J Eng Math (2010) 68:5–14
DOI 10.1007/s10665-010-9365-0

Crack-bridging in a unidirectionally fibre-reinforced plate

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Received: 12 August 2009 / Accepted: 25 February 2010 / Published online: 16 March 2010

Abstract The mechanics of a matrix crack situated in a unidirectional fibre-reinforced elastic plate where the fibres have experienced debonding over an identifiable zone is examined. This results in a condition referred to as crack-bridging or flaw-bridging; the mechanics of the matrix crack is influenced by the kinematic constraints exerted by the bridging actions of the fibres. The problem is of interest when examining the role of delaminations in ideal fibre-reinforced solids in the presence of fibre continuity. The boundary-integral equation is applied to determine the influence of the elastic stiffness of the bridging fibres on the stress-intensity factor at the tip of the crack, which governs the energy release for the extension of the crack. The paper presents the computational approach and provides examples that clearly demonstrate the significant role of the bridging action and the fibre elasticity in suppressing crack extension in a self-similar fashion.

Keywords Boundary-integral equations · Bridged edge-crack · Bridged Griffith cracks · Computational methods · Delaminated zones · Fibre-reinforced composites · Stress-intensity factors

1 Introduction

Unidirectional reinforcement is specific to a functional requirement of a fibre-reinforced material. It is usually regarded as an idealization, since unidirectional reinforcement is used as an engineering solution to enhance the load-carrying capacity only in special circumstances. The study of the mechanics of unidirectionally reinforced fibre composites is an important development in the modern theory of advanced materials and these developments have always led to treatments that involve more complicated forms of fibre reinforcement arrangements. Seminal studies in this area relevant to the classical theory of ideal fibre-reinforced solids can be found in the works of Spencer [1, pp. 1–9], [2,3], [4, pp. 3–10] and applications dealing with the theory of composites can be found in the works of Hill [5], Hashin and Rosen [6], Hale [7], Christensen [8, Chap. 4], and Neilsen [9, Chap. 2]. Common engineering applications invariably involve multi-directional reinforcement to accommodate the directional variability of the loading of fibre-reinforced materials; nonetheless, the study of unidirectionally reinforced materials sheds important light on micro-mechanical features that can influence the load-transfer mechanisms at these scales.

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Ideally, a fibre-reinforced material is expected to be defect-free in its fabricated condition. This is largely a matter of definition, since even perfect fibre reinforcement can result in micromechanical defects during curing processes and under certain conditions of use, particularly in response to localized loads, extreme temperatures and impact. The effects of damage can be severe when brittle matrices with low stiffness are reinforced with stronger fibres possessing higher elasticity. When the disparity in the tensile strength and the elasticity is large, the weaker brittle matrix can exhibit cracking and in the process lead to defects in the reinforced solid. The integrity of fibre-reinforced materials can therefore be compromised by the development of features such as fibre breakage, fibre pullout, matrix fracture, fibre–matrix interface delamination, matrix void growth, etc. The importance of damage to the structural integrity of fibre-reinforced materials was discussed several decades ago by a number of researchers including Beaumont and Harris [10], Aveston and Kelly [11], Bowling and Groves [12] and Backlund [13]. The topic of flaw- or crack-bridging in unidirectional fibre-reinforced composites was discussed by Kelly [14], Sih [15] and Beaumont [16]. A key development with applicability to flaw-bridging is due to Atkinson [17] who examined the axisymmetric problem for a penny-shaped crack with displacement-dependent tractions on the crack surfaces. The initial investigations dealing with the modelling of flaw-bridging in composites were presented by Selvadurai [18,19], followed by the work of Stang [20], Rose [21], McCartney [22], Budiansky et al. [23] and Budiansky and Amazigo [24] who investigated various aspects of the elastostatic problem of the bridging-induced behaviour of flaws in unidirectional fibre-reinforced materials. Recently, Movchan and Willis [25,26] examined crack-tip effects in bridged cracks using asymptotic methods for the analysis of the associated integral equations. The topic of crack-bridging has also been extended to cover axisymmetric bridged external circular cracks in orthotropic elastic solids that are loaded by a doublet of Kelvin forces [27].

