented as

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}s}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{55}$$

and the dependence of the saturation on pressure $s(p)$ for negative pressure $p < 0$ is given. Note that $s(p)$ has the following properties: $s(p) = 1$ for positive pressure $p \geq 0$, i.e. when the porous medium is fully saturated, the pressure is non-negative. Also, $(\mathrm{d}s/\mathrm{d}p) = 0$ if $s = 1$ and $(\mathrm{d}s/\mathrm{d}p) = 0$ for $s \to 0$. The typical form of the function $s(p)$ is shown in Figure 3.

Using (55), the void occupancy Equation (53) can be rewritten as

$$\begin{aligned} &nsv_{l,l} + \left(1 - \frac{K_D}{K_s}\right)s\frac{\mathrm{d}\varepsilon_V}{\mathrm{d}t} + \frac{ns}{K_f}\frac{\mathrm{d}p}{\mathrm{d}t} + \left(1 - \frac{K_D}{K_s} - n\right)\frac{s^2}{K_s}\frac{\mathrm{d}p}{\mathrm{d}t} + n\frac{\mathrm{d}s}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t}. \\ &= ns3\alpha_f\frac{\mathrm{d}T}{\mathrm{d}t} + \left(1 - \frac{K_D}{K_s} - n\right)s3\alpha_s\frac{\mathrm{d}T}{\mathrm{d}t} \end{aligned} \tag{56}$$

4.3. *Darcy's Law*

For the case of a partially saturated medium, Darcy's Law can be written as

$$ns\,v_l = -\frac{k\,k_{rel}(s)}{\gamma}p_{,l}, \tag{57}$$

where $k_{rel}(s)$ is the relative permeability, $0 \leq k_{rel} \leq 1$ with the property that $k_{rel}(1) = 1$ and $k_{rel}(0) = 0$. Substituting Darcy's Law (57) into the void occupancy, Equation (56) gives the governing equation for the displacement components and the pressure:

$$-ns\left(\frac{kk_{rel}(s)}{\gamma ns}p_{,l}\right)_{,l} + \alpha s\frac{\mathrm{d}u_{k,k}}{\mathrm{d}t} + \frac{ns}{K_f}\frac{\mathrm{d}p}{\mathrm{d}t} + (\alpha - n)\frac{s^2}{K_s}\frac{\mathrm{d}p}{\mathrm{d}t} + n\frac{\mathrm{d}s}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}t} = ns3\alpha_f\frac{\mathrm{d}T}{\mathrm{d}t} + (\alpha - n)s3\alpha_s\frac{\mathrm{d}T}{\mathrm{d}t}, \tag{58}$$

where the dependent variables are the displacement components u_i, the pressure p and the temperature T. Note that to solve the governing equations for the case of a partially saturated medium, it is necessary to provide dependence of saturation on pressure $s(p)$ for negative values of pressures, i.e. when $0 \leq s < 1$.

Finally, the heat conduction equation can be solved to obtain the variation of temperature T on a macroscale:

$$c_p^* \frac{\partial T}{\partial t} - k_c^* T_{,kk} = 0, \tag{59}$$

where c_p^* is the effective specific heat of the porous medium and k_c^* is the effective thermal conductivity. This equation can be derived from Fourier's law (7) and the thermal energy balance equation. Both c_p^* and k_c^* are evaluated for a partially saturated porous medium, i.e. $c_p^* = snc_f\rho_f + (1-n)c_s\rho_s$, $k_c^* = snk_f + (1-n)k_s$ if the density and thermal conductivity of the air phase are neglected. Note that definition of k_c^* is consistent with the fact that the temperature field is assumed equal in both the pore space and the solid phase.

One of the advantages of using the eigenstrain-stress approach is the possibility to compute the averaged strains and stresses in the pores (local fields) knowing the overall strains and stresses. Consider the case of a fully saturated porous medium. Insertion of the eigenstrain $\frac{\Delta m_f}{m_f}$ found from (51) into (41) gives the averaged volumetric strain in the pore space ε_{Vf} as

$$\varepsilon_{Vf} = \frac{\alpha}{n}\varepsilon_V + \frac{\alpha - n}{n}\frac{p}{K_s} - \frac{\alpha - n}{n} 3\alpha_s T. \tag{60}$$

Note that all quantities on the right side of this equation are the macro-scale fields found by solving equations of poroelasticity.

