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On the hydraulic intake shape factor for a circular opening located at an impervious boundary: Influence of inclined stratification

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SUMMARY

In this paper, we develop an *exact closed form solution* for a circular entry point located at the interface between an impermeable material and a stratified porous medium, where the principal plane of hydraulic isotropy is inclined to the interface. Copyright © 2010 John Wiley & Sons, Ltd.

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KEY WORDS: intake shape factor; disk-shaped intake; circular entry point; intake at an interface; hydraulic transverse isotropy; mixed boundary value problem; potential theory; elliptical intake

1. INTRODUCTION

Darcy flow in porous media forms a cornerstone in the development of the subject of geomechanics. Advanced treatises on the subject are due to [1–4] and pedagogical studies of flow in porous media are dealt in the texts by References [5–9]. In the context of modern environmental geomechanics, the flow of fluids through porous media continues to be the dominant process that determines the transport of contaminants through porous media [10–15].

Directional hydraulic properties are characteristic of sedimentary geologic materials, including varved clays, inter-bedded sands and silts in fluvial and lacustrine environments. In general, these materials tend to display hydraulic anisotropy resulting from both micro-stratifications due to particle shape and macro-stratification due to periodic deposition. In general, the hydraulic conductivity tensor is second-order and symmetric [16–21] characterized by either six independent components or the three principal values and their directions. More advanced concepts that examine the influence of permeability anisotropy on Darcy's law and its relationship to Brinkman equations are given by Payne *et al.* [22], who also investigate issues related to continuous dependence, anisotropy and stability. These authors also present an extensive review of Darcy's law that addresses issues related to convection.

In instances where the frame of reference describing a problem and the principal directions of anisotropy coincide, the hydraulic properties can be described more conveniently by appeal to an orthotropic hydraulic conductivity model, which requires the specification of the three principal components of hydraulic conductivity. With stratified media, the hydraulic conductivity can be adequately described by the transversely isotropic model, which usually involves the values of hydraulic conductivity perpendicular to (k_n) and along (k_t) the bedding planes. When stratification

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results from the accretion of discrete layers, the point measures for describing the hydraulic conductivity are characterized by inhomogeneity. Averaged estimates can be obtained through *Voigt* and *Reuss* bounds that can be obtained from elementary considerations: The equivalent hydraulic conductivity normal to the bedding planes of an arrangement of n layers is given by the weighted harmonic mean of the hydraulic conductivities of the individual layers as

$$k_n = \left(\sum_{i=1}^{n} t_i\right) \Bigg/ \left(\sum_{i=1}^{n} \frac{t_i}{k_i}\right) \tag{1}$$

where t_i and k_i are, respectively, the uniform thickness and hydraulic conductivity of the ith layer. The equivalent hydraulic conductivity along the bedding planes k_t (i.e. the plane of transverse isotropy) is similarly given by

$$k_t = \left(\sum_{i=1}^{n} k_i t_i\right) \Bigg/ \left(\sum_{i=1}^{n} t_i\right) \tag{2}$$

Consider the development of hydraulic transverse isotropy within a sequence of inter-bedded silty sand and unweathered clay of equal thickness and with *isotropic* individual hydraulic conductivities estimated to be in the order of 10^{-6} and 10^{-12} m/s, respectively. The equivalent hydraulic conductivity k_n normal to the bedding planes of the entire sequence will be approximately 2×10^{-12} and the equivalent hydraulic conductivity along the bedding plane will be approximately 0.5×10^{-6} m/s. It is clear that significant hydraulic transverse isotropy in the scale of a representative volume element, with dimensions significantly larger than the individual layer thicknesses, can easily materialize even with plausible choices of hydraulic conductivities of the individual layers. Similar considerations apply to rock masses that are heavily fractured due to geologic stresses induced by tectonic action and stress relief. The hydraulic conductivity of such fractured media is governed both by the matrix hydraulic conductivity of the intact rock and the hydraulic conductivity of the fractures. In situations where the parent rock is relatively impervious, the directional properties can differ by up to 10 orders of magnitude ([23]–[25]).

It is often stated that the property of hydraulic conductivity is the easiest of geomaterial properties to define and the most difficult to determine, experimentally. Estimates of the hydraulic conductivity can be influenced by scale, ranging from *crustal scales* of 0.5–5.0 km, to *borehole scales* ranging from 30 to 300 m, to *laboratory scales* of 5–15 cm. The '*bulk hydraulic conductivity*' of geological media will be influenced by the choice of scale, and factors contributing to such variations can occur due to the abundance of fractures, fissures, inclusions and other inhomogeneities. Laboratory evaluations of hydraulic conductivity even at the size of large samples [26, 27] provide estimates only of the intact hydraulic conductivity and recourse must therefore be made to *in situ* investigations to determine the hydraulic conductivity characteristics at appreciable scales of interest.

