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# Macroscopic constitutive equations of thermo-poroviscoelasticity derived using eigenstrains

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## ABSTRACT

Macroscopic constitutive equations for thermo-viscoelastic processes in a fully saturated porous medium are re-derived from basic principles of micromechanics applicable to solid multi-phase materials such as composites. Simple derivations of the constitutive relations and the void occupancy relationship are presented. The derivations use the notion of eigenstrain or, equivalently, eigenstress applied to the separate phases of a porous medium. Governing coupled equations for the displacement components and the fluid pressure are also obtained.

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## 1. Introduction

The theory of poroelasticity is an extension of the classical theory of elasticity for a continuum that accounts for both a continuous pore structure in the medium and the presence of fluids occupying the pore space. The classical theory in its complete form was first proposed by Biot (1941) and formed the basis for the three-dimensional theory of consolidation of a fully saturated soil with an elastic skeleton. Reviews of the subject and recent developments in the area are given by Selvadurai (1996, 2007), Coussy (2004), Gueguen and Bouteca (2004), Verruijt (2005), Dormieux et al. (2006), and Schanz (2009). Further applications include bone mechanics (Cowin, 2001), soft tissue mechanics (Cimrman and Rohan, 2007), energy resources extraction and recovery (Abousleiman and Ekbote, 2005) and geoenvironmental applications related to the geologic disposal of heat emitting nuclear fuel wastes (Berkowitz et al., 1988; Selvadurai and Nguyen, 1995).

Biot's theory was extended by several researchers to include the effects of compressibility of both the fluid and solid phases (Rice and Cleary, 1976). Extensions of the theory to include thermal effects were given by McTigue (1986), Savvidou and Booker (1989), Selvadurai and Nguyen (1995), and Khalili and Selvadurai (2003). In these studies, the constitutive relations for the thermo-poroelastic response of the medium, including the thermal expansion of solid grains and liquid, are taken into consideration. Analytical solutions for the poroviscoelastic medium can be found, for example, in the papers by Freudenthal and Spillers (1962), Marvin (1972), Vgenopoulou and Beskos (1992), Janno and Wolfersdorf (1998), and Schanz and Cheng (2001). Chateau and Dormieux (2002) and Dormieux et al. (2006) derived the equations of poroelasticity by treating the fluid pressure

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Nomenclature			
		$\alpha_f$	linear thermal expansion coefficient of liquid
		$\varepsilon_V$	volumetric strain of soil
$n$	porosity, ratio of volume of pores to the total volume of soil	$\varepsilon_{ij}$	strain components
		$\sigma_{ij}$	stress components
$K_D$	bulk modulus of drained soil	$p$	pressure
$G_D$	shear modulus of drained soil	$k$	permeability
$K_s$	bulk modulus of solid grains	$\gamma$	viscosity of liquid
$K_f$	bulk modulus of fluid	$v$	fluid velocity
$\alpha_s$	linear thermal expansion coefficient of solid grains	$T$	temperature change

as a prestress applied to the skeleton and using Levin's (1967) formula to evaluate the contribution of this prestress to the overall stress. The complete set of the poroelasticity equations can be found in the paper by Norris (1992).

One of the objectives of this paper is to link the micromechanics eigenstrain theory (ET), used primarily in studies related to solid heterogeneous materials, with the macroscopic theory of poroelasticity of the fully saturated medium. The eigenstrain approach was not employed in the classical studies of Rice and Cleary (1976), Burridge and Keller (1981) and Norris (1992); it relies on the combination of the definitions of eigenstresses or eigenstrains, as given by Iwakuma and Nemat-Nasser (1983), Mura (1987) and Dvorak et al. (1994), and the general principles of micromechanics. In particular, the eigenstrain is defined as the stress-free transformation strain that a body would undergo if the stresses were equal to zero, while the eigenstress is defined as the stress that a body would experience if the displacements were set to zero. Also, if the stiffness of a body is zero, then any stress existing in the body is an eigenstress; similarly, if a body is too rigid, then any strain is an eigenstrain. Thus, the fluid pressure can be considered as an eigenstress applied to the "empty" pores of the drained poroelastic medium or the skeleton.

By using the eigenstrain theory, we extend the classical equations of poroelasticity to *thermo-poroviscoelastic* materials. The thermal and viscous deformations are also treated here as the eigenstrains applied to the phases of the poroelastic medium. Other eigenstrains can also be taken into consideration; these include drying and moisture induced stresses (Mihoubi and Bellagi, 2009), chemical degradation of the material caused by exposure to water containing sulfate ions (Bary, 2008), plastic deformations (Dvorak et al., 1994), loss or gain of mass and eigenstrains associated with the change in the molecular structure of the phases (melting, freezing).

We note that the eigenstrain theory is sometimes referred to as the transformation field analysis. Recent developments on the subject can be found in Michel and Suquet (2003).

## 2. The constitutive relationships for thermo-poroviscoelasticity

Consider a porous medium consisting of a solid phase, or solid grains and pore space occupied by a liquid. The porous medium is characterized by the porosity  $n$ , the ratio of the volume of the pores  $V_p$  to the total volume of the medium  $V$ , i.e.,  $n = V_p/V$ .

Constitutive relations are often expressed in terms of the elastic properties of the material under drained conditions, i.e., skeleton response. Hence, all eigenstresses or eigenstrains existing in the porous media must be considered as being applied to the skeleton. By applying these eigenfields, the behaviour of the porous medium under general conditions (not necessarily drained) can be simulated. Following the general form of the overall stress–strain relation for heterogeneous materials in the presence of eigenfields (see Eq. (A.3)), the total stress in the isotropic porous medium can be expressed as

$$\sigma_{ij} = 2G_D \varepsilon_{ij} + \left( K_D - \frac{2G_D}{3} \right) \varepsilon_V \delta_{ij} + \lambda_{ij}^T + \lambda_{ij}^p + \lambda_{ij}^m \quad (1)$$

where  $\varepsilon_{ij}$  are the strain components,  $K_D$  and  $G_D$  are the bulk and shear modulus, respectively, of the soil under drained conditions, and  $\lambda_{ij}^T$ ,  $\lambda_{ij}^p$ ,  $\lambda_{ij}^m$  are the overall eigenstresses due to thermal, fluid pressure, and inelastic effects. All fields in (1) are, in general, functions of position  $\mathbf{x}$  and time  $t$ .

We now need to obtain  $\lambda_{ij}^T$ . When a temperature change  $T_s$  is applied to the solid phase, the thermal (volumetric) eigenstrain in the solid phase is  $\mu_s = 3\alpha_s T_s$ , where  $\alpha_s$  is the linear thermal expansion coefficient of the solid phase. It follows from the relationship between eigenstresses and eigenstrains (A.2) that the thermal eigenstress in the solid phase is  $\lambda_s = -K_s 3\alpha_s T_s$ , where  $K_s$  is the bulk modulus of the solid phase. To convert this local eigenstress to the overall eigenstress applied to the drained medium, we employ the result of Levin (1967) (see (A.6)),

$$\lambda^T = (1-n)A_s \lambda_s = (1-n) \left( \frac{1}{1-n} \frac{K_D}{K_s} \right) (-K_s 3\alpha_s T_s) = -K_D 3\alpha_s T_s \quad (2)$$

where  $A_s$  is the (volumetric) strain concentration factor for the solid grains and  $n$  is the porosity. The temperature change applied to the fluid phase  $T_f$  gives rise to the local eigenstress in the fluid, given by  $\lambda_f = -K_f 3\alpha_f T_f$ . This eigenstress

contributes to the total eigenstress  $-p$  existing in the fluid phase, where  $p$  is the fluid pressure, and it will enter the constitutive equations through  $\lambda^p$ .