Equation above can be generalized so that all strain components in the pore space (phase 1) can be found in terms of the components of the overall strain, pressure and temperature fields:

$$\varepsilon_{ij}^{(1)} = A_{ijkl}^{(1)}\varepsilon_{kl} + (I_{ijkl} - A_{ijkl}^{(1)})\alpha_s T\delta_{kl} - (I_{ijkl} - A_{ijkl}^{(1)})M_{klmn}^{(2)} p\delta_{mn}, \tag{61}$$

where $A_{ijkl}^{(1)}$ is the strain concentration factor for the pore space under drained conditions, $M_{ijkl}^{(2)}$ is the elastic compliance matrix for the solid phase. The strain concentration factor can be estimated by using one of the effective medium methods, such as Mori–Tanaka method.

It is important to note that the constitutive Equation (37) also can be written in terms of the bulk and shear moduli of the undrained porous material, i.e. K_u, defined in (4), and $G_u = G_D$, respectively. Substitution of the fluid pressure from (51) into the stress–strain Equation (37) leads to

$$\sigma_{ij} = 2G_D\varepsilon_{ij} + \left(K_u - \frac{2G_D}{3}\right)\varepsilon_V\delta_{ij} - (K_u - Q\alpha n)3\alpha_s T\delta_{ij} - Q\alpha n\left[3\alpha_f T + \frac{\Delta m_f}{m_f}\right]\delta_{ij}, \tag{62}$$

where $Q = 1/(\frac{n}{K_f} + \frac{\alpha - n}{K_s})$. Here $(3\alpha_f T + \frac{\Delta m_f}{m_f})$ is the volumetric eigenstrain applied to the liquid phase of the porous medium in the undrained state, and $3\alpha_s T$ is the volumetric eigenstrain in the solid phase. According to the eigenstrain-stress theory, temperature-related terms in the last equation can be reduced to $-3K_u\alpha_u^* T\delta_{ij}$,

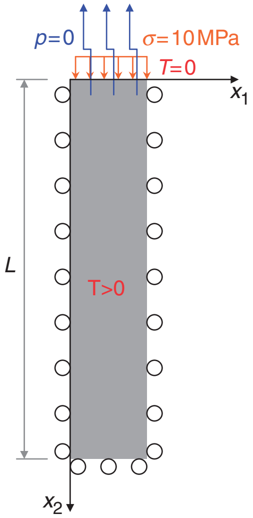

Figure 4. Column of geomaterial subjected to an initial temperature rise and surface tractions.

where α_u^* is the overall thermal expansion coefficient of a fluid-saturated porous medium. Now we can define $\alpha_u^* = K_u^{-1}\,[(K_u - Q\alpha n)\alpha_s + Q\alpha n\,\alpha_f]$.

5. Example

Consider the one-dimensional deformation of a column of a fully-saturated geomaterial which is at an initial temperature of T_0 over the entire length $0 \le x_2 \le L$. At time $t = 0$, the surface $x_2 = 0$ is subjected to a total normal stress $\sigma_{22}(0, t) = \sigma_0 H(t)$, where σ_0 is a constant and $H(t)$ is the Heaviside step function of time (Figure 4). The surface $x_2 = 0$ is also maintained in a drained condition.

Due to the one-dimensional nature of the problem and noting that $s = 1$, Equations (37), (40) and (58) can be reduced to

$$\sigma_{22} = \left(K_D + \frac{4G_D}{3}\right)u_{2,2} - 3K_D\alpha_s T - \left(1 - \frac{K_D}{K_s}\right)p \tag{63}$$

$$\left(K_D + \frac{4G_D}{3}\right)u_{2,22} - K_D 3\alpha_s T_{,2} - \left(1 - \frac{K_D}{K_s}\right)p_{,2} = 0 \tag{64}$$

$$\begin{aligned} -\frac{k}{\gamma}p_{,22} + \left(1 - \frac{K_D}{K_s}\right)\frac{\mathrm{d}u_{2,2}}{\mathrm{d}t} + \frac{n}{K_f}\frac{\mathrm{d}p}{\mathrm{d}t} + \left(1 - \frac{K_D}{K_s} - n\right)\frac{1}{K_s}\frac{\mathrm{d}p}{\mathrm{d}t} \\ = n3\alpha_f\frac{\mathrm{d}T}{\mathrm{d}t} + \left(1 - \frac{K_D}{K_s} - n\right)3\alpha_s\frac{\mathrm{d}T}{\mathrm{d}t}. \end{aligned} \tag{65}$$