A technique used for such purposes is the *cased borehole test* where either the transient water level rise or water level fall during an extraction or a recharge test is used to interpret the hydraulic conductivity characteristics of the geomaterials. The rate of water entry to the casing depends on the geometrical arrangement of the base of the borehole casing or the geometrical characteristics of the entry point. The most common of these is a formed cylindrical region at the base of the borehole with a diameter roughly equal to that of the casing and a length that can be a variable quantity. To facilitate the rapid rise of the water level in the casing to equalization with the ground water level, a piezometric standpipe of smaller diameter can be used with a bentonite sealing of the base of the casing. The definitive work dealing with the theory and practice of *in situ* measurement of hydraulic conductivity characteristics of soils in particular is due to Hvorslev [28], who proposed that, for situations where there is no consolidation of the soil region around the intake and for stationary groundwater conditions, the flow rate to the entry region is given by

$$q = F_0 k H \tag{3}$$

where H is the head inducing the flow, k is the effective hydraulic conductivity in the vicinity of the intake and F_0 is the intake shape factor, with dimensions of length, that is specific to the

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intake geometry. Based on results of the potential theory related to a prolate spheroid, Hvorslev proposed a relationship for the cylindrical intake shape factor for a cylindrical intake of diameter D and length L; the resulting expression for the intake shape factor F_0 is given in the form

$$F_0 = \frac{2\pi L}{\ln\left[\dfrac{L}{D} + \sqrt{1+\left(\dfrac{L}{D}\right)^2}\right]} \tag{4}$$

The determination of intake shape factors for various shapes of intakes has been investigated quite extensively in the geomechanics and water resources engineering literature and extensive reviews of these advances are given by Selvadurai [29, 30]. Many of the studies dealing with intake shape factors for entry points deal with isotropic porous geomaterials and studies that deal with entry points located in porous media with general anisotropic hydraulic conductivity characteristics are virtually non-existent. Some limited progress has been made in developing intake shape factors for hydraulically orthotropic media where the principal axes of hydraulic conductivity coincide with the geometrical axes of the intake. Intake shape factors for spherical and disk-shaped entry points located in hydraulically transversely isotropic have been evaluated in exact closed form by Selvadurai [29, 30] and further results concerning intake shape factors for partial sealing of the surface of halfspace regions can be deduced from the results given by [31–35]. Again, in general, these results have limited applicability in situations where the principal axes of hydraulic conductivity do not coincide with the geometric axes of the intake. The most effective approach for determining the intake shape factor for entry points of arbitrary profile located in porous media with general hydraulic anisotropy is by appeal to computational approaches. Such techniques can be effectively adopted to examine characteristics of entry points located in both anisotropic and inhomogeneous geomaterials. At the same time, it is instructive to develop analytical estimates, wherever possible, which can establish the accuracy of computational methods, particularly when the hydraulic conductivity mismatch between k_n and k_t is significant. The mismatch can have a bearing on the size of the domain that needs to be modeled to adequately represent infinite and semi-infinite regions.

This paper deals with the problem of a disk-shaped entry point that is located at the interface between a hydraulically transversely isotropic porous medium and an impervious boundary. In particular, the principal directions of transverse isotropy are no longer parallel to the boundary but inclined at an angle α (Figure 1). The case where the principal axes of hydraulic transverse isotropy

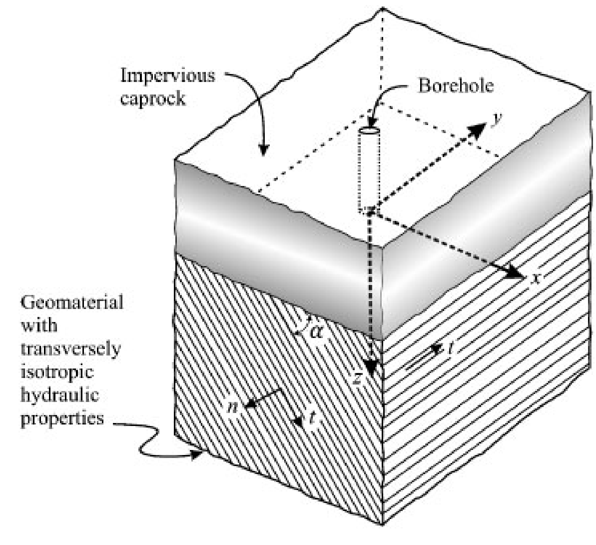

Figure 1. Borehole entry point at an interface with inclined stratification of the permeable medium.