Next, consider the action of the fluid pressure  $p$ . The stress in the pore space is equal to  $-p$ , assuming that the compressive fluid pressure is positive. Since the stiffness of the pores under drained conditions is equal to zero, the stress  $-p$  can also be considered as an eigenstress applied to the pore space of the skeleton, according to (A.1). To convert this local eigenstress to the overall eigenstress, applied to the skeleton, we again make use of the result (A.6), which gives

$$\lambda^p = nA_p(-p) = -n\left(\frac{1}{n}\left(1 - \frac{K_D}{K_s}\right)\right)p = -\alpha p \quad (3)$$

where  $A_p$  is the (volumetric) strain concentration factor for the pore, and the Biot parameter  $\alpha$  is defined as

$$\alpha = 1 - \frac{K_D}{K_s} \quad (4)$$

It is worthwhile noting that the local eigenstress in the pore space,  $-p$ , in (3), and the local thermal eigenstress of the solid phase,  $-K_s\alpha_s T_s$ , in (2) are not uniform on the scale of the pore, and the solid grain, respectively. Therefore, using Levin's formula (A.6) to find a corresponding overall eigenstress gives only an approximate value of the overall eigenstress.

Substituting (2) and (3) into (1) leads to the thermo-poroelastic constitutive relation for the porous medium as

$$\sigma_{ij} = 2G_D \varepsilon_{ij} + \left(K_D - \frac{2G_D}{3}\right) \varepsilon_V \delta_{ij} - K_D 3\alpha_s T_s \delta_{ij} + \lambda_{ij}^{in} - \left(1 - \frac{K_D}{K_s}\right) p \delta_{ij} \quad (5)$$

Let us define the effective stress acting on the solid skeleton or the porous medium under drained conditions as

$$\sigma'_{ij} = 2G_D \varepsilon_{ij} + \left(K_D - \frac{2G_D}{3}\right) \varepsilon_V \delta_{ij} - K_D 3\alpha_s T_s \delta_{ij} + \lambda_{ij}^{in} \quad (6)$$

Thus, the total stress in the porous material is simply  $\sigma_{ij} = \sigma'_{ij} - \alpha p \delta_{ij}$ .

We now consider eigenstresses due to viscous behaviour of the solid skeleton. We can identify the eigenstress  $\lambda_{ij}^{in}$  with the relaxation stresses in the skeleton, i.e.,  $\lambda_{ij}^{in} = \sigma_{ij}^{rel}$ . In order to define the viscous behaviour of the solid skeleton, we need to consider the decomposition of the effective stress in the skeleton into its isotropic and deviatoric parts, i.e.,  $\sigma'_{ij} = \sigma'_i \delta_{ij} + S'_{ij}$  and the decomposition of the total strain into isotropic and deviatoric parts,  $\varepsilon_{ij} = (\varepsilon_V/3)\delta_{ij} + e_{ij}$ . Here, the isotropic stress is  $\sigma'_i = \sigma'_{ii}/3$  and the volumetric strain is  $\varepsilon_V = \varepsilon_{ii}$ . The relaxation stress of the skeleton can also be decomposed into its isotropic and deviatoric parts, i.e.,  $\sigma_{ij}^{rel} = \sigma_i^{rel} \delta_{ij} + S_{ij}^{rel}$ . Within the framework of linear viscoelasticity, the isotropic part of the relaxation stress  $\sigma_i^{rel}$  is defined in terms of the hydrostatic relaxation function  $K^{rel}(t)$  as (see e.g. Selvadurai, 1992; Christensen, 2003)

$$\sigma_i^{rel} = \int_0^t \dot{K}^{rel}(t-t') \{\varepsilon_V(t') - 3\alpha_s T_s(t')\} dt' \quad (7)$$

and the deviatoric part of the relaxation stress is defined in terms of the shear relaxation function  $G^{rel}$ ,

$$S_{ij}^{rel} = \int_0^t \dot{G}^{rel}(t-t') \varepsilon_{ij}(t') dt' \quad (8)$$

where the dot signifies differentiation with respect to time. The total eigenstress due to inelastic behaviour is given, therefore, by  $\lambda_{ij}^{in} = \sigma_{ij}^{rel} = \sigma_i^{rel} \delta_{ij} + S_{ij}^{rel}$ .

In results (7) and (8) we used the stress relaxation functions, and not the creep compliance functions to characterize the viscoelastic behaviour of a body in order to conveniently input the data into finite element codes such as ABAQUS (2008). In (7) and (8) the influence of the temperature change on the relaxation functions  $K^{rel}(t)$ ,  $G^{rel}(t)$  can also be taken into account. For thermorheologically simple materials (Selvadurai, 1992), the effect of the temperature change on the material behaviour is introduced by defining the reduced time variable as  $\xi(t) = \int_0^t d\zeta/A(T(\zeta))$ , where  $A(T)$  is a given function of the temperature change  $T$ , and by replacing  $K^{rel}(t)$  with  $K^{rel}(\xi(t))$ , and  $G^{rel}(t)$  with  $G^{rel}(\xi(t))$ . Consequently,  $\dot{K}^{rel}(t)$  is to be replaced with  $(dK^{rel}(\xi)/d\xi)\dot{\xi}(t)$ ,  $\dot{G}^{rel}(t)$  with  $(dG^{rel}(\xi)/d\xi)\dot{\xi}(t)$ . If the temperature change  $T$  is constant, i.e.,  $T_0$ , then  $\xi(t) = t/A(T_0)$  and therefore,  $\dot{\xi}(t) = 1/A(T_0)$ —constant.

To obtain the first governing equation, the standard equations of equilibrium and the strain–displacement relations are used. The equilibrium equations are

$$\sigma_{ij,j} + \rho g_i = 0 \quad (9)$$

where  $\rho = n\rho_f + (1-n)\rho_s$  is the overall density of the porous media, and  $g_i$  are components of the gravitational acceleration vector.

The infinitesimal strain components  $\varepsilon_{ij}$  are related to the displacement components  $u_i$  by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (10)$$

with  $\varepsilon_V = \varepsilon_{ii} = u_{i,i}$ .