Suppose that the initial temperature within the column is T_0, i.e.

$$T(x_2, t = 0) = T_0. \tag{66}$$

The temperature of the upper surface is reduced to zero at time $t = 0$ and the heat flux at the base of the column $x_2 = L$ is kept equal to zero, i.e.

$$T(x_2 = 0, t) = 0 \quad \frac{\partial T}{\partial x_2}(x_2 = L, t) = 0. \tag{67}$$

The temperature field satisfying the heat conduction Equation (59), the boundary conditions (67), and the initial condition (66) is given by [39,40]

$$T(x_2, t) = T_0 \sum_{m=1,3,5} \frac{4}{m\pi} \sin\left(\frac{m\pi}{2L} x_2\right) \exp(-\kappa_m^2 t) \quad \kappa_m^2 = \frac{m^2\pi^2}{4L^2} \frac{k_c^*}{c_p^*}. \tag{68}$$

Suppose that the tractions applied on the upper surface are equal to a constant σ_0, i.e.

$$\sigma_{22}(0, t) = \sigma_0. \tag{69}$$

The equations of equilibrium (38) for this one-dimensional problem reduce to $\sigma_{22,2} = 0$ and, hence, the total stress σ_{22} is uniform and equal to σ_0 for all points within the column.

Then, from Equation (63), the strain within the column can be determined as

$$\varepsilon_{22}(x_2) = u_{2,2} = \frac{1}{K_D + \frac{4}{3}G_D}\left(\sigma_0 + \left(1 - \frac{K_D}{K_s}\right)p + 3K_D\alpha_s T\right). \tag{70}$$

Substitution of this strain into the governing Equation (65) leads to the differential equation for the pressure:

$$\begin{aligned} &-\frac{k}{\gamma}p_{,22} + \left[\frac{n}{K_f} + (\alpha - n)\frac{1}{K_s} + \frac{\alpha^2}{K_D + \frac{4}{3}G_D}\right]\frac{\mathrm{d}p}{\mathrm{d}t} \\ &= n(3\alpha_f)\frac{\mathrm{d}T}{\mathrm{d}t} + \left(\alpha - n - \alpha\frac{K_D}{K_D + \frac{4}{3}G_D}\right)(3\alpha_s)\frac{\mathrm{d}T}{\mathrm{d}t}. \end{aligned} \tag{71}$$

Since the fluid can flow freely in and out through the upper surface, the pressure on the upper surface $x_2 = 0$ is taken as equal to zero:

$$p(x_2 = 0, t) = 0. \tag{72}$$

At the base $x_2 = L$, the velocity is set equal to zero:

$$\frac{k}{\gamma}\frac{\partial p}{\partial x_2}(x_2 = L, t) = 0. \tag{73}$$

We assume that both the general solution of the homogeneous version of (71) and a particular solution of (71) satisfy these boundary conditions.

Solution of the homogeneous equation takes the form:

$$
\begin{aligned}
p^H(x_2, t) &= \sum_{m=1,3,5} C_m \sin\left(\frac{m\pi}{2L} x_2\right) \exp(-\omega_m^2 t) \\
\omega_m^2 &= \frac{k}{\gamma} \frac{m^2\pi^2}{4L^2} \frac{K_D + \frac{4}{3} G_D}{(\frac{n}{K_f} + (\alpha - n)\frac{1}{K_s})(K_D + \frac{4}{3} G_D) + \alpha^2},
\end{aligned}
\tag{74}
$$

where C_m are constants to be determined.

The particular solution of (71) is given as

$$
\begin{aligned}
p^P(x_2, t) &= \sum_{m=1,3,5} A_m \frac{4}{m\pi} \sin\left(\frac{m\pi}{2L} x_2\right) \exp(-\kappa_m^2 t) \\
\kappa_m^2 &= \frac{m^2\pi^2}{4L^2} \frac{k_c^*}{c_p^*},
\end{aligned}
\tag{75}
$$

where A_m are unknown constants. The solution of (71) is the sum $p = p^H + p^P$. The constants A_m can be found by substitution of p^P (75) and the temperature field (68) into the equation for the pressure (71). This gives

$$
A_m = -T_0 \frac{(n3\alpha_f + (\alpha - n)3\alpha_s - \frac{\alpha K_D}{K_D + 4G_D/3} 3\alpha_s)}{\frac{k}{\gamma}\frac{c_p^*}{k_c^*} - (\frac{n}{K_f} + (\alpha - n)\frac{1}{K_s} + \frac{\alpha^2}{K_D + 4G_D/3})}. \tag{76}
$$

Thus, it has been shown that $A_m (= A)$ is independent of m.

