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On the Mode I stress intensity factor for an external circular crack with fibre bridging

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ABSTRACT

This paper presents a theoretical analysis of an external matrix crack located in a unidirectional fibre-reinforced elastic solid modelled as a transversely isotropic material. The presence of matrix cracking with fibre continuity introduces bridging action that has an influence on the stress intensity factors at the crack tip of the external crack. This paper presents a model for the bridged crack, where the fibre ligaments induce a constant displacement-dependent traction constraint over the external crack. This gives rise to a Fredholm integral equation of the second kind, which can be solved in an approximate fashion. We examine the specific problem where the bridged external circular crack is loaded by a doublet of concentrated forces. Numerical results are presented to illustrate the influence of the fibre–matrix modular ratio and the location of the loading on the bridged-crack opening mode stress intensity factor.

1. Introduction

The more general applications of fibre reinforcement invariably involve multidirectional reinforcement to address directional variability in the loading of fibre-reinforced materials. Unidirectional reinforcement usually results from a highly specific functional requirement in the use of fibre reinforcement, where the dominant loading direction can be identified a priori. In spite of this limitation, the study of unidirectionally reinforced materials provides useful insight into micro-mechanical features that can influence the load transfer between the fibres and the matrix. In its fabricated condition, a fibre-reinforced material is expected to be defect free. The notion of a defect free fibre-reinforced material is largely a matter of definition, since even perfect fibre reinforcement can exhibit micro-mechanical defects during curing processes and under certain loading and environmental conditions resulting from localized loads, extreme temperatures and impact. The integrity of a fibre-reinforced material can therefore be influenced by the development of features such as fibre breakage, fibre pullout, matrix fracture, fibre–matrix interface delamination, matrix damage, matrix void growth, etc. The influence of damage and defects on the structural integrity of fibre-reinforced materials was discussed several decades ago by a number of researchers including Beaumont and Harris [1], Aveston and Kelly [2], Bowling and Groves [3] and Backlund [4]. In particular, flaw- or crack bridging in unidirectional fibre-reinforced composites was discussed by Kelly [5], Aveston and Kelly [6] and Sih [7]. The initial investigations dealing with the modelling of flaw bridging in composites were presented by Selvadurai [8,9], followed by the work of Stang [10], Rose [11], McCartney [12], Budiansky et al. [13], Budiansky and Amazigo [14] and Selvadurai [15] who investigated various aspects of the elastostatic problem of bridging-induced behaviour of flaws in unidirectional fibre-reinforced materials including frictional load transfer across single fibres bridging cracks. Recently, Movchan and Willis [16,17] have examined the crack tip effects in bridged cracks using asymptotic methods for the analysis of the associated integral equations. When there is complete continuity in the load transfer between the unidirectional fibres and the matrix, the composite will exhibit either orthotropy or transverse isotropy in its mechanical behaviour. The equivalent elasticity properties of the unidirectionally reinforced composite can be determined by considering the theories of effective elastic properties developed in the literature. Unidirectionally reinforced brittle elastic composites can exhibit various forms of structural damage and fracture depending on the types of loading contributing to such phenomena. Examples can include impact or dynamic loadings, thermal loadings, localized inhomogeneities introduced due to fabrication effects, etc. With unidirectionally reinforced brittle matrices, damage and fracture is usually restricted to either the brittle matrix or the interface between the fibre and the matrix. Interface failure can be minimized by the use of bond enhancing agents, and, as a result, the fracture and damage is largely restricted to the brittle matrix. A model of a fractured unidirectionally reinforced composite region can be visualized as a fractured matrix through which the fibres can exhibit continuity. The resulting defect in the composite is referred to as a bridged flaw, where the intact fibres exert a constraint on the mechanics of the flaw. Bridged-flaws can vary