Using (5) and (10), the equilibrium equation (9) now becomes

$$\left(K_D + \frac{G_D}{3}\right)u_{k,ki} + G_D u_{i,kk} - K_D 3\alpha_s T_{s,i} - \left(1 - \frac{K_D}{K_s}\right)p_{,i} + \sigma_{i,i}^{rel} + s_{ij,j}^{rel} + \rho g_i = 0 \quad (11)$$

The dependent variables are the displacement components  $u_i$ , the pressure  $p$  and the temperature of the solid phase  $T_s$ . *Void occupancy equation (VOE)*. The volumetric strain in the fluid is equal to

$$\varepsilon_{vf} = -\frac{p}{K_f} + 3\alpha_f T_f + \mu_m \quad (12)$$

where  $T_f$  is the temperature change in the fluid phase,  $3\alpha_f T_f$  is the thermal eigenstrain in the fluid, and  $\mu_m$  is the eigenstrain associated with either a loss or gain of liquid mass. The latter eigenstrain is equal to

$$\mu_m = \frac{\Delta m_f}{m_f} \quad (13)$$

where  $\Delta m_f$  is the change in the liquid mass. Therefore,

$$\varepsilon_{vf} = -\frac{p}{K_f} + 3\alpha_f T_f + \frac{\Delta m_f}{m_f} \quad (14)$$

The volumetric strain in the solid phase is

$$\varepsilon_{vs} = \frac{\sigma_{is}}{K_s} + 3\alpha_s T_s + \varepsilon_{vs}^{cr} \quad (15)$$

where  $\sigma_{is}$  is the average of the isotropic stress in the solid phase,  $\sigma_{is} = \sigma_{ii}^{(s)}/3$ , and  $\varepsilon_{vs}^{cr}$  is the creep (viscous) strain in the solid phase.

The averaged strain of the entire porous medium is equal to the sum of averages of the strains in the fluid and solid phases: i.e.,

$$\varepsilon_V = n\varepsilon_{vf} + (1-n)\varepsilon_{vs} \quad (16)$$

Using the expressions for the volumetric strain of the solid grains (15) and the volumetric strain in the liquid (14), we can reduce (16) to the result,

$$\varepsilon_V = (1-n)\left(\frac{\sigma_{is}}{K_s} + 3\alpha_s T_s + \varepsilon_{vs}^{cr}\right) + n\left(-\frac{p}{K_f} + 3\alpha_f T_f + \frac{\Delta m_f}{m_f}\right) \quad (17)$$

Now we eliminate the isotropic stress of the solid phase  $\sigma_{is}$  in (17). The total (isotropic) stress of the porous medium  $\sigma_I$  is

$$\sigma_I = -np + (1-n)\sigma_{is} \quad (18)$$

but from the overall stress–strain relationship (5), this stress can be written as

$$\sigma_I = K_D \varepsilon_V - 3K_D \alpha_s T_s + \sigma_I^{rel} - \alpha p \quad (19)$$

By equating (18) and (19), the stress of the solid phase  $\sigma_{is}$  can be found. On substituting this stress into (17) and performing simplifications we find

$$-n\frac{\Delta m_f}{m_f} + \alpha \varepsilon_V + n\frac{p}{K_f} + (\alpha-n)\frac{p}{K_s} = n3\alpha_f T_f + (\alpha-n)3\alpha_s T_s + (1-\alpha)\frac{\sigma_I^{rel}}{K_D} + (1-n)\varepsilon_{vs}^{cr} \quad (20)$$

where the Biot parameter  $\alpha$  is defined in (4).

The change in the fluid mass can be expressed in terms of the fluid velocity,

$$\frac{\Delta m_f}{m_f} = -\int_0^t v_{i,l} dt \quad (21)$$

where  $v_{i,l}$  is the divergence of the velocity. Note that the divergence of the velocity in (21) cannot be obtained by taking the time derivative of the fluid strain (14). The velocity in (21) is the velocity of the fluid entering and exiting a control volume, which can be taken as the meso-sized cube centered at point  $\mathbf{x}$ . Substitution of (21) into the void occupancy equation (20) and subsequent differentiation with respect to time gives

$$nv_{i,l} + \alpha \dot{\varepsilon}_V + n\frac{\dot{p}}{K_f} + (\alpha-n)\frac{\dot{p}}{K_s} = n3\alpha_f \dot{T}_f + (\alpha-n)3\alpha_s \dot{T}_s + (1-\alpha)\frac{\dot{\sigma}_I^{rel}}{K_D} + (1-n)\dot{\varepsilon}_{vs}^{cr} \quad (22)$$

In deriving (22) from (20), we used the fact that

$$\frac{d}{dt}(n * \text{any strain}) = n * \frac{d(\text{strain})}{dt} + \text{strain} * \frac{dn}{dt} \approx n * \frac{d(\text{strain})}{dt}$$

since the time-dependent change in the porosity  $n$  multiplied by the strain can be neglected. For example,  $d/dt(np/K_f) \approx n\dot{p}/K_f$ , and the strain here is  $p/K_f$ .

Darcy's Law: Darcy's Law for an isotropic porous medium can be written as

$$nv_i = -\frac{k}{\gamma}(p_{,i} - \rho_f g_i) \quad (23)$$

where  $k$  is the permeability,  $\gamma$  is the viscosity of liquid and  $g_i$  is the  $i$ -th component of the gravitational acceleration vector. Eq. (23) can be made more precise if the velocity of the fluid in relation to the velocity of the porous skeleton is used; however, in most practical applications, the velocity of the porous solid can be neglected in relation to the pore fluid velocity.

Recently Selvadurai and Selvadurai (2010) examined in detail, both experimentally and computationally, the effective permeability of a porous medium that has an inhomogeneous or multiphase structure. These authors have presented measures for effective permeability that are based on the geometric mean of a statistical distribution.

Substituting Darcy's Law (23) into (22) gives the governing equation for the displacement components and the pressure,

$$-\left[\frac{k}{\gamma}(p_{,i} - \rho_f g_i)\right]_{,i} + \alpha \dot{\epsilon}_V + \frac{n}{K_f} \dot{p} + \frac{\alpha - n}{K_s} \dot{p} = n3\alpha_f \dot{T}_f + (\alpha - n)3\alpha_s \dot{T}_s + (1 - \alpha) \frac{\dot{\sigma}_I^{rel}}{K_D} + (1 - n) \dot{\epsilon}_{Vs}^{cr} \quad (24)$$

*Heat conduction equation:* We assume that the heat transfer in the porous medium occurs by conduction. The heat conduction equation can be solved to obtain a macroscopic variation of temperature  $T$  in the porous media,

$$c_p^* \frac{\partial T}{\partial t} - k_c^* T_{,kk} = 0 \quad (25)$$

where  $c_p^*$  is the effective specific heat of the porous medium and  $k_c^*$  is the effective thermal conductivity of the porous medium. This equation can be derived from Fourier's Law and the thermal energy balance equation. Both  $c_p^*$  and  $k_c^*$  signify the effective properties of the fully saturated porous medium.

We also need to determine the fluid temperature  $T_f$  and the temperature of the solid  $T_s$  knowing the overall temperature  $T$ . One simple approximation, used in the general-purpose finite element program ABAQUS, is to assume that  $T_f = T_s = T$ . An effective conductivity of the porous medium, which is consistent with this choice of the functions  $T_f$ ,  $T_s$ , is then given by a simple volume average of phase conductivities, i.e.,

$$k_c^* = nk_{cf} + (1 - n)k_{cs} \quad (26)$$

where  $k_{cf}$  and  $k_{cs}$  are the thermal conductivities of the fluid and solid phase, respectively. Khalili and Selvadurai (2003) derived equations that included both the temperature of the solid phase  $T_s$  and the overall temperature  $T$  as dependent variables.