This paper presents the application of a boundary-integral equation technique to study crack-bridging in a unidirectional fibre-reinforced material. Attention is restricted to the study of crack-bridging in a plate problem, although the procedures can be extended to include three-dimensional crack geometries. The boundary-integral equation approach is better suited for applications where either the finite geometry of the domain or the complicated configurations of the crack or unilateral constraints imposed by the fibre cannot be handled through classical formulations. The paper describes the boundary-integral equation approach used to model the domain and the reduction of the crack-bridging problem to a matrix equation for the displacements. The technique is applied to several problems where Griffith cracks are located within fibre de-laminated zones or at the interior of a fibre-reinforced plate and at the free boundary of a stretched fibre-reinforced plate.

2 Governing equations

We consider a thin plate consisting of an elastic matrix reinforced with unidirectionally aligned closely spaced fibres. The linear elastic stress analysis of such a composite can be performed by considering the plate as a transversely isotropic elastic composite where the elastic constants can be expressed in terms of the properties of the constituents [5–8]. Following Green and Zerna [28, Chap. 5] and Lekhnitskii [29, Chap. 1], the linear elastic stress–strain relationships governing the orthotropic form are

$$\sigma_{xx} = c_{11}\varepsilon_{xx} + c_{12}\varepsilon_{yy}; \quad \sigma_{yy} = c_{12}\varepsilon_{xx} + c_{22}\varepsilon_{yy}; \quad \sigma_{xy} = 2c_{66}\varepsilon_{xy}, \tag{1}$$

where σ_{ij} and ε_{ij} are the stress and strain tensors and c_{ij} are the elastic constants. It can be shown [6] that c_{ij} can be expressed in terms of the elastic constants of the matrix and fibre materials and

$$c_{11} = \bar{K} + \bar{G}; \quad c_{12} = 2\bar{\nu}\bar{K}; \quad c_{22} = \bar{E} + 4\bar{\nu}^2\bar{K}; \quad c_{66} = \bar{G}^*, \tag{2}$$

where the constants $\bar{K}$, $\bar{G}$, $\bar{E}$, $\bar{\nu}$ and $\bar{G}^*$ depend on the elastic constants of the fibre (f) and the matrix (m); the volume fractions of the fibres (V_f) and the matrix (V_m) and specific expressions are given in Appendix 1. In the absence of body forces, the displacement equations of equilibrium for the orthotropic elastic material are given by

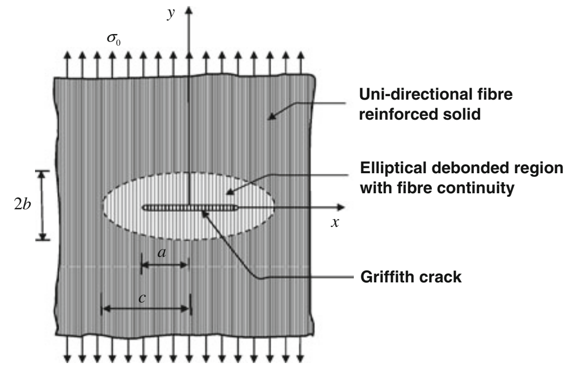


Fig. 1 The bridged Griffith crack with elliptical debonded region

$$c_{11}\frac{\partial^2 u_x}{\partial x^2}+(c_{12}+c_{66})\frac{\partial^2 u_y}{\partial x\partial y}+c_{66}\frac{\partial^2 u_x}{\partial y^2}=0, \tag{3}$$

$$c_{22}\frac{\partial^2 u_y}{\partial y^2}+(c_{12}+c_{66})\frac{\partial^2 u_x}{\partial x\partial y}+c_{66}\frac{\partial^2 u_y}{\partial x^2}=0, \tag{4}$$

where u_x and u_y are the displacement components.