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are parallel to the impervious boundary has been examined in the literature [29] and the estimate for the entry point can be obtained in an *exact closed form*. The presence of hydraulic transverse isotropy with an inclination to the interface makes the problem more interesting and of general interest, since geologic features can either outcrop or terminate at an impervious boundary at orientations other than horizontal and the determination of hydraulic conductivity through surface testing will necessitate the evaluation of intake shape factors for the general case. The objective of the paper is to show that the conventional approach to the study of the potential problem at a circular entry point located at the interface between an impervious boundary and a porous medium with a principal plane of transverse isotropy that is oriented to the interface leads to an unwieldy mixed boundary value problem that has no straightforward solution. It is shown that the conventional reduction of the problem to a pseudo-Laplacian approach can be used to generate an exact closed form solution for the intake shape factor.

2. THE PROBLEM FORMULATION

We consider the problem of a hydraulically transversely isotropic porous medium where the plane of transverse isotropy is inclined at an angle α to the x–y plane. This assumption is quite general and the reference coordinate system can be configured in this manner since the orientation in relation to the circular entry point is defined uniquely by the single angle. The hydraulic conductivity matrix for the transversely isotropic material $[\mathbf{k}^{\mathrm{P}}]$, referred to the principal directions aligned along the normal (n) and tangential (t) directions, is given by

$$[\mathbf{k}^{\mathrm{P}}]=\begin{bmatrix} k_t & 0 & 0 \\ 0 & k_t & 0 \\ 0 & 0 & k_n \end{bmatrix} \tag{5}$$

The hydraulic conductivity matrix $[\mathbf{k}^{\mathrm{RC}}]$ referred to the rectangular Cartesian coordinates (x, y, z) is given by

$$[\mathbf{k}^{\mathrm{RC}}]=[\mathbf{H}]^{\mathrm{T}}[\mathbf{k}^{\mathrm{P}}][\mathbf{H}] \tag{6}$$

where

$$[\mathbf{H}]=\begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \tag{7}$$

and $[\mathbf{H}]^{\mathrm{T}}$ denotes the transpose.

We can similarly construct the hydraulic conductivity matrix $[\mathbf{k}^{\mathrm{CP}}]$ referred to the cylindrical polar coordinate system as follows:

$$[\mathbf{k}^{\mathrm{CP}}]=[\mathbf{D}]^{\mathrm{T}}[\mathbf{k}^{\mathrm{RC}}][\mathbf{D}]=[\mathbf{D}]^{\mathrm{T}}[\mathbf{H}]^{\mathrm{T}}[\mathbf{k}^{\mathrm{P}}][\mathbf{H}][\mathbf{D}] \tag{8}$$

where

$$[\mathbf{D}]=\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

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and $[\mathbf{D}]^{\mathrm{T}}$ is the transpose . The components of $[\mathbf{k}^{\mathrm{CP}}]$ can be evaluated as follows:

$$\begin{aligned}
k_{rr} &= (k_t\cos^2\alpha + k_n\sin^2\alpha)\cos^2\theta + k_t\sin^2\theta\\
k_{\theta\theta} &= (k_t\cos^2\alpha + k_n\sin^2\alpha)\sin^2\theta + k_t\cos^2\theta\\
k_{zz} &= k_t\sin^2\alpha + k_n\cos^2\alpha\\
k_{r\theta} &= (k_t-k_n)\sin^2\alpha\ \sin\theta\cos\theta\\
k_{rz} &= (k_t-k_n)\sin\alpha\cos\alpha\cos\theta\\
k_{\theta z} &= -(k_t-k_n)\sin\alpha\cos\alpha\sin\theta
\end{aligned} \tag{10}$$

which is a fully populated matrix that satisfies the appropriate invariance requirements applicable to the symmetric, positive-definite, second-order tensors, $[\mathbf{k}^{\mathrm{P}}]$, $[\mathbf{k}^{\mathrm{RC}}]$ and $[\mathbf{k}^{\mathrm{CP}}]$. We can now express Darcy's law applicable to the transversely isotropic medium but referred to the cylindrical polar coordinate system. We have

$$[\mathbf{v}^{\mathrm{CP}}] = -[\mathbf{k}^{\mathrm{CP}}][\nabla\varphi] \tag{11}$$

where $\varphi(r,\theta,z)$ is the flow potential and $[\mathbf{v}^{\mathrm{CP}}]$ and $[\nabla\varphi]$ are, respectively, the velocity vector and the hydraulic gradients operator referred to the cylindrical polar coordinate system and given by

$$[\mathbf{v}^{\mathrm{CP}}] = [v_r, v_\theta, v_z]^{\mathrm{T}}, \qquad [\nabla\varphi] = \left[\frac{\partial\varphi}{\partial r}, \frac{1}{r}\frac{\partial\varphi}{\partial\theta}, \frac{\partial\varphi}{\partial z}\right]^{\mathrm{T}} \tag{12}$$