It is now necessary to find the unknown constants C_m, which can be determined from the initial condition on the pressure field:

$$
p(x_2, 0) = p_0, \tag{77}
$$

where p_0 is an as yet unknown initial pressure.

Since

$$
\sum_{m=1,3,5} \frac{4}{m\pi} \sin\left(\frac{m\pi}{2L} x_2\right) = 1 \tag{78}
$$

and since $p = p^H + p^P$, we can establish that

$$
p_0 \frac{4}{m\pi} = C_m + A_m \frac{4}{m\pi}, \tag{79}
$$

where p_0 is the initial pressure. Equation (79) allows us to find the constants C_m.

Therefore, the total solution of (71) is

$$
\begin{aligned}
p(x_2, t) = p_0 &\sum_{m=1,3,5,} \frac{4}{m\pi} \sin\left(\frac{m\pi}{2L} x_2\right) \exp(-\omega_m^2 t) - \sum_{m=1,3,5,} A_m \frac{4}{m\pi} \sin\left(\frac{m\pi}{2L} x_2\right) \exp(-\omega_m^2 t) \\
&+ \sum_{m=1,3,5} A_m \frac{4}{m\pi} \sin\left(\frac{m\pi}{2L} x_2\right) \exp(-\kappa_m^2 t).
\end{aligned}
\tag{80}
$$

To find the initial pressure p_0 we use the constitutive Equation (63) and the void occupancy Equation (51). Equating the normal stresses σ_{22} to the applied stress σ_0, we obtain

$$\left(K_D+\frac{4}{3}G_D\right)\varepsilon_V-K_D3\alpha_sT_0-\alpha p_0=\sigma_0$$
$$-n\frac{\Delta m_f}{m_f}+\left(1-\frac{K_D}{K_s}\right)\varepsilon_V+n\frac{p_0}{K_f}+\left(1-\frac{K_D}{K_s}-n\right)\frac{p_0}{K_s}=n3\alpha_fT_0+\left(1-\frac{K_D}{K_s}-n\right)3\alpha_sT_0. \tag{81}$$

By setting the initial change in the fluid mass equal to zero $\left(\Delta m_f\right)_{t=0}=0$, we can obtain the initial pressure from (81):

$$p_0=-\frac{\alpha\sigma_0}{\alpha^2+(\frac{n}{K_f}+\frac{\alpha-n}{K_s})(K_D+\frac{4}{3}G_D)}+\frac{\left(K_D+\frac{4}{3}G_D\right)[(\alpha-n)3\alpha_sT_0+n\,3\alpha_fT_0]-K_D3\alpha_sT_0\alpha}{\alpha^2+(\frac{n}{K_f}+\frac{\alpha-n}{K_s})(K_D+\frac{4}{3}G_D)}. \tag{82}$$

This formula is simplified if the solid phase is incompressible, i.e. $K_s\to\infty$, $\alpha=1$. In this case, the initial pressure becomes

$$p_0=-\frac{\sigma_0}{1+\frac{n}{K_f}(K_D+\frac{4}{3}G_D)}+\frac{(K_D+\frac{4}{3}G_D)[(1-n)3\alpha_sT_0+n\,3\alpha_fT_0]-K_D3\alpha_sT_0}{1+\frac{n}{K_f}(K_D+\frac{4}{3}G_D)}. \tag{83}$$

The vertical displacement can be found by integrating the normal strain (70):

$$u_2=\int\frac{\sigma_0+\alpha p+3K_D\alpha_sT}{K_D+4G_D/3}\mathrm{d}x_2+U, \tag{84}$$

where U is a constant of integration which can be found from the boundary condition at the base:

$$u_2(L,0)=0. \tag{85}$$

Substitution of the pressure field (80) and the temperature field (68) into Equation (84) gives the vertical displacement:

$$\begin{aligned}u_2=\frac{1}{K_D+4G_D/3}\Bigg[&-K_D3\alpha_sT_0\sum_{m=1,3,5\ldots}\frac{8L}{m^2\pi^2}\cos\left(\frac{m\pi}{2L}x_2\right)\exp(-\kappa_m^2t)\\&-\alpha p_0\sum_{m=1,3,5,}\frac{8L}{m^2\pi^2}\cos\left(\frac{m\pi}{2L}x_2\right)\exp(-\omega_m^2t)\\&+\alpha\sum_{m=1,3,5\ldots}A_m\frac{8L}{m^2\pi^2}\cos\left(\frac{m\pi}{2L}x_2\right)\exp(-\omega_m^2t)\\&-\alpha\sum_{m=1,3,5\ldots}A_m\frac{8L}{m^2\pi^2}\cos\left(\frac{m\pi}{2L}x_2\right)\exp(-\kappa_m^2t)+\sigma_0(x_2-L)\Bigg],\end{aligned} \tag{86}$$

where the initial pressure is given by (83), the coefficient ω_m by (74), the coefficient κ_m by (75), and the coefficient A_m is found from (76). Note that the initial displacement

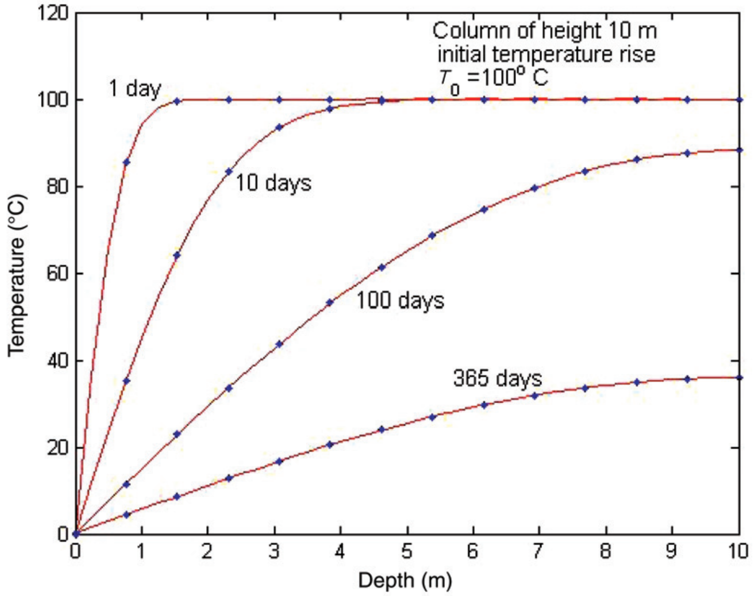

Figure 5. Temperature distribution within a one-dimensional column subjected to an initial temperature change of 100°C and subsequent heat dissipation due to the reduction of temperature at the upper surface to zero.

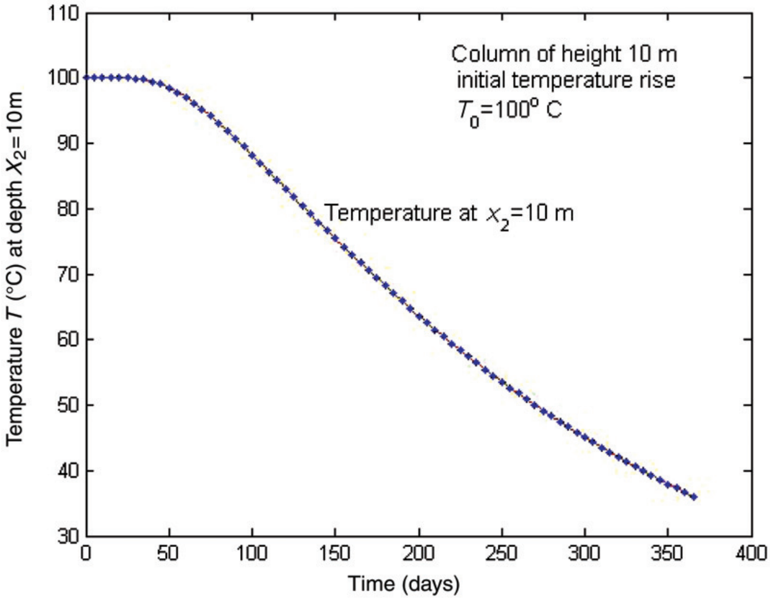

Figure 6. Evolution of temperature with time at the bottom surface of a one-dimensional column subjected to an initial temperature change 100°C and subsequent heat dissipation.