3 The bridged-crack problem

Consider the generalized plane-stress problem related to a unidirectionally reinforced elastic plate that contains a bridged plane matrix crack within a fibre-debonded region (Fig. 1). The bridging fibres impose displacement-dependent traction constraints in the vicinity of the crack. We assume that the debonded fibres can be modelled as a set of densely spaced spring elements or Winkler ligaments [30, Chap. 2], of variable length. The effective length of the bridging fibres can be identified in relation to the dimensions of the debonded region. The Winkler ligaments are assumed to exert a unilateral constraint at the boundary of the debonded region and we also assume that the fibres are incapable of exerting any traction constraint normal to its orientation. The traction boundary conditions applied to the boundary of the bridging region can be written as

$$t_y+\frac{E_f V_f \Delta u_y}{2y_0(x)}=0,\quad t_x=0, \tag{5}$$

where E_f and V_f are, respectively, the Young's modulus and volume fraction of the fibres, $2y_0(x)$ is the length of the debonded fibre at the location x and Δu_y is the jump in the displacement across two sides of the fibre-debonded region. If the fibre-debonded region is symmetrically placed across the crack, the first equation of (5) can be written as

$$t_y+\frac{E_f V_f \Delta u_y}{y_0(x)}=0. \tag{6}$$

The second equation of (5) implies that the fibres are incapable of exerting a traction constraint normal to its orientation.

4 The boundary-element method

To examine the crack-bridging problem dealing with the unidirectionally fibre-reinforced solid, we adopt a boundary-integral equation approach. Prior to formulating the boundary-integral equation applicable to the class of plane problems in elasticity, we consider the displacement and traction fundamental solutions, D_{ij} and S_{ij} respectively, which can be deduced from the original works of Green [31]. Here we use the reduced forms presented by Rizzo and Shippy [32]. In matrix form, D_{ij} is given by

$$D_{ij} = \begin{pmatrix} \left[\sqrt{\alpha_1} A_2^2 \log r_1 - \sqrt{\alpha_2} A_1^2 \log r_2\right] & -A_1 A_2(\theta_1 - \theta_2) \\ -A_1 A_2(\theta_1 - \theta_2) & -\left[\frac{1}{\sqrt{\alpha_1}} A_1^2 \log r_1 - \frac{1}{\sqrt{\alpha_1}} A_2^2 \log r_2\right] \end{pmatrix}, \tag{7}$$

where

$$(\alpha_1 + \alpha_2) = (2c_{12} + c_{66})/c_{22}; \quad \alpha_1\alpha_2 = c_{11}/c_{22}; \quad A_i = c_{12} - \alpha_i c_{22} \tag{8}$$

and

$$r_i^2 = x^2 + \frac{y^2}{\alpha_i}; \quad \theta_i = \tan^{-1}\left(\frac{y}{x\sqrt{\alpha_i}}\right). \tag{9}$$

Similarly, in matrix form, S_{ij} is given by

$$S_{ij} = \begin{pmatrix} \left(xn_x + yn_y\right)\left(\frac{A_1}{r_2^2\sqrt{\alpha_2}} - \frac{A_2}{r_1^2\sqrt{\alpha_1}}\right) & \left(\frac{M_1 A_1}{\alpha_1 r_1^2} - \frac{M_2 A_2}{\alpha_2 r_2^2}\right) \\ \left(\frac{M_1 A_2}{r_1^2} - \frac{M_2 A_1}{r_2^2}\right) & \left(xn_x + yn_y\right)\left(\frac{A_1}{r_1^2\sqrt{\alpha_1}} - \frac{A_2}{r_2^2\sqrt{\alpha_2}}\right) \end{pmatrix}, \tag{10}$$

where

$$M_1 = \left(xn_y\sqrt{\alpha_1} - \frac{yn_x}{\sqrt{\alpha_1}}\right), \quad M_2 = \left(xn_y\sqrt{\alpha_2} - \frac{yn_x}{\sqrt{\alpha_2}}\right) \tag{11}$$

and n_x and n_y are the direction cosines to the outward unit normal to a surface S. In developing these fundamental solutions, it is assumed that α_i are both real and positive, a condition that is satisfied by most fibre-reinforced materials and those considered in this study for evaluation of numerical results.