Our numerical results suggest that the effective specific heat can be approximated by

$$c_p^* = n\rho_f c_f + (1 - n)\rho_s c_s \quad (27)$$

Here  $c_f$  and  $c_s$  are the specific heats of the fluid phase and the solid phase, respectively,  $\rho_f$  and  $\rho_s$  are their densities. More accurate expression for the effective specific heat of the isotropic two-phase medium can be found in Christensen (1979).

### 3. Numerical results

To validate the eigenstrain theory (ET), in which the thermal and creep (viscous) strains are considered as eigenstrains applied to the poroelastic material, solution of a simple thermo-poroviscoelastic problem will be presented. This solution will then be compared with other available solutions obtained analytically with the help of the linear correspondence principle and numerically by using the ABAQUS software.

Consider the one-dimensional deformation of a column of a fully saturated porous material with a viscoelastic skeleton. Displacement of the base of the column is set to zero, but the movement of the upper surface is unconstrained (Fig. 1). The external surface of the column is insulated against flow of fluid. The upper surface remains traction free at all times.

We also assume that, as a result of heating, in the long term the temperature change inside the column becomes uniform and equal to  $T_0 > 0$ .

The solid skeleton is assumed to be linear viscoelastic. The hydrostatic and shear stress relaxation functions in (7) and (8) are assumed to be of the form

$$K^{rel}(t) = K_D[1 - \chi\{1 - \exp(-t/\tau)\}], \quad G^{rel}(t) = G_D[1 - \chi\{1 - \exp(-t/\tau)\}] \quad (28)$$

where  $\tau$  is the relaxation time,  $\chi$  is a material parameter. If  $\chi = 1$ , the relaxation functions (28) correspond to a viscoelastic body with a Maxwell material type response in shear and hydrostatic tension/compression. If the material is thermorheologically simple, the time variable  $t$  in (28) must be replaced by the reduced time  $\xi(t)$ , which, in turn, depends on the temperature change  $T(t)$ . However, we will not consider the dependence of the relaxation functions on the temperature change since ultimately we are concerned with the long-term response of the material for which  $\xi(t) = t/A(T_0)$  as  $t \rightarrow \infty$ .

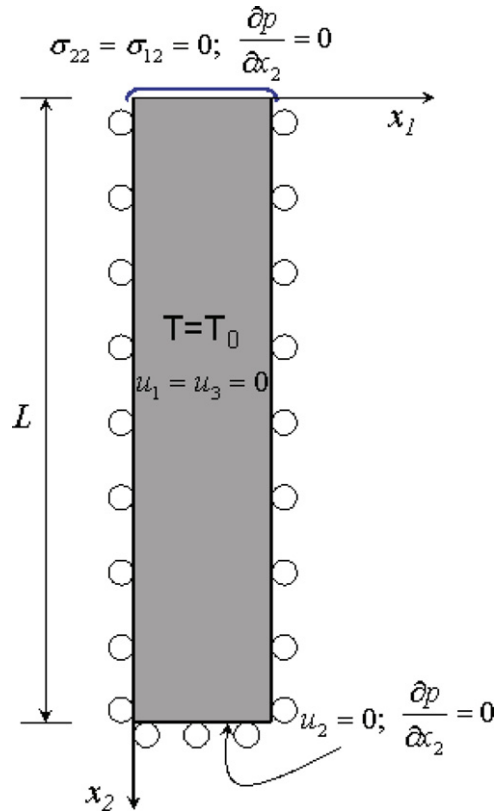


Fig. 1. Geometry of poroviscoelastic column subjected to temperature change  $T_0$ . No fluid escapes or enters the column (undrained column).

To obtain the analytical solution, we use the stress–strain equation (5) and the void occupancy equation (20), which, for the given problem, take the form

$$\left(K_D + \frac{4G_D}{3}\right)u_{2,2} - K_D 3\alpha_s T - \alpha p + \int_0^t \dot{K}^{rel}(t-t')(u_{2,2}(t') - 3\alpha_s T(t')) dt' + \frac{4}{3} \int_0^t \dot{G}^{rel}(t-t')u_{2,2}(t') dt' = 0 \quad (29)$$

$$\alpha u_{2,2} + p \left(\frac{n}{K_f} + \frac{\alpha - n}{K_s}\right) = n 3\alpha_f T + (\alpha - n) 3\alpha_s T + (1 - n)e_{Vs}^{ct} + \frac{1}{K_s} \int_0^t \dot{K}^{rel}(t-t')(u_{2,2}(t') - 3\alpha_s T(t')) dt' \quad (30)$$

In Eq. (30), since there is no loss or gain of liquid mass in the column, the change in the mass of the liquid  $\Delta m_f$  is zero. In Eq. (29), we make use of the fact that the total stress  $\sigma_{22}$  in the longitudinal direction is zero (no applied stresses). The only nonzero displacement is  $u_2$ . To simplify the problem further, we assume that the thermal expansion of the solid phase  $\alpha_s$  is much smaller than the thermal expansion of the liquid  $\alpha_f$ . Consequently, we neglect all terms containing the thermal expansion of the solid phase  $\alpha_s$ . Using all these assumptions, Eqs. (29) and (30) can be reduced to

$$\left(K_D + \frac{4G_D}{3}\right)u_{2,2} - \alpha p + \int_0^t \dot{K}^{rel}(t-t')u_{2,2}(t') dt' + \frac{4}{3} \int_0^t \dot{G}^{rel}(t-t')u_{2,2}(t') dt' = 0 \quad (31)$$

$$\alpha u_{2,2} + p \left(\frac{n}{K_f} + \frac{\alpha - n}{K_s}\right) = n 3\alpha_f T + (1 - n)e_{Vs}^{ct} + \frac{1}{K_s} \int_0^t \dot{K}^{rel}(t-t')u_{2,2}(t') dt' \quad (32)$$

By applying the Laplace transform to Eqs. (31) and (32) and by using the final value theorem, we can derive the final (uniform) value of the fluid pressure and displacement at the upper surface  $x_2=0$ , as predicted by the eigenstrain theory (ET),

$$p^{ET}(t = \infty) = \frac{n 3\alpha_f T_0 + (1 - n)e_{Vs}^{ct}}{\frac{n}{K_f} + \frac{\alpha - n}{K_s} + \frac{\alpha^2 + \alpha\chi(1 - \alpha)}{(1 - \chi)(K_D + 4G_D/3)}}$$

$$u_2^{ET}(0, t = \infty) = \alpha L \frac{n3\alpha_f T_0 + (1-n)e_{V_s}^{cr}}{\left(\frac{n}{K_f} + \frac{\alpha-n}{K_s}\right)(1-\chi)(K_D + 4G_D/3) + \alpha^2 + \alpha\chi(1-\alpha)} \quad (33)$$

where  $L$  is the length of the column. Notice that for the Maxwell model, where  $\chi = 1$ , the final value of the fluid pressure is zero.