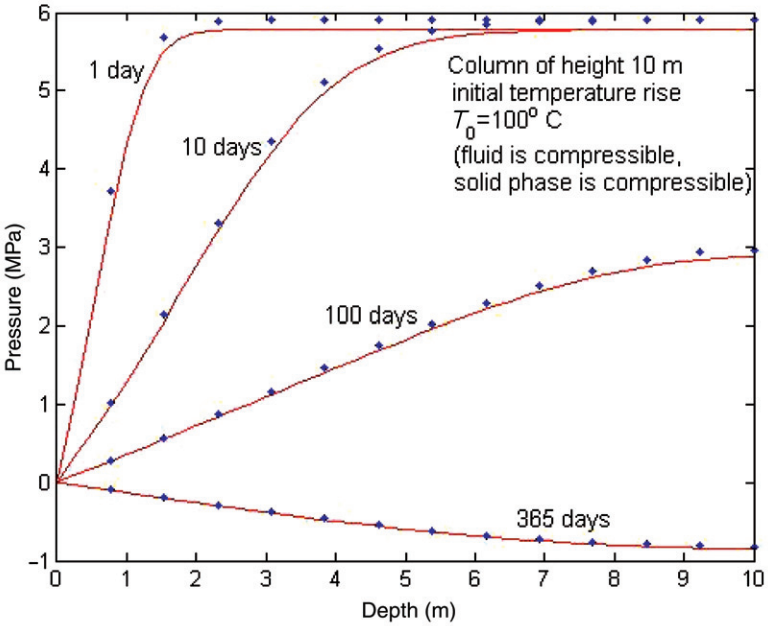

Figure 7. Pressure distribution within a one-dimensional column subjected to the temperature change shown in Figure 5. Pressure at the upper surface is zero, as is the fluid velocity at the bottom surface.

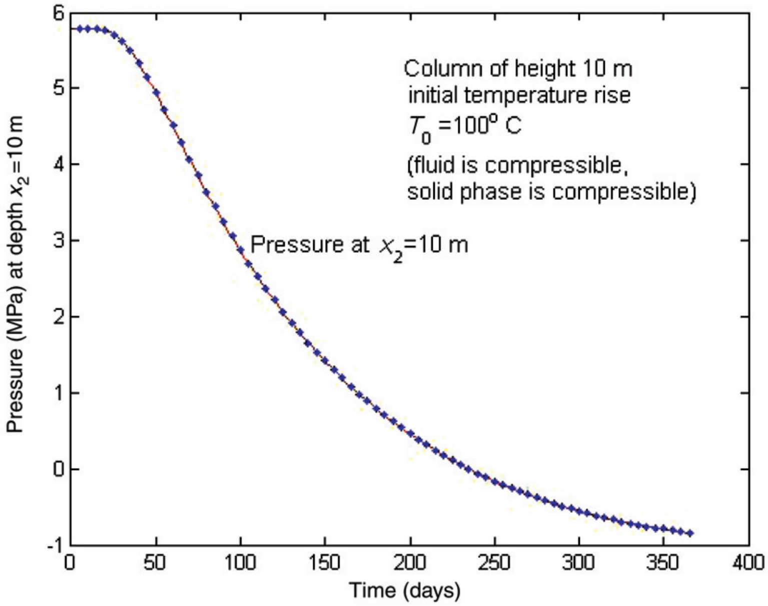

Figure 8. Pressure evolution at a 10-m depth in a one-dimensional column subjected to the temperature change shown in Figure 5.