The mass conservation during flow through the non-deformable porous medium gives the result

$$\nabla\cdot\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z} = 0 \tag{13}$$

Combining (11)–(13), we obtain the partial differential equation governing fluid flow through the transversely isotropic porous medium and referred to the cylindrical polar coordinate system as follows:

$$\begin{aligned}
&\frac{1}{r^2}\left[\frac{1}{4}k_t\left\{4r^2\sin^2\alpha\frac{\partial^2\varphi}{\partial z^2} - 8\cos\theta\sin^2\alpha\sin\theta\frac{\partial\varphi}{\partial\theta} + (3+\cos 2\alpha + 2\cos 2\theta\sin^2\alpha)\frac{\partial^2\varphi}{\partial\theta^2}\right.\right.\\
&\quad + r\left((3+\cos 2\alpha + 2\cos 2\theta\sin^2\alpha)\frac{\partial\varphi}{\partial r} + 4\sin 2\alpha\left(-\sin\theta\frac{\partial^2\varphi}{\partial\theta\partial z} + r\cos\theta\frac{\partial^2\varphi}{\partial r\partial z}\right)\right.\\
&\quad \left.\left. + 4\sin^2\alpha\sin 2\theta\frac{\partial^2\varphi}{\partial r\partial\theta} + r(3+\cos 2\alpha - 2\cos 2\theta\sin^2\alpha)\frac{\partial^2\varphi}{\partial r^2}\right)\right\}\\
&\quad + k_n\left\{r^2\cos^2\alpha\frac{\partial^2\varphi}{\partial z^2} + r\sin 2\alpha\sin\theta\frac{\partial^2\varphi}{\partial\theta\partial z} + \sin\alpha\left(\sin\alpha\sin 2\theta\frac{\partial\varphi}{\partial\theta} + \sin\alpha\sin^2\theta\frac{\partial^2\varphi}{\partial\theta^2}\right.\right.\\
&\quad \left.\left.\left. + r\left(\sin\alpha\sin^2\theta\frac{\partial\varphi}{\partial r} + \cos\theta\left(-2r\cos\alpha\frac{\partial^2\varphi}{\partial r\partial z} + \sin\alpha\left(-2\sin\theta\frac{\partial^2\varphi}{\partial r\partial\theta} + r\cos\theta\frac{\partial^2\varphi}{\partial r^2}\right)\right)\right)\right)\right\}\right] = 0
\end{aligned} \tag{14}$$

It can be verified that when $k_n = k_t$, the dependency on the orientation α vanishes and (14) reduces to Laplace's equation referred to the cylindrical polar coordinate system.

$$\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial z^2}\right) = 0 \tag{15}$$

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Similar reductions can be obtained for the case where $\alpha=0$, when (14) reduces to

$$k_t\left(\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\right)+k_n\frac{\partial^2\varphi}{\partial z^2}=0 \tag{16}$$

Consider the mixed boundary value problem that is posed by the general case where a circular entry point of radius a is located at the interface between the *impermeable medium* and the transversely isotropic medium with the principal plane inclined at an angle α to the interface (Figure 1). The circular entry region is subjected to a Dirichlet boundary condition of constant potential φ_0 and the exterior interface region is subjected to null Neumann boundary conditions. In addition, the three-dimensional nature of the problem implies that far-field regularity conditions can be imposed on the flow potential. We have

$$\varphi(r,\theta,z)=\varphi_0 \qquad \text{for } z=0 \quad \forall r\in(0,a) \quad \forall\theta\in(0,2\pi) \tag{17}$$

$$\frac{\partial\varphi}{\partial\widehat{n}}=\widehat{\mathbf{n}}\cdot\nabla\varphi=0 \quad \text{for } z=0 \quad \forall r\in(a,\infty) \quad \forall\theta\in(0,2\pi) \tag{18}$$

where $\widehat{\mathbf{n}}$ is the unit outward normal to the impervious boundary. In addition, the flow potential should satisfy the regularity conditions

$$\varphi(r,\theta,z)\to 0 \quad \text{as } (r,z)\to\infty \quad \text{for } \forall\theta\in(0,2\pi) \tag{19}$$