The boundary-integral equation for an orthotropic elastic continuum can be written in several forms; for conciseness we adopt the indicial form

$$u_j(P) = \frac{1}{\beta}\int_S \left\{t_i(Q)D_{ij}(P,Q) - u_i(Q)S_{ij}(P,Q)\right\} \mathrm{d}S, \tag{12}$$

where P is the point under consideration and Q is a general point; S is the boundary of the composite region; u_i are components of the displacement, $t_i (i = x, y)$ are the components of the traction and D_{ij} and S_{ij} are defined, respectively, by (7) and (10) and

$$\beta = 2\pi(\alpha_1 - \alpha_2)/c_{22}. \tag{13}$$

The boundary S of the elastic domain is discretized into M elements such that, for a single element, the geometry is described by

$$[x\ ;\ y] = \sum_{\beta=1}^{3} N_{(\zeta)}^{(\beta)}\left[x^{(\beta)}\ ;\ y^{(\beta)}\right] \tag{14}$$

and the variables u_i and t_i can be written in the form

$$[u_i\ ;\ t_i] = \sum_{\beta=1}^{3} N_{(\zeta)}^{(\beta)}\left[u_i^{(\beta)}\ ;\ t_i^{(\beta)}\right] = [N(\zeta)]\left[\{u_i\}\ ;\ \{t_i\}\right]. \tag{15}$$

The interpolation functions are

$$N_{(\zeta)}^{(1)} = \frac{\zeta(\zeta-1)}{2}, \quad N_{(\zeta)}^{(2)} = \left(1-\zeta^2\right), \quad N_{(\zeta)}^{(3)} = \frac{\zeta(\zeta+1)}{2} \tag{16}$$

with $-1 \le \zeta \le 1$. Following discretization of the boundary into M elements, the boundary-integral equation (12) can be expressed in the form

$$\beta u_k + \sum_{e=1}^{M}\int_{-1}^{1} S_{lk}[N(\zeta)]|J|\mathrm{d}\zeta\{u_k\}^e = \sum_{e=1}^{M}\int_{-1}^{1} V_{lk}[N(\zeta)]|J|\mathrm{d}\zeta\{t_k\}^e, \tag{17}$$

where e is the total number of elements in the discretization and $|J|$ is the Jacobian given by

$$|J| = \left[\left(\frac{\partial x}{\partial \zeta} \right)^2 + \left(\frac{\partial y}{\partial \zeta} \right)^2 \right]^{1/2}. \tag{18}$$

Performing the integrations of the coefficients occurring in (16), the complete system of equations for each location P can be written in the form of a matrix equation

$$[\mathbf{H}]\{\mathbf{u}\} = [\mathbf{G}]\{\mathbf{t}\}, \tag{19}$$

where $[\mathbf{H}]$ and $[\mathbf{G}]$ are the corresponding coefficients matrices obtained from the fundamental solutions.

Incorporating the boundary condition (6) in (19), we obtain

$$\left[\mathbf{H}_1 \left(\mathbf{H}_2 + \mathbf{G}_2 \frac{E_f V_f}{y_0} \right) \right] \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{Bmatrix} = [\mathbf{G}_1]\{\mathbf{t}_1\}, \tag{20}$$

where the subscript 2 refers to the part of the boundary prescribed by (6) and the subscript 1 refers to the remainder of the boundary.

5 Crack-tip modelling

The boundary-element developments described thus far can be used to model all parts of the boundaries involved except those locations that contain crack extremities. In linear elastic fracture mechanics, the stress state at the crack tip located within either an isotropic or anisotropic but homogeneous elastic solid should exhibit the classical $1/\sqrt{r}$-type singularity in the stress field [33, Vol. II, Chap. 1], [34, Vol. 2, pp. XV–LIII], [35, Chap. 3]. If the crack tip is located at a bimaterial boundary, the stress field should exhibit an oscillatory form that depends on the elasticity mismatch between the two regions [36,37]. This condition can arise when the crack tip is located at the boundary of the delaminated zone, irrespective of the extent of the debonded region. In this study, we restrict attention to the category of problem where the crack tip is located in a homogeneous region and, in the special situation where the crack is oriented along the x-direction, the stress σ_{yy} is given by

$$\sigma_{yy}(r, \theta) \approx \frac{K_\mathrm{I}}{\sqrt{2\pi r}} f_{yy}(\theta); \tag{21}$$

here (r, θ) are the local coordinates at the crack tip, $f_{yy}(\theta)$ is an angular distribution function with $f_{yy}(0) = 1$ and K_I is the stress-intensity factor. As can be seen, the stress σ_{yy} exhibits a $1/\sqrt{r}$-type stress singularity. To account for the stress singularity, a quarter-point boundary element developed by Cruse and Wilson [38] can be used. The displacements and tractions take the forms

$$u_i = b_0 + b_1\sqrt{r} + b_2 r; \quad t_i = \frac{c_0}{\sqrt{r}} + c_1 + c_2\sqrt{r}, \tag{22}$$