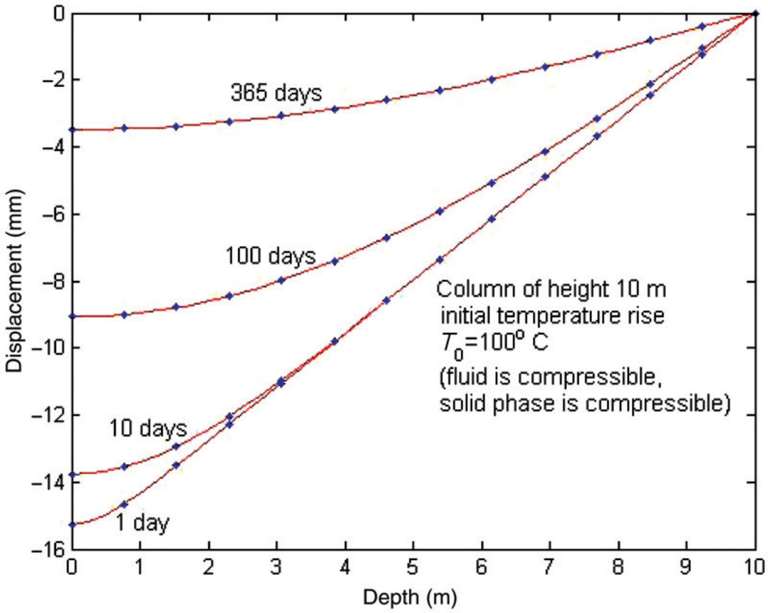

Figure 9. Displacement distribution inside a one-dimensional column subjected to the temperature change shown in Figure 5. Displacement is zero at the bottom surface. Negative displacements signify expansion of the column.

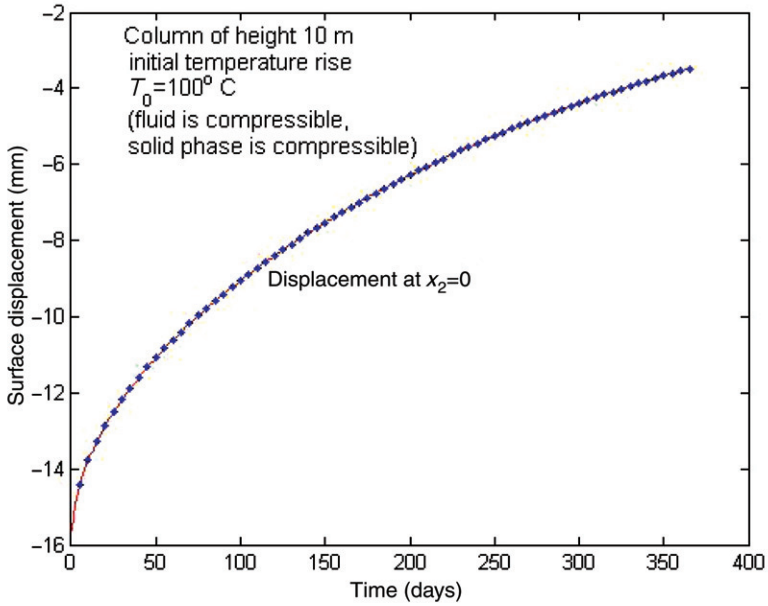

Figure 10. Surface displacement of a one-dimensional column subjected to the temperature change shown in Figure 5.

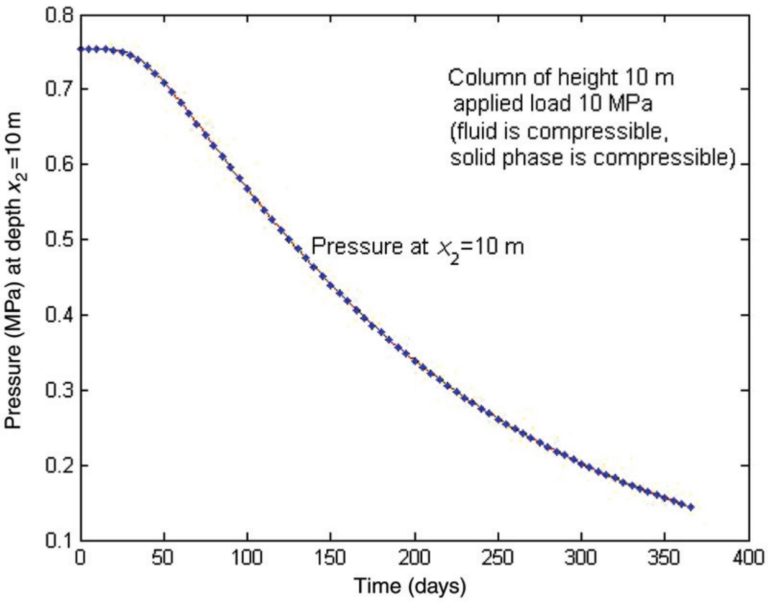

Figure 11. Pressure distribution at a 10-m depth in a one-dimensional column subjected to a compressive load of 10 MPa at the upper surface.