To obtain the volumetric creep (viscous) strain in the solid phase  $\varepsilon_{V_s}^{cr}$ , we assume that the solid phase is linear viscoelastic and its stress relaxation functions can be written as

$$\begin{aligned} K_s^{rel}(t) &= K_s[1 - \chi_s\{1 - \exp(-t/\tau)\}] \\ G_s^{rel}(t) &= G_s[1 - \chi_s\{1 - \exp(-t/\tau)\}] \end{aligned} \quad (34)$$

where  $\chi_s, \tau$  are material parameters chosen to be identical for the hydrostatic and shear stress relaxation functions, as in (28). For simplicity, the relaxation time  $\tau$  for the solid phase is set equal to the relaxation time for the skeleton.

The volumetric creep strain in the solid phase can thus be represented in an integral form, similar to (7),

$$\varepsilon_{V_s}^{cr} = -\frac{\sigma_{V_s}^{rel}}{K_s} = -\frac{1}{K_s} \int_0^t \dot{K}_s^{rel}(t-t') \varepsilon_{V_s}(t') dt' \quad (35)$$

where the thermal expansion coefficient of the solid phase  $\alpha_s$  was neglected. From (14) and (16) the total volumetric strain in the solid phase  $\varepsilon_{V_s}$  can be expressed in terms of the volumetric strain of the skeleton as

$$\varepsilon_{V_s} = \frac{1}{1-n} \left[ \varepsilon_V - n \left( -\frac{p}{K_f} + 3\alpha_f T + \frac{\Delta m_f}{m_f} \right) \right] \quad (36)$$

Then, substituting (36) into (35) gives

$$\varepsilon_{V_s}^{cr} = -\frac{1}{K_s(1-n)} \int_0^t \dot{K}_s^{rel}(t-t') \left[ \varepsilon_V(t') + n \frac{p(t')}{K_f} - n3\alpha_f T(t') \right] dt' \quad (37)$$

since  $\Delta m_f = 0$ . To determine the stress relaxation function for the solid phase  $K_s^{rel}(t)$  in (34), in particular the parameter  $\chi_s$ , we establish a connection between the stress relaxation functions of the skeleton  $K^{rel}(t)$  (supposedly known) and the stress relaxation function of the solid phase  $K_s^{rel}(t)$ . We examine the problem of a porous medium possessing a specific microstructure, which permits this connection to be established.

As an example, consider a composite sphere assemblage and assume that each sphere has a fluid-filled inclusion. The bulk and shear moduli of the solid phase-shell are denoted by  $K_s$  and  $G_s$ , respectively, and the bulk modulus of the fluid is  $K_f$ . All the spheres in the assemblage are topologically equivalent.

The bulk modulus of the undrained composite sphere assemblage is given by Christensen (1979) and Milton (2002)

$$K_u = K_f + (1-n)(K_s - K_f) \frac{3K_f + 4G_s}{4G_s + 3nK_s + 3K_f(1-n)} \quad (38)$$

By setting  $K_f=0$  in (38) we can find the overall bulk modulus of the assemblage under drained conditions or, equivalently, the bulk modulus of the skeleton, as

$$K_D = (1-n) \frac{K_s}{1 + \frac{3nK_s}{4G_s}} \quad (39)$$

Then, by applying, for example, the linear correspondence principle and the final value theorem, we can derive, using (39), the connection between the long-term values of the stress relaxation functions of the skeleton and solid phase,

$$K_D(1-\chi) = (1-n) \frac{K_s(1-\chi_s)}{1 + \frac{3nK_s(1-\chi_s)}{4G_s(1-\chi_s)}} \quad (40)$$

which gives

$$\chi_s = \chi \quad (41)$$

A different result would be obtained if, for example, the solid phase were elastic in hydrostatic tension/compression, and viscoelastic only in shear, or vice versa.

The long term value of the creep strain in the solid phase can now be found by applying the Laplace transform and the final value theorem to (37), and using the fact that the stress relaxation parameters for the solid phase  $\chi_s$  and for the skeleton  $\chi$  are equal. This gives

$$\varepsilon_{V_s}^{cr}(t = \infty) = \frac{\chi}{1-n} \left( \varepsilon_V + \frac{np}{K_f} - n3\alpha_f T \right) \quad (42)$$

The creep strain (42) can now be substituted into the expressions (33). Alternatively, we also can substitute (42) into the Laplace transform of the constitutive equations (31) and (32) and find the pressure and the displacement in the column in

this way. This gives

$$p^{ET}(t = \infty) = \frac{n3\alpha_f T_0}{\frac{n}{K_f} + \frac{\alpha-n}{K_s(1-\chi)} + \frac{\alpha^2}{(1-\chi)(K_D+4G_D/3)}}$$

$$u_2^{ET}(0, t = \infty) = \alpha L \frac{n3\alpha_f T_0}{\left(\frac{n}{K_f} + \frac{\alpha-n}{K_s(1-\chi)}\right)(1-\chi)(K_D+4G_D/3) + \alpha^2} \quad (43)$$

To test the accuracy of the eigenstrain theory solution (Eqs. (33) and (43)), we performed the finite element computations with the help of the ABAQUS software. The analytical and numerical results were compared for different values of the parameter  $\chi$  in (28) and zero and non-zero values of the averaged volumetric creep strain  $\epsilon_{Vs}^{cr}$  in the solid phase. Table 1 lists the specific properties of the geomaterial and its constituents along with the applied loadings used in the computations.

By default, the ABAQUS code assumes that the solid phase is elastic, which makes the averaged volumetric creep strain in the solid phase equal to zero, i.e.,  $\epsilon_{Vs}^{cr} = 0$ . The ABAQUS solution coincides with the linear correspondence principle solution that can be obtained from the stress–strain relation (5) and the void occupancy equation (20), reduced to the elastic case, by replacing  $K_D$  with  $K_D(1-\chi)$ ,  $G_D$  with  $G_D(1-\chi)$ ,  $K_s$  with  $K_s$  (no change),

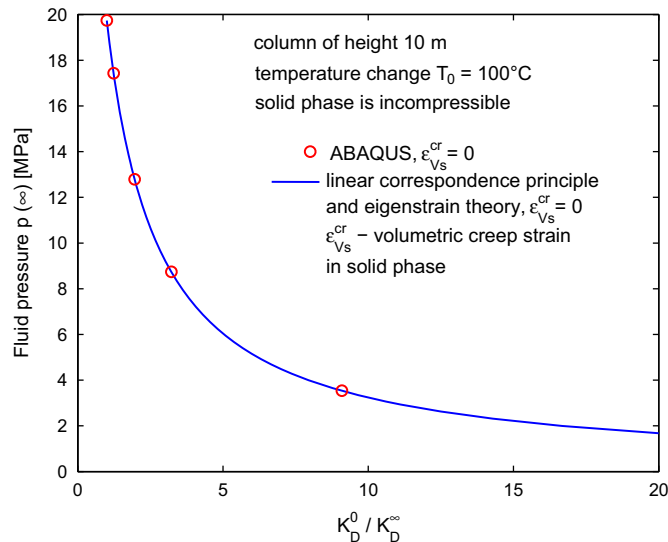
$$p^{AB}(t = \infty) = \frac{n3\alpha_f T_0}{\frac{n}{K_f} + \frac{\alpha-n+\chi(1-\alpha)}{K_s} + \frac{(\alpha+\chi(1-\alpha))^2}{(1-\chi)(K_D+4G_D/3)}}$$

$$u_2^{AB}(0, t = \infty) = L \frac{n3\alpha_f T_0(\alpha+\chi(1-\alpha))}{\left(\frac{n}{K_f} + \frac{\alpha-n+\chi(1-\alpha)}{K_s}\right)(1-\chi)(K_D+4G_D/3) + (\alpha+\chi(1-\alpha))^2} \quad (44)$$

**Table 1**

Properties of soil and applied loads.