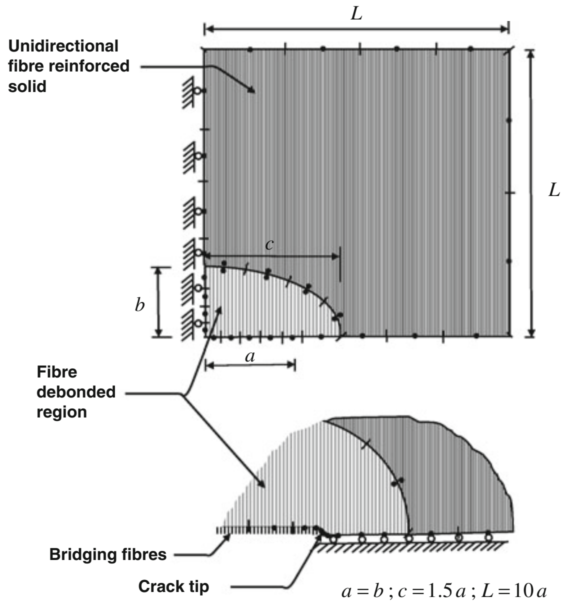
where b_i and c_i are constants. The accuracy of boundary-element techniques that incorporate singular elements is well documented by Blandford et al. [39] and Aliabadi [40]. The stress-intensity factor at the crack tip, which is of interest to fracture-mechanics, can be calculated by approximating the tractions in the vicinity of the crack tip, i.e.,

$$K_\mathrm{I} = \operatorname*{Lim}_{r \to 0} \left\{ -\sqrt{2\pi r}\,(2r)\,\frac{\mathrm{d}\sigma_{yy}}{\mathrm{d}r} \right\} \tag{23}$$

6 Elliptical debonded zones around interior and edge-bridged cracks

We consider the problem of a unidirectional fibre-reinforced composite that contains a Griffith crack of length $2a$, located normal to the fibre direction. The process of crack-bridging occurs as a result of fibre debonding over an elliptical region (major axis $2c$ and minor axis $2b$), which is symmetrically located around the Griffith crack.