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depending on the mode of initiation of matrix cracking: In this paper, we examine the influence of flaw bridging in a unidirectionally reinforced composite by considering an axisymmetric problem where an external crack in a composite is loaded by a doublet of concentrated forces P located at a finite distance from the bridged external crack (Fig. 1). It is, however, important to emphasize the fact that analytical treatments of this nature are intended to examine localized flaws that can exist in regions of unidirectionally reinforced solids. If the assumption of axial symmetry is removed then other conditions have to be stipulated to ensure that the net stresses in the bridging fibres are always tensile. The absence of such a guarantee will make the problem inordinately complicated requiring an iterative incremental technique to identify regions where the bridging actions are present. Non-axisymmetric problems are better handled via computational approaches, which have issues of their own.

2. Governing equations

When complete continuity exists between the reinforcing fibres and the defect free matrix, the elastic behaviour of the unidirectionally reinforced composite can be modelled by appeal to the theory of transversely isotropic elastic solids and the effective elasticity properties can be determined by the theory of composite materials developed in the classical studies by Hashin and Rosen [18], Hill [19], Broutman and Krock [20], Hale [21] and Christensen [22]. We consider the mechanics of a transversely isotropic elastic material where the axis of elastic symmetry coincides with the fibre direction. As has been shown by Elliott [23] and Shield [24], the displacement and stress fields in the resulting transversely isotropic elastic material can be expressed in terms of two functions $\varphi_i(r,z)$, $(i = 1,2)$, which are solutions of

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z_i^2}\right)\varphi_i(r,z)=0 \tag{1}$$

where $z_i = z/\sqrt{v_i}$, and v_i are roots of the equation

$$c_{11}c_{44}v^2+[c_{13}(2c_{44}+c_{13})-c_{11}c_{33}]v+c_{33}c_{44}=0 \tag{2}$$

and c_{ij} are the elastic constants of the transversely isotropic elastic material. In general,

$$c_{ij}=c_{ij}(E_f,E_m,v_f,v_m,V_f,V_m,C) \tag{3}$$

where the material parameters of the composite depend on the constituent properties, the volume fractions and a contiguity factor, C. Explicit expressions for c_{ij} are given, for example, in [18,22] (see Appendix A). The displacement and stress fields relevant to the problems considered here are given by

$$\begin{Bmatrix} u_z(r,z)\\ \sigma_{zz}(r,z)\\ \sigma_{rz}(r,z)\end{Bmatrix}=\sum_{i=1}^{2}\begin{Bmatrix}\frac{\partial}{\partial z}k_i\varphi_i\\ (k_ic_{33}-v_ic_{13})\frac{\partial^2\varphi_i}{\partial z^2}\\ c_{44}(1+k_i)\frac{\partial^2\varphi_i}{\partial r\partial z}\end{Bmatrix} \tag{4}$$

where

$$k_i=\frac{(c_{11}v_i-c_{44})}{(c_{13}+c_{44})} \tag{5}$$

2.1. The loading of the intact fibre-reinforced solid by a doublet of forces

Prior to examining the bridged external circular crack problem we consider the axisymmetric loading of the *intact fibre-reinforced solid* by *symmetrically placed body forces*. Forces of magnitude P act at the locations $z = \pm c$ along the z-axis. These are directed in the positive and negative z-directions, respectively, such that a state of tension exists in the plane of symmetry $z = 0$. The state of stress σ_{ij}^B due to this doublet of forces is such that

$$\sigma_{rz}^B(r,0)=0 \tag{6}$$

and

$$\sigma_{zz}^B(r,0)=p_0\left[\frac{\zeta_1a^2c_1}{(r^2+c_1^2)^{3/2}}-\frac{\zeta_2a^2c_2}{(r^2+c_2^2)^{3/2}}\right]=p^*(r) \tag{7}$$

where

$$p_0=\frac{P}{\pi a^2};\quad c_i=\frac{c}{\sqrt{v_i}} \tag{8}$$

and

$$\zeta_i=\frac{(c_{13}+c_{44})(k_ic_{33}-v_ic_{13})v_1v_2}{2c_{33}c_{44}(v_1-v_2)v_i} \tag{9}$$