Property or parameter	Value
Porosity, $n$	0.25
Young's modulus, $E_D$	$5 \times 10^9$ Pa
Poisson's ratio, $\nu_D$	0.3
Linear thermal expansion of solid phase, $\alpha_s$	$8.3 \times 10^{-6}$ (1/°C) (neglected)
Linear thermal expansion of liquid, $\alpha_f$	$69 \times 10^{-6}$ (1/°C)
Fluid bulk modulus, $K_f$	$2.2 \times 10^9$ Pa
Solid phase bulk modulus, $K_s$	$10.41 \times 10^9$ Pa
Temperature change, $T_0$	100 °C
Applied surface tractions	0 Pa



**Fig. 2.** Long-term fluid pressure in the poroviscoelastic undrained column subjected to a positive temperature change of 100 °C. The solid phase is incompressible and assumed to be elastic.



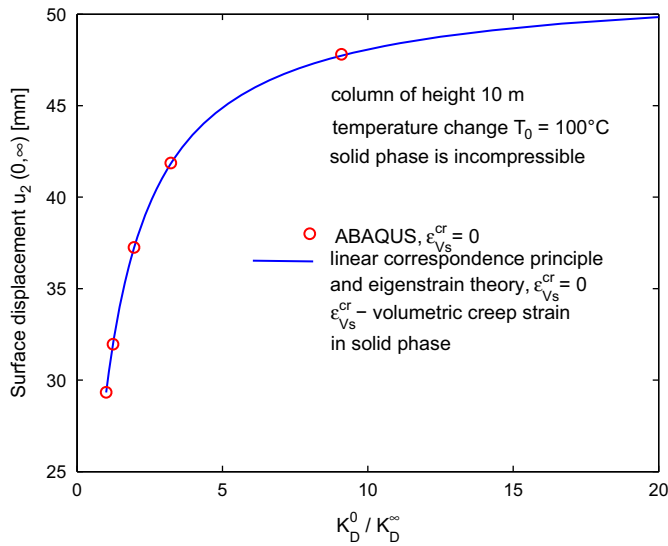
Figs. 2–7 present the results of the computations. The long-term values of the pressure and axial displacement of the column are plotted for different values of the ratio of the instantaneous bulk modulus to the long term bulk modulus, i.e.,

$$\frac{K_D^0}{K_D^\infty} = \frac{G_D^0}{G_D^\infty} = \frac{1}{1-\chi}$$

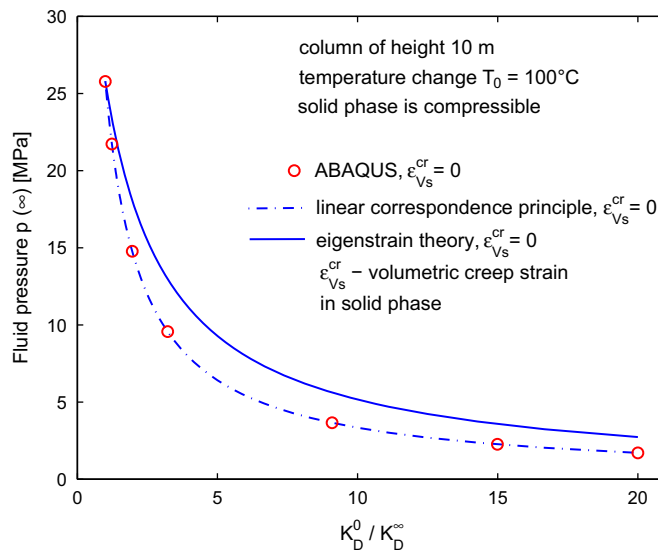
$K_D^0/K_D^\infty = 1$  corresponds to the elastic solution, while  $K_D^0/K_D^\infty = \infty$  implies complete loss of stiffness of the skeleton in the long term, which is possible only for the Maxwell material when  $\chi = 1$ .

In Figs. 2 and 3 we compare the ABAQUS result (44) with the eigenstrain theory solution (33). To match the ABAQUS results, we set the averaged volumetric creep strain of the solid phase  $\epsilon_{Vs}^{cr}$  equal to zero in Eq. (33). The solid phase is assumed to be incompressible. In this case, the ET solution (33) and the ABAQUS result (44) coincide.

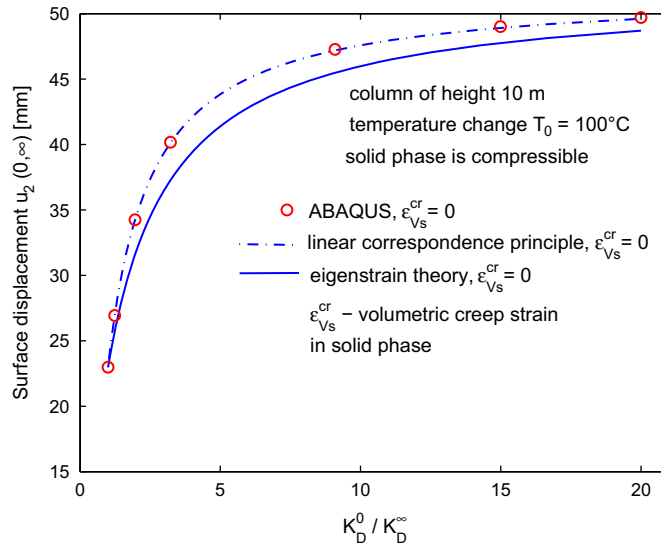
Figs. 4 and 5 show similar results for the porous medium with a compressible solid phase; Biot’s parameter is equal to  $\alpha = 0.6$  (from Table 1). As before, the averaged volumetric creep strain in the solid phase  $\epsilon_{Vs}^{cr}$  is assumed to be zero. In this case, the finite element results (44) coincide with the solution obtained by using the linear correspondence principle;



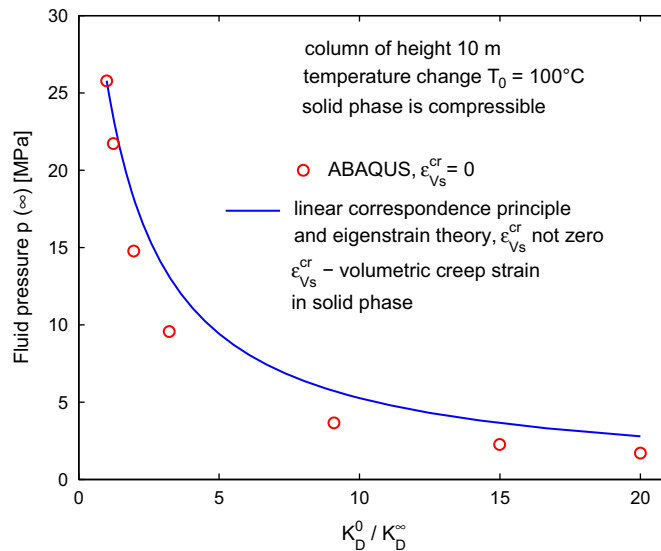
**Fig. 3.** Long-term displacement (absolute value) at the upper surface of the poroviscoelastic undrained column, shown in Fig. 1, subjected to a positive temperature change of 100 °C. The solid phase is incompressible and assumed to be elastic, i.e., the averaged volumetric creep strain  $\epsilon_{Vs}^{cr}$  in the solid phase is zero.