The boundary conditions (17) and (18) give rise to a mixed boundary value problem, which can be formulated when solutions of the governing partial differential equation are available. The general nature of the governing partial differential equation is such that there appears to be no manageable solutions of (14) that can be used to solve the mixed boundary value problem posed by (17)–(19). The conventional product decomposition, variables separable solutions give rise to sets of ordinary differential equations of unmanageable algebraic complexity, even with the advantages offered by symbolic mathematical manipulation techniques in codes such as MATHEMATICA® and MAPLE®. Furthermore, at best, the analysis that involves product decomposition solutions will invariably give rise to the solution of the *mixed boundary value problem* in a *dual series form*, which does not lend itself to convenient generation of results of practical value.

3. AN ALTERNATIVE FORMULATION

An alternative approach is to re-cast the mixed boundary value problem by changing the coordinate system such that pseudo-isotropy is established. This, however, changes the geometry of the entry point, which necessitates the use of an appropriate coordinate system. Referring to Figure 2(a), we select the coordinate system $(\bar{x},\bar{y},\bar{z})$, which is aligned with the principal directions of hydraulic conductivity (t_1,t_2,n). Referring to the $(\bar{x},\bar{y},\bar{z})$ coordinate system, Laplace's equation takes the form

$$k_t\left(\frac{\partial^2\varphi}{\partial\bar{x}^2}+\frac{\partial^2\varphi}{\partial\bar{y}^2}\right)+k_n\frac{\partial^2\varphi}{\partial\bar{z}^2}=0 \tag{20}$$

At the circular intake of radius a

$$\bar{x}=a\cos\alpha, \quad \bar{y}=a, \quad \bar{z}=a\sin\alpha \tag{21}$$

Now we make the usual transformation that converts the pseudo-Laplacian equation to a Laplacian form, by introducing the transformations

$$\bar{x}=X, \quad \bar{y}=Y, \quad \bar{z}=Z\sqrt{\frac{k_n}{k_t}} \tag{22}$$

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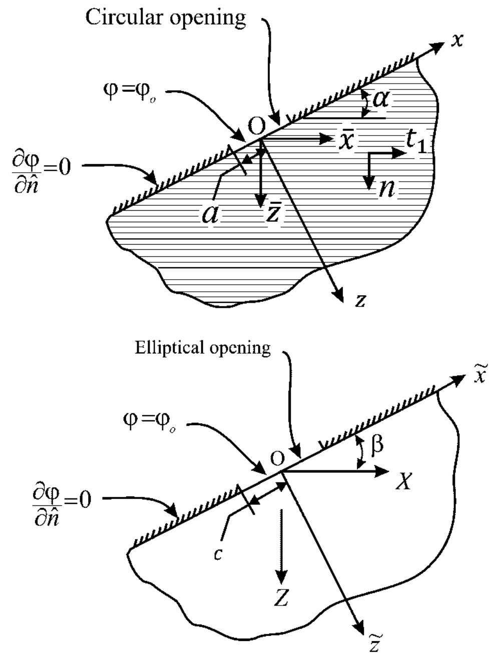

Figure 2. Transformation of the hydraulically transversely isotropic porous medium to a pseudo-isotropic medium.

which reduces (20) to the Laplacian form

$$\frac{\partial^2 \varphi}{\partial X^2}+\frac{\partial^2 \varphi}{\partial Y^2}+\frac{\partial^2 \varphi}{\partial Z^2}=0 \tag{23}$$

These substitutions, however, transform the circular entry point to an elliptical one, with axes a and c, and the inclination of the plane of the elliptical entry point β and at the elliptical entry point region (Figure 2(b))

$$X=c\ \cos\beta, \quad Y=a, \quad Z=c\ \sin\beta \tag{24}$$

From (21) and (24), we have

$$c^2=a^2\left\{\cos^2\alpha+\left(\frac{k_t}{k_n}\right)\sin^2\alpha\right\} \tag{25}$$

and the usual reductions follow for the limit of isotropy and when $\alpha=0$, where the circular entry point will remain circular.