Fig. 2 Boundary-element discretization



In the intact region, the elastic plate has orthotropic properties that are governed by the elasticity properties of the fibres and the matrix and the volume fraction $\left(E_f,\, E_m,\, \nu_f,\, \nu_m,\, V_f\,(\text{or } V_m)\right)$. In the fibre-debonded region, the elastic plate has orthotropic properties governed by the elasticity properties of the matrix $\left(E_m,\, \nu_m,\, V_m(\text{or } V_f)\right)$ and the unilateral bridging action of the debonded fibres is governed by the elasticity properties of the fibres and the volume fraction $(E_f,\, V_f)$. The fibres that are continuous over the debonded region exert an elastic constraint on the mechanics of the crack opening and thus on the stress-intensity factor at the crack tip under the action of a homogeneous far-field stress state $\sigma_{yy} = \sigma_0$ (Fig. 1). The parameters influencing the stress-intensity factor include the fibre–matrix modular ratio (E_f/E_m), the Poisson ratios for the fibre and the matrix phases $(\nu_f,\, \nu_m)$, the volume fraction of the fibre or matrix phases (V_f or V_m), the geometry of the elliptical fibre-debonded region (b/c) and the extent of matrix cracking in the fibre-debonded region (c/a). For the purposes of the numerical treatments, we assume that $\nu_f = \nu_m = 0.2$ and the volume fraction of fibres $V_f = 0.30$. For ease of presentation and for establishing the influence of the crack-bridging action on the stress-intensity factor at the bridged crack, it is convenient to normalize the computational results with respect to the Mode I stress-intensity factor at the crack tip in the absence of both flaw bridging and fibre debonding. This corresponds to the Mode I stress-intensity factor for a Griffith crack located in an orthotropic plate of infinite extent, where the crack is aligned with the axis of elastic symmetry (i.e., $K_{\rm I}^0 = \sigma_0\sqrt{\pi a}$). The boundary-element discretization used in the computational modeling is shown in Fig. 2 and the dimensions of the domain are assigned in relation to the length of the Griffith crack. Figures 3 and 4 illustrate the influence of the bridging action on the attenuation of the normalized stress-intensity factor $K_{\rm I}/K_{\rm I}^0$ at the crack tip. The computational results indicate that as the fibre–matrix modular ratio E_f/E_m approaches values in excess of 10^5, the stress-intensity factor at the crack tip is significantly reduced due to the bridging action. Similarly, when the fibre–matrix modular ratio is of the order of 10^{-3}, the bridging has no influence on the attenuation of the normalized stress-intensity factor. Within the range $(E_f/E_m) \in \left(10^{-3},\ 10^{-5}\right)$, the magnitude of the normalized stress-intensity factor is influenced by both the relative geometric parameters c/a and b/c, the latter relating to the aspect ratio of the elliptical debonded region.

Computations were also performed to examine the influence of a semi-elliptical fibre-debonded zone around a bridged edge-crack located in a unidirectional fibre-reinforced plate subjected to an axial stress σ_0 (Fig. 5). The problem of the uniform axial loading of a plate containing an unbridged plane crack was investigated by Rooke

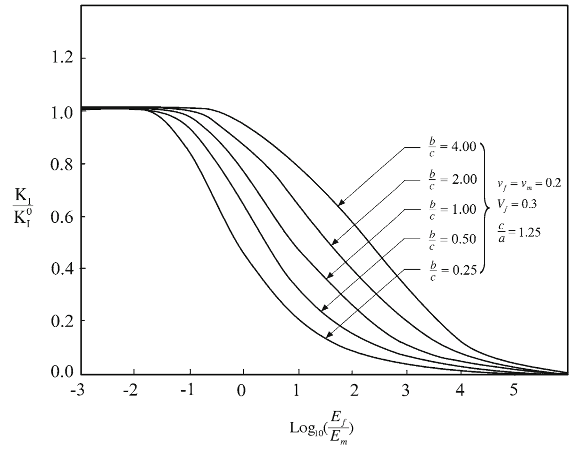


Fig. 3 Normalized Mode I stress-intensity factor at the tip of a bridged Griffith crack with an elliptical fibre-debonded region [$c/a = 1.25$]

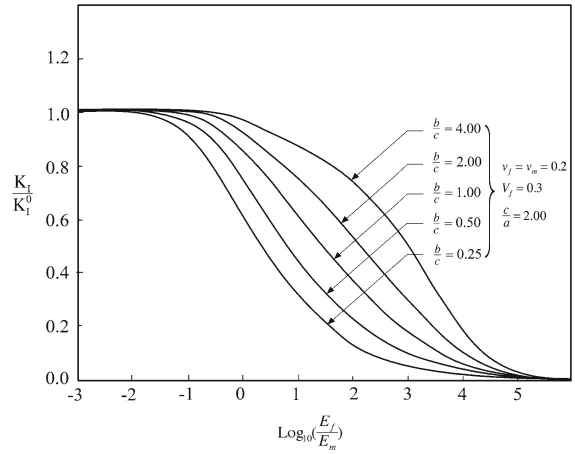


Fig. 4 Normalized Mode I stress-intensity factor at the tip of a bridged Griffith crack with an elliptical fibre-debonded region [$c/a = 2.00$]