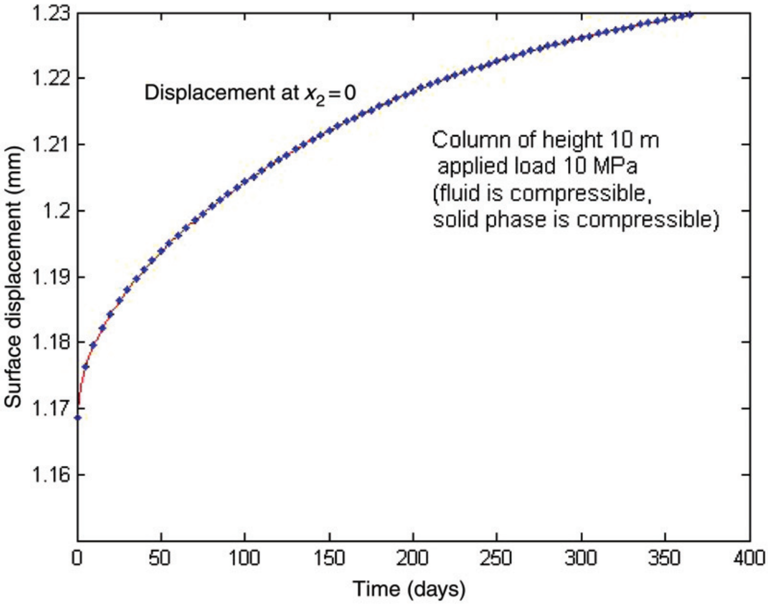

Figure 12. Surface displacement of a one-dimensional column subjected to a compressive load of 10 MPa at the upper surface. Positive displacements signify contraction of the column.

Table 1. Specific values of the geometric and physical parameters of a geomaterial whose transient response is shown in Figures 5–12.

Name of variable or parameter	Value	Name of variable or parameter	Value
L	10 m	$\left(K_D + \frac{4}{3} G_D\right)/K_f$	36.7133
K_D	50 GPa	Biot parameter α	0.75
$\alpha_s = \alpha_f$	8.3 E-6 1/C	Porosity n	0.25
σ_0	−10 MPa	$\frac{k_c^*}{c_p^*} \frac{1}{L^2}$	1.6227 E-8 [1/sec]
T_0	100 C	$\frac{k}{\gamma} \frac{(K_D + 4G_D/3)}{L^2}$	2.4231 E-7 [1/sec]
$\left(K_D + \frac{4}{3} G_D\right)/K_D$	1.6154		

is given by

$$u_2(x_2, t = 0) = \frac{L}{K_D + 4G_D/3}\left(1 - \frac{x_2}{L}\right)[-K_D 3\alpha_s T_0 - \alpha p_0 - \sigma_0] \tag{87}$$

since

$$\sum_{m=1,3,5,} \frac{8L}{m^2 \pi^2} = L.$$

The *time-dependent or transient* response of a poroelastic column of height L subjected to the initial temperature change T_0 and constant surface tractions σ_0 (Figure 4) is illustrated in Figures 5–12. The specific values of the height, bulk modulus under drained conditions, the applied traction and temperature change appear in Table 1; the relationship between the properties of the poroelastic column is also shown in Table 1.

In Figures 5–12, the analytical solution is shown with a solid line, and the solution obtained by using the finite element program COMSOL is shown as a dotted line (the same results can be obtained by using the finite element program ABAQUS). When using either ABAQUS or COMSOL, it is not necessary to specify the initial pressure as it can be set equal to zero.

6. Conclusions

An alternative derivation of the constitutive equations for a porous thermoelastic medium is presented. The derivation is based on the eigenstrain-stress approach and basic principles of micromechanics of a heterogeneous material conventionally identified with composite materials. The constitutive equations can be extended to include other forms of eigenstresses or eigenstrains, besides the thermal eigenstrain, existing in the porous medium (for example, swelling, eigenstrains associated with the change of molecular structure of the phases, inelastic deformation).

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Key symbols

n	porosity, ratio of volume of pores to the total volume of soil V_p/V
K_D	bulk modulus of drained soil
G_D	shear modulus of drained soil
K_s	bulk modulus of solid grains
K_f	bulk modulus of fluid
α_s	linear thermal expansion coefficient of solid grains
α_f	linear thermal expansion coefficient of liquid
ε_V	volumetric strain of soil
ε_{ij}	strain components
σ_{ij}	stress components
p	pressure
k	hydraulic conductivity
γ	unit weight of liquid
v	fluid velocity
T	temperature change

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