Since the representation of the potential problem in the (X, Y, Z) space is isotropic, the reference coordinate system needed to analyze the elliptical entry point can be chosen in any fashion and it is convenient to select the coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ as shown in Figure 2(b). Referring to this coordinate system, the elliptical entry region corresponding to the transformed problem is defined by

$$\left(\frac{\tilde{x}}{c}\right)^2+\left(\frac{\tilde{y}}{a}\right)^2\leqslant 1 \tag{26}$$

where c will be the major axis if $(k_t/k_n)>1$, and a minor axis if $(k_t/k_n)<1$. For most naturally occurring stratified geologic media, $k_t>k_n$. We define the elliptical region 'entry point' by S_i and the exterior impervious region on the plane $\tilde{z}=0$ is defined by S_e. The mixed boundary value

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governing the elliptical intake region now takes the form

$$\varphi(\tilde{x}, \tilde{y}, 0) = \varphi_0, \quad (\tilde{x}, \tilde{y}) \in S_i \tag{27}$$

$$\left(\frac{\partial \varphi}{\partial \tilde{z}}\right)_{\tilde{z}=0} = 0, \quad (\tilde{x}, \tilde{y}) \in S_e \tag{28}$$

The solution of the mixed boundary value problem posed by (27) and (28) can be accomplished by employing generalized ellipsoidal coordinate systems and selecting a limiting case where the ellipsoid reduces to an elliptical plan form. The solution of the problem is also encountered in fluid flow and potential theory [36–38] and was used by Sir Horace Lamb [38] in connection with his classical study of the motion of a perfect fluid through an elliptical aperture. The result has also been used by Green and Sneddon [39] in their study of elliptical cracks in elastic solids and in the analysis of disk inclusion problems related to isotropic and transversely isotropic elastic solids [40, 41]. The solution is most conveniently presented in terms of ellipsoidal coordinates (ξ, η, ζ) of the point $(\tilde{x}, \tilde{y}, \tilde{z})$, which are the roots of

$$\frac{\tilde{x}^2}{(c^2+\theta)} + \frac{\tilde{y}^2}{(a^2+\theta)} + \frac{\tilde{z}^2}{\theta} - 1 = 0 \tag{29}$$

In the (ξ, η, ζ) coordinate system, S_i corresponds to $\xi = 0$ and S_e corresponds to $\eta = 0$. The mixed boundary conditions (27) and (28) can be explicitly satisfied by the harmonic function

$$\varphi(\tilde{x}, \tilde{y}, \tilde{z}) = \frac{c\ \varphi_0}{K(\rho)} \int_{\xi}^{\infty} \frac{\mathrm{d}s}{\sqrt{s(c^2+s)(a^2+s)}} \tag{30}$$

where

$$\xi = c^2(\mathrm{sn}^{-2}u - 1) \tag{31}$$

and sn$u$ represents the Jacobian elliptic function. Also, $K(\rho)$ is the complete elliptic integral of the first kind, defined by

$$K(\rho) = \int_0^{\pi/2} \frac{\mathrm{d}\varsigma}{\sqrt{1-\rho^2 \sin^2 \varsigma}} \tag{32}$$

and

$$\rho = \left(\frac{c^2 - a^2}{c^2}\right)^{1/2} = \sqrt{\frac{(k_t - k_n)\ \sin^2 \alpha}{k_t\ \sin^2 \alpha + k_n\ \cos^2 \alpha}} \tag{33}$$

The fluid velocity at the elliptical opening is given by

$$v_{\tilde{z}}(\tilde{x}, \tilde{y}, 0) = \sqrt{k_t k_n}\ \left(\frac{\partial \varphi}{\partial \tilde{z}}\right)_{\tilde{z}=0} = \frac{\sqrt{k_t\ k_n}\varphi_0}{aK(\rho)} \frac{1}{\sqrt{1 - \dfrac{\tilde{x}^2}{c^2} - \dfrac{\tilde{y}^2}{a^2}}} \tag{34}$$

The flow rate through the elliptical aperture is given by

$$Q = \frac{\sqrt{k_t\ k_n}\varphi_0}{a\ K(\rho)} \int\int_{S_i} \frac{\mathrm{d}x\ \mathrm{d}y}{\sqrt{1 - \frac{\tilde{x}^2}{c^2} - \frac{\tilde{y}^2}{a^2}}} = \frac{2\ \pi c\ \varphi_0 \sqrt{k_t\ k_n}}{K(\rho)} \tag{35}$$

We can express (35) in the conventional form

$$Q = F k_{eq} \varphi_0 \tag{36}$$

where $k_{eq} = \sqrt{k_t k_n}$ and

$$F = \frac{2\pi a}{K(\rho)} \left(\cos^2 \alpha + \frac{k_t}{k_n} \sin^2 \alpha \right)^{1/2} \tag{37}$$

As far as the author is aware, this *exact closed-form* result for the intake shape factor F given by (37) does not appear to be available in the literature, and it can be conveniently evaluated for the choice of the inclination α and the values of k_n and k_t.