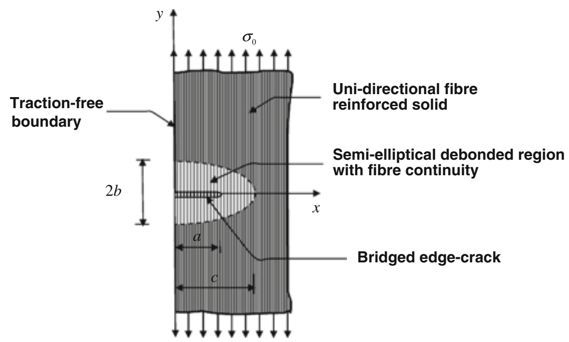


Fig. 5 The bridged edge-crack with a semi-elliptical debonded region

and Cartwright [41, pp. 300–310], and this result provides a check on the accuracy of the computational scheme in the absence of the bridging action. Figures 6 and 7 illustrate the variation in the normalized stress-intensity factor $K_{\mathrm{I}}/K_{\mathrm{I}}^{0}$ for the edge-crack as a function of the fibre–matrix modular ratio and the relative geometric parameters c/a and b/c. The computations conducted for the bridged edge-crack display trends similar to those observed for the Griffith crack, with flaw-bridging. Also, in this instance, the Mode I stress-intensity factor is suppressed by the fibre-bridging action when the modular ratio $(E_f/E_m) > 10^5$ and the fibre-bridging has no influence on the Mode I stress-intensity factor when $(E_f/E_m) < 10^{-3}$. These observations are, however, specific to the fibre volume fraction (V_f) and Poisson's ratios ν_f and ν_m used in developing the computational results.

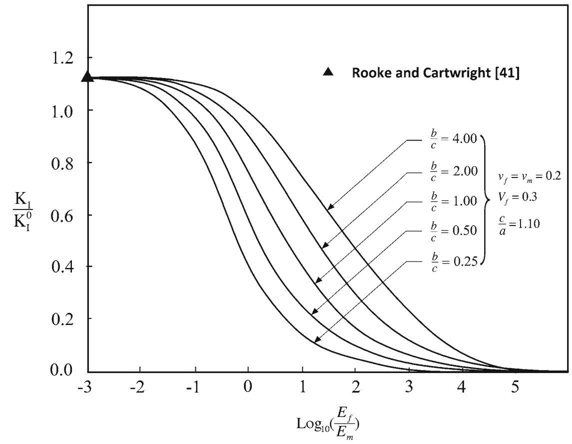


Fig. 6 Normalized Mode I stress-intensity factor at the tip of a bridged edge-crack with a semi-elliptical fibre-debonded region $[c/a = 1.50]$

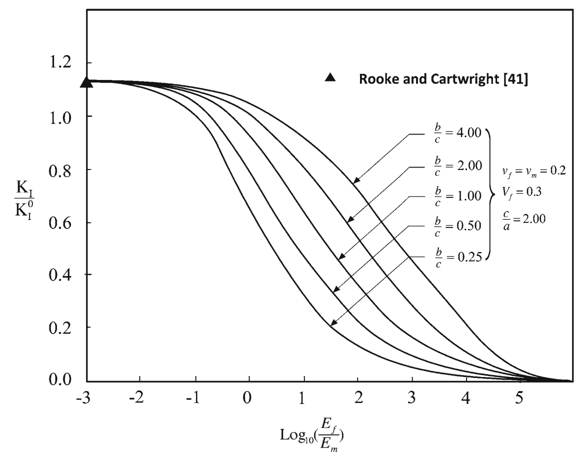


Fig. 7 Normalized Mode I stress-intensity factor at the tip of a bridged edge-crack with a semi-elliptical fibre-debonded region $[c/a = 2.00]$