**Fig. 4.** Long-term fluid pressure in the poroviscoelastic undrained column subjected to a positive temperature change of 100 °C. The solid phase is compressible and assumed to be elastic.



**Fig. 5.** Long-term displacement (absolute value) at the upper surface of the poroviscoelastic undrained column, shown in Fig. 1, subjected to a positive temperature change of 100 °C. The solid phase is compressible and assumed to be elastic.



**Fig. 6.** Long-term fluid pressure in the poroviscoelastic undrained column subjected to a positive temperature change of 100 °C. The solid phase is compressible. The averaged volumetric creep strain in the solid phase is not zero.

however, the eigenstrain theory solution (33) is different but the agreement between the two methods is still good. It should be noted that the eigenstrain theory predicts a somewhat stiffer response of the porous medium (i.e., smaller displacements) because the concentration factors for the solid and fluid phases  $A_s$ ,  $A_p$  are assumed constant. On the contrary, when using the linear correspondence principle the concentration factors are not constant, i.e., in the stress–strain relation (5) and the void occupancy equation (20),  $\alpha$  is replaced with  $1 - K_D(1 - \chi)/K_s$  in all terms.

Figs. 6 and 7 show the results for the viscoelastic porous medium in which the averaged volumetric creep strain in the solid phase  $\varepsilon_{Vs}^{cr}$  is not zero. The solid phase is compressible, as in Figs. 4 and 5. Here, the eigenstrain theory solution is given by (43), and it can be shown that the linear correspondence principle provides the same solution. (The linear correspondence principle solution can be obtained from (5) and (20), reduced to the elastic case, by replacing  $K_D$  with  $K_D(1 - \chi)$ ,  $G_D$  with  $G_D(1 - \chi)$ ,  $K_s$  with  $K_s(1 - \chi)$ .) We also can see that the difference between this solution and the ABAQUS result (44), in which  $\varepsilon_{Vs}^{cr} = 0$ , is not very large.

It should be noted that from Levin's relation (A.9) the overall creep strain of the skeleton can be expressed in terms of the local creep strain in the solid phase as  $\varepsilon_V^{cr} = (1 - n)B_s\varepsilon_{Vs}^{cr}$ ; thus  $\varepsilon_V^{cr} = \varepsilon_{Vs}^{cr}$  since  $B_s = 1/(1 - n)$ . By applying the Laplace

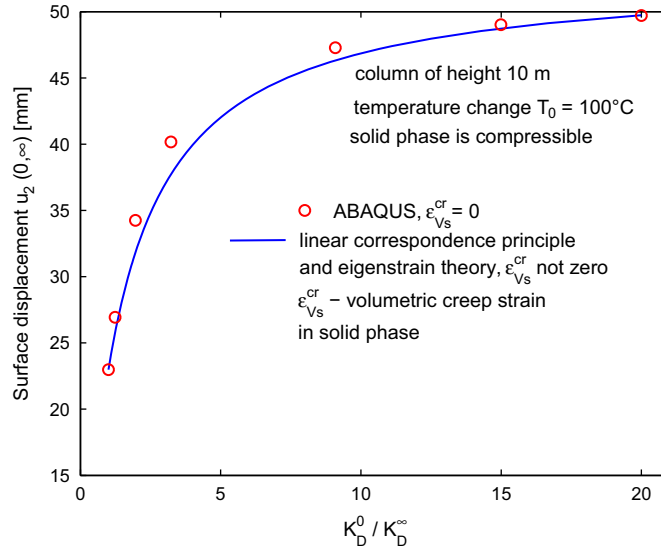


Fig. 7. Long-term displacement (absolute value) at the upper surface of the poroviscoelastic undrained column, shown in Fig. 1, subjected to a positive temperature change of 100 °C. The solid phase is compressible. The averaged volumetric creep strain in the solid phase is not zero.

transform and the final value theorem to (31) and (32), it can be shown that the equality  $\epsilon_{Vs}^{cr} = \epsilon_V^{cr}$  leads to the following solution for the pressure and displacement in the column:

$$p^{LV}(t = \infty) = \frac{n3\alpha_f T_0}{\frac{n}{K_f} + \frac{\alpha-n}{K_s} + \frac{\alpha^2 - \alpha\chi(\alpha-n)}{(1-\chi)(K_D + 4G_D/3)}}$$

$$u_2^{LV}(0, t = \infty) = L \frac{\alpha n 3\alpha_f T_0}{\left(\frac{n}{K_f} + \frac{\alpha-n}{K_s}\right) (1-\chi)(K_D + 4G_D/3) + \alpha^2 - \alpha\chi(\alpha-n)} \tag{45}$$

However, this solution is very different from the one shown in Figs. 6 and 7; in particular, for the Maxwell viscoelastic material,  $\chi = 1$ , the displacement of the upper surface  $u_2^{LV} = L3\alpha_f T$ , while the ABAQUS result, obtained from (44), is  $u_2^{AB} = nL3\alpha_f T \ll u_2^{LV}$ . This difference can be explained by the nonuniformity of the components of the creep strain in the solid phase in the given viscoelastic composite sphere assemblage, which makes application of Levin’s formula (A.9) unjustified.

#### 4. Conclusions

An alternative derivation of constitutive equations for a thermo-rheologically simple poroviscoelastic medium is presented. The derivation is based on applying the thermal and creep (viscous) eigenstrains to the poroelastic medium and using the basic principles of micromechanics of a heterogeneous material. The constitutive equations can be extended to include other forms of eigenstresses or eigenstrains, besides the thermal and viscous eigenstrains, existing in the porous medium (for example, swelling, eigenstrains associated with the change of molecular structure of the phases, and plastic eigenstrains).

From the results, it was shown that the constraint that the solid phase remains locally elastic while the overall behaviour of the skeleton is inelastic can be removed. For the simple case of a viscoelastic composite sphere assemblage we have demonstrated that the difference in the solutions for the porous medium with either (locally) an elastic or viscoelastic solid phase is not very large. Nevertheless it is suggested that the influence of the local inelastic strain on the solution be examined whenever the (overall) volumetric inelastic strain of the skeleton is non-zero.