4. LIMITING CASES

Since the intake shape factor can be evaluated in an *exact closed form*, it is unnecessary to provide extensive numerical results. Some limiting cases, however, could be examined.

(i) *Stratification parallel to the boundary:* In this instance, $\alpha = 0$ and $\rho = 0$, giving $K(0) = \pi/2$ and (33) reduces to $F = 4a$, which is in agreement with the result obtained by Selvadurai [29] for the problem of a disk-shaped entry point located in a hydraulically transversely isotropic porous medium, using a dual integral equation formulation of the mixed boundary value problem. (N.B. The result given by Selvadurai [29] is for a disk-shaped entry point located within a hydraulically transversely isotropic porous medium of infinite extent, with the plane of the entry point aligned with the plane of transverse isotropy.)

(ii) *Isotropy of the porous medium:* In the limit of material isotropy, $k_n = k_t = k$ and $\rho = 0$, giving $K(0) = \pi/2$ and (37) reduces to $F = 4a$.

(iii) *Limit of impermeable stratification:* This limiting case deals with the situation where, for example, the stratifications can result in the hydraulic conductivity of a layered porous medium normal to the stratifications being zero. An example would be varved soils consisting of layered deposits of alternating layers of clay and sand. If we assume that $k_n = 0$, then (20) yields a potential flow problem that is two dimensional. In the two-dimensional case, the general solution depends on $\log_e r$, where $r = (x^2 + y^2)^{1/2}$. As such, the solution to the problem, particularly for the definition of an intake shape factor, exists if and only if r is bounded. The intake shape factor developed for the three-dimensional problem assumes a region of semi-infinite extent and is therefore inapplicable for situations involving '*extreme*' hydraulic conductivity properties. (This is akin to problems for strongly anisotropic elastic solids, which can exhibit loss of ellipticity of the governing partial differential equations and the propagation of strong discontinuities, such as boundary layers of stress channeling, in the limit of inextensibility of the reinforcing fibers [42, 43]). This deficiency will also materialize if flow through fractured media of either infinite or semi-infinite extent is modeled, where the hydraulic conductivity properties of the fracture is several orders of magnitude greater than the hydraulic conductivity properties of the matrix.

(iv) *Stratification orthogonal to the interface:* It can be shown that when the orientation of the stratification is such that $\alpha = \pi/2$, (14) reduces to

$$(k_n \cos^2 \theta + k_t \sin^2 \theta) \frac{\partial^2 \varphi}{\partial r^2} + (k_n \sin^2 \theta + k_t \cos^2 \theta) \left(\frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} \right)$$
$$+ (k_n - k_t) \sin 2\theta \left(\frac{1}{r^2} \frac{\partial \varphi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \varphi}{\partial r \partial \theta} \right) + k_t \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{38}$$

Again no significant simplification occurs in the governing partial differential equation that would permit the development of an analytical result in the compact form as evaluated in (37).

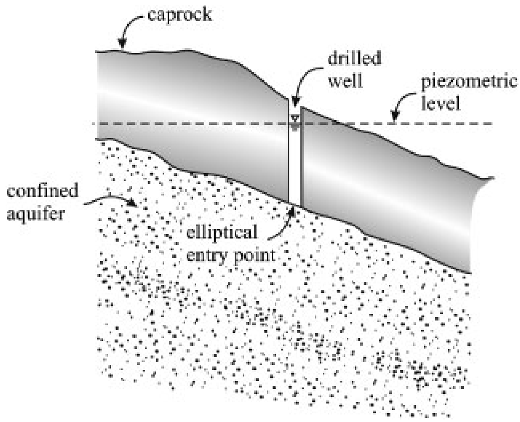

Figure 3. Borehole entry point that intersects an inclined interface.

(v) *The elliptical intake*
The results derived here can also be used to develop an intake shape factor for an elliptic entry point located at the interface of a porous medium with isotropic hydraulic conductivity. This situation can arise [44] when a borehole terminates at an interface that is inclined to the axis of the borehole (Figure 3). The mixed boundary value problem associated with the elliptical entry point is identical to that defined by (27) and (28) and omitting details, the intake shape factor F_{ell} corresponding to an elliptical entry point of semi-major axis a_0 and semi-minor axis b_0 is given by

$$F_{\text{ell}} = \frac{2\pi a_0}{K(e_0)} \tag{39}$$

where $K(e_0)$ is the complete elliptic integral of the first kind defined by (32), with the argument given by

$$e_0 = \sqrt{1 - \left(\frac{b_0}{a_0}\right)^2}, \quad \left(\frac{b_0}{a_0}\right) \leqslant 1 \tag{40}$$

In the limit as $(b_0/a_0) \to 1$, we recover the intake shape factor for the circular intake. The intake with the elliptical plan form can be used in conjunction with an 'equivalent area' concept to develop approximate solutions for intake regions with a rectangular and other irregular shapes.