7 Concluding remarks

The process of flaw-bridging occurs when fibres exert continuity over a defect such as a crack or a void in a fibre-reinforced material. The transfer of load from the matrix to the fibres at a crack can result in fibre-debonding in the vicinity of the crack. Modelling of progressive delamination at a fibre-debonded region requires incremental nonlinear micromechanical models of matrix–fibre interaction. The problems examined in the paper deal with the global influences of fibre–matrix interface debonding in the presence of fibre continuity at a crack located in a unidirectional fibre-reinforced solid. In particular, the delamination is assumed to occur over an elliptical zone surrounding either a Griffith crack or an edge-crack. This results in the problem of a crack located in an orthotropic elastic plate corresponding to a matrix with aligned voids surrounded by a fibre-reinforced orthotropic plate region with fibre continuity exerting an elastic constraint at the boundary of the fibre-debonded zone. The resulting problems are examined using a boundary-integral equation approach, although it is foreseeable that the problems could equally well be examined more elegantly using complex-variable techniques. The computations indicate the considerable importance of the bridging action in suppressing the Mode I stress-intensity factor at the crack tips, for both the Griffith crack and the edge-crack. In particular the modular ratio and the volume fraction of the fibres are expected to have a dominant influence on the attenuation of the crack-opening-mode stress-intensity factor. In the limit of fibre inextensibility, the approach adopted in the numerical treatment is likely to break down due to stress channeling and boundary-layer effects that can appear at the crack tips. The ultimate idealized model in this regard is that of the ideal fibre-reinforced solid where an incompressible elastic matrix is reinforced everywhere by inextensible fibres. This gives rise to highly concentrated stress fields in which the fibres at the crack tips are required to carry finite forces. England and Rogers [42] have considered a variety of problems for plane-strain deformations of this class of materials. Their solutions illustrate the very different physical effects that can occur when these models are taken to extreme limits. Other examples that demonstrate similar phenomena are given by Everstine and Pipkin [43], Morland [44] and Spencer [45]. When considering the normal range of material properties encountered in many fibre-reinforced materials, the computational procedures outlined in this paper can also be used to examine more general types of problems where distributions of cracks with damage zones occur in plate regions and to examine fibre-bridging that exerts only a unilateral constraint.

Acknowledgements The work described in this paper was supported by the Max Planck Forschungspreis in the Engineering Sciences awarded by the Max Planck Gesellschaft, Germany, and by a Discovery Research Grant awarded by the Natural Sciences and Engineering Research Council of Canada. The author is grateful to the reviewers for their suggestions, which led to improvements in the presentation of the paper.

Appendix 1

The expressions for the constants $\bar{K}$, $\bar{G}$, $\bar{E}$, $\bar{\nu}$ and $\bar{G}^*$ are given by Hashin and Rosen [6] as follows:

$$\bar{K} = \left\{ \frac{\xi_0(1 + 2\nu_m V_f) + 2\nu_m V_m}{\xi_0 V_m + V_f + 2\nu_m} \right\} (\lambda_m + G_m), \quad \bar{G} = \left\{ \frac{(\alpha + \beta_m V_f)\left(1 + \rho V_f^3\right) - 3V_f V_m^2 \beta_m^2}{(\alpha - V_f)\left(1 + \rho V_f^3\right) - 3V_f V_m^2 \beta_m^2} \right\} G_m,$$

$$\bar{\nu} = \left\{ \frac{V_f E_f L_1 + V_m E_m L_2 \nu_m}{V_f E_f L_3 + V_m E_m L_2} \right\}, \quad \bar{G}^* = \left\{ \frac{\eta(1 + V_f) + V_m}{\eta V_m + V_f + 1} \right\} G_m$$

and

$$\bar{E} = V_f E_f + V_m E_m,$$

where

$$L_1 = 2\nu_f \left(1 - \nu_m^2\right) V_f + \nu_m(1 + \nu_m)V_m, \quad L_2 = 2\left(1 - \nu_f^2\right) V_f, \quad L_3 = 2\left(1 - \nu_m^2\right) V_f + (1 + \nu_m)V_m$$

$$\xi_0 = \left\{ \frac{\lambda_f + G_f}{\lambda_m + G_m} \right\}, \quad \alpha = \left\{ \frac{\eta + \beta_m}{\eta - 1} \right\}, \quad \rho = \left\{ \frac{\beta_m - \eta\beta_f}{1 + \eta\beta_f} \right\}, \quad \eta = \frac{G_f}{G_m}, \quad V_m + V_f = 1,$$

$$G_i = \frac{E_i}{2(1 + \nu_i)}, \quad \lambda_i = \frac{\nu_i E_i}{(1 + \nu_i)(1 - 2\nu_i)}, \quad \beta_i = \frac{1}{(3 - 4\nu_i)}, \quad (i = m, f).$$

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