We also showed that Levin’s formula applied to relate non-uniform local viscoelastic eigenfields to the overall viscoelastic eigenfields can yield results that are very different from those obtained, for example, by the linear correspondence principle; i.e., it is important to recognize that Levin’s formula was proven to be valid only for local eigenstrains which are uniform on the microscale.

It is worthwhile mentioning that no difference in the solutions is expected for the porous medium with a skeleton that is viscoelastic in shear, but elastic in hydrostatic tension or compression.

Finally, it should be noted that the stress–strain constitutive equation (5) can be written in terms of the bulk and shear moduli of the undrained porous medium (Norris, 1992). To include the effects of the eigenstrains into this constitutive

equation, the concentration factors for the undrained material should be used. The eigenstrains applied to the undrained porous medium are the thermal eigenstrains  $3\alpha_f T_f$ ,  $3\alpha_s T_s$ , the eigenstrain due to the change in the fluid mass  $\Delta m_f/m_f$ , and the viscoelastic eigenstrain.

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## Appendix A. Summary of the micromechanics theory

Consider a fully saturated porous medium with porosity  $n$ . The local (phase) constitutive equations at point  $\mathbf{x}$  can be written as

$$\begin{aligned}\sigma_{ij}(\mathbf{x}) &= L_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}) + \lambda_{ij}(\mathbf{x}) \\ \varepsilon_{ij}(\mathbf{x}) &= M_{ijkl}(\mathbf{x})\sigma_{kl}(\mathbf{x}) + \mu_{ij}(\mathbf{x})\end{aligned}\quad (\text{A.1})$$

where  $L_{ijkl}$  is the local (phase) stiffness,  $M_{ijkl}$  is the local compliance,  $\lambda_{ij}$ ,  $\mu_{ij}$  are the phase eigenstress and the eigenstrain, respectively. It can be shown from (A.1) that the eigenstrain and eigenstress are related by the following:

$$\lambda_{ij}(\mathbf{x}) = -L_{ijkl}\mu_{kl}(\mathbf{x}), \quad \mu_{ij}(\mathbf{x}) = -M_{ijkl}\lambda_{kl}(\mathbf{x})\quad (\text{A.2})$$

The overall constitutive equations can be written as

$$\begin{aligned}\sigma_{ij}^*(\mathbf{x}) &= L_{ijkl}^*\varepsilon_{ij}^*(\mathbf{x}) + \lambda_{ij}^*(\mathbf{x}) \\ \varepsilon_{ij}^*(\mathbf{x}) &= M_{ijkl}^*\sigma_{ij}^*(\mathbf{x}) + \mu_{ij}^*(\mathbf{x})\end{aligned}\quad (\text{A.3})$$

where  $L_{ijkl}^*$  and  $M_{ijkl}^*$  are the overall stiffness and compliance tensors, and  $\lambda_{ij}^*(\mathbf{x})$  and  $\mu_{ij}^*(\mathbf{x})$  are, respectively, the overall eigenstress and eigenstrain.

Consider a heterogeneous body, subjected only to mechanical loads, and all eigenstresses and eigenstrains are prescribed as zero. Define the mechanical strain concentration factors (Hill, 1963, 1965) for the two phases as

$$\bar{\varepsilon}_{ij}^{(1)}(\mathbf{x}) = A_{ijkl}^{(1)}\varepsilon_{kl}^*(\mathbf{x}), \quad \bar{\varepsilon}_{ij}^{(2)}(\mathbf{x}) = A_{ijkl}^{(2)}\varepsilon_{kl}^*(\mathbf{x})\quad (\text{A.4})$$

which connect the averaged strain of the heterogeneous body  $\bar{\varepsilon}_{ij}^*(\mathbf{x})$  to the averages of the strains in two phases.

It can be shown (Levin, 1967; Levin and Alvarez-Tostado, 2006) that the overall stiffness tensor  $L_{ijkl}^*$  and the overall eigenstress  $\lambda_{ij}^*$  can be expressed in terms of the strain concentration factors as follows:

$$L_{ijkl}^* = nL_{ijmn}^{(1)}A_{mnkl}^{(1)} + (1-n)L_{ijmn}^{(2)}A_{mnkl}^{(2)}\quad (\text{A.5})$$

$$\lambda_{ij}^* = n\bar{\lambda}_{kl}^{(1)}A_{kl ij}^{(1)} + (1-n)\bar{\lambda}_{kl}^{(2)}A_{kl ij}^{(2)}\quad (\text{A.6})$$

Similarly, we can define the mechanical stress concentration factors as

$$\bar{\sigma}_{ij}^{(1)}(\mathbf{x}) = B_{ijkl}^{(1)}\sigma_{kl}^*(\mathbf{x}), \quad \bar{\sigma}_{ij}^{(2)}(\mathbf{x}) = B_{ijkl}^{(2)}\sigma_{kl}^*(\mathbf{x})\quad (\text{A.7})$$

Then the overall compliance  $M_{ijkl}^*$  and the overall eigenstrain  $\mu_{ij}^*(\mathbf{x})$  can be obtained as

$$M_{ijkl}^* = nM_{ijmn}^{(1)}B_{mnkl}^{(1)} + (1-n)M_{ijmn}^{(2)}B_{mnkl}^{(2)}\quad (\text{A.8})$$

$$\mu_{ij}^* = n\bar{\mu}_{kl}^{(1)}B_{kl ij}^{(1)} + (1-n)\bar{\mu}_{kl}^{(2)}B_{kl ij}^{(2)}\quad (\text{A.9})$$

It is important to emphasize that Levin's formulae (A.6) and (A.9) are rigorously valid only if the local (phase) eigenstrains or eigenstresses are uniform on a microscale (on a scale of the pore and the solid grain).

It can be shown that the volumetric strain concentration factors for the solid phase and the pore space in the skeleton (or the drained porous medium) are

$$A_s = \frac{1}{1-n} \frac{K_D}{K_s}, \quad A_p = \frac{1}{n} \left(1 - \frac{K_D}{K_s}\right)\quad (\text{A.10})$$

The isotropic stress concentration factors for the phases of the drained porous medium are given by

$$B_s = \frac{1}{1-n}, \quad B_p = \lim_{K_f \rightarrow 0} \left[ \frac{1}{n} \frac{K_f}{K_D} \left(1 - \frac{K_D}{K_s}\right) \right]\quad (\text{A.11})$$

For completeness, we present the concentration factors for the phases in the undrained porous medium,

$$B_p = \frac{K_u - K_D}{\alpha K_u}, \quad B_s = \frac{(\alpha - n)K_u + nK_D}{\alpha(1 - n)K_u}, \quad A_p = B_p \frac{K_u}{K_f}, \quad A_s = B_s \frac{K_u}{K_s} \quad (\text{A.12})$$

where the undrained bulk modulus  $K_u$  is given by

$$K_u = K_D + \frac{\alpha^2}{n/K_f + (\alpha - n)/K_s} \quad (\text{A.13})$$

These concentration factors can be used to derive the stress–strain constitutive equation in terms of the undrained bulk modulus  $K_u$  and the undrained shear modulus  $G_u = G_D$ .

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