The result (36), which defines the flow rate to the borehole has $\sqrt{k_t k_n}$ as a normalizing parameter and results for any limiting cases of the intake shape factor should be approached bearing in mind this multiplier. The influence of the magnitude of the transverse isotropy, measured by k_t/k_n and the orientation of the stratifications denoted by α, on the intake shape factor is illustrated in Figure 4. These results clearly indicate the influence of large values of the hydraulic conductivity mismatch k_t/k_n on the intake shape factor. Figure 5 presents a three-dimensional representation of the numerical values for the intake shape factor as a function of the hydraulic conductivity mismatch k_t/k_n and the inclination of the stratification.

5. CONCLUDING REMARKS

This paper determines the intake shape factor for a borehole that terminates at the boundary of an impervious interface and a stratified medium. The principal axes of hydraulic transverse isotropy are such that the stratification is inclined to the interface. Since the borehole is circular, the orientation can be described by a single angle. The intake shape factor for the borehole is obtained in an exact

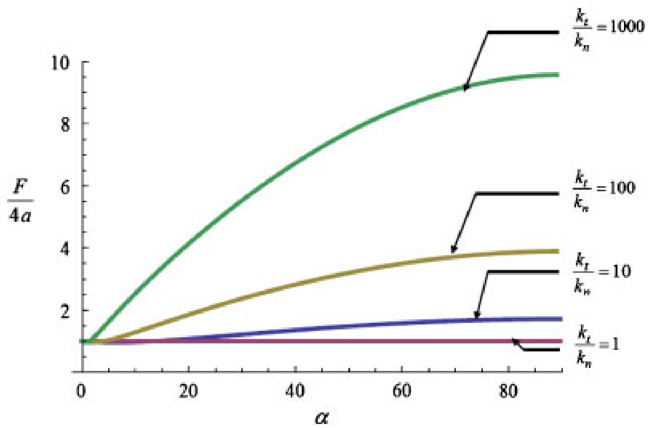

Figure 4. Influence of the orientation of the stratified porous medium on the intake shape factor.

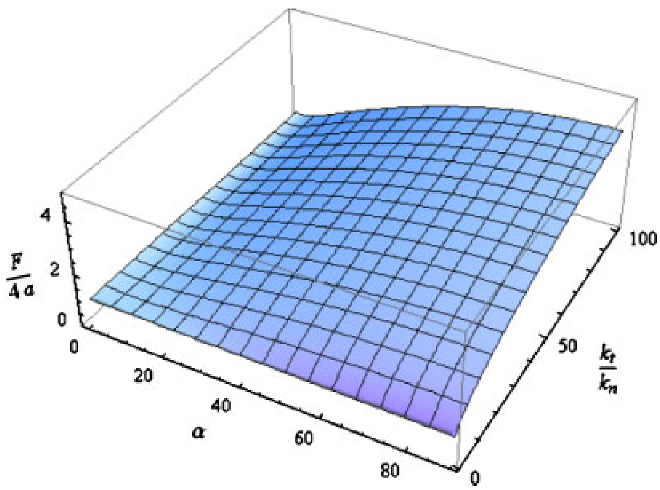

Figure 5. Influence of the orientation of the stratified porous medium and the hydraulic conductivity mismatch on the intake shape factor.

closed form. The development of the result is facilitated by a suitable coordinate transformation that reduces the problem to a mixed boundary value problem in potential theory for an ellipse. The analytical developments should not detract from the ultimate objective of the evaluation of the intake shape factor; i.e. to use the results to determine the hydraulic conductivities of the stratified porous medium as defined by k_n and k_t, and the orientation α, all of which are generally unknowns. Site investigations can usually point to the orientation of the stratification. This still requires two independent tests to evaluate k_n and k_t. The bore hole with a flat base is one test that can be used in an experimental configuration. A further test should involve an intake that has an entirely different configuration. Cylindrical intakes are certainly possible; these however will require a computational approach to estimate the intake shape factor. An alternative is to use a formed hemi-spherical intake at the boundary as the second experiment. The intake shape factor for this case is, by virtue of symmetry and invariance, exactly half of that for the spherical intake. The relevant results are given by Selvadurai [29, 30, 45].

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