2.2. The boundary value problem for the external bridged crack

We now consider the axisymmetric problem of an external circular crack of radius a located in a unidirectionally reinforced composite. The external crack is bridged by fibres of effective length $2l$ that are present due to the bridging action. Unlike the classical formulations of crack problems, the cracked region is assumed to have a finite "*effective length*" $2l$ of exposed fibres. This "*effective length*" can be enhanced by the debonding of fibres resulting from the "*load transfer*" associated with matrix cracking and the ensuing lateral contraction of exposed regions due to a *Poisson's ratio effect*.

The unidirectionally reinforced composite containing the bridged external crack is loaded by a doublet of axial forces that are located at distances c from the intact region (Fig. 1). The analysis of the axisymmetric mixed boundary value problem governing this problem requires that

$$u_z(r,0)=0;\quad r\in(0,a) \tag{10}$$

$$\sigma_{zz}(r,0)=-p^*(r)+\left(\frac{V_fE_f}{l}\right)u_z(r,0);\quad r\in(a,\infty) \tag{11}$$

$$\sigma_{rz}(r,0)=0;\quad r\in(0,\infty) \tag{12}$$

where E_f and V_f are respectively the fibre elastic modulus and volume fraction and $p^*(r)$ is the stress state in the plane of the intact solid due to the doublet of forces. As is evident from (11), in the present analysis the length of the bridging fibres (l) is assumed to be a constant over the externally cracked region. This assumption has been used by other authors (see e.g. [1–10]) and others in their

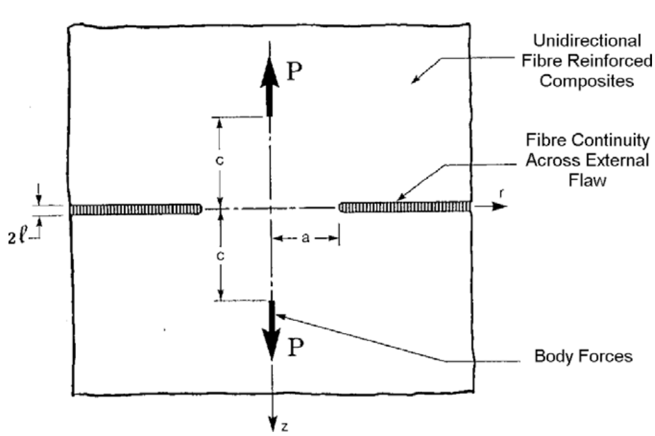

Fig. 1. The bridged external circular crack.

investigation of bridged cracks. If a computational approach is adopted this restriction can be replaced by a bridging region that can vary spatially. Owing to the symmetry of the bridged-crack problem about the plane $z = 0$, the problem can be formulated as a mixed boundary value problem related to a transversely isotropic halfspace region. The solutions of (1), applicable to the region $z \geqslant 0$, can be written in the integral form

$$\varphi_i(r,z) = \frac{1}{a^2}\int_0^\infty \xi A_i(\xi)e^{-\lambda_i z}J_0(\xi r/a)d\xi; \quad (i=1,2) \tag{13}$$

where $A_i(\xi)$ are arbitrary functions and $\lambda_i = \xi/a\sqrt{\nu_i}$.

By making use of the Hankel integral solutions (3) and the relationships (4) and (5), it can be shown that the mixed boundary conditions (11) and (12) are equivalent to the system of *dual integral equations*

$$\int_0^\infty \xi B(\xi)F(\xi)J_0(\xi r/a)d\xi = \frac{p^*(r)}{2\mu^*}; \quad a < r < \infty \tag{14}$$

$$\int_0^\infty B(\xi)J_0(\xi r/a)d\xi = 0; \quad 0 \leqslant r \leqslant a \tag{15}$$

where

$$A_1(\xi) = -\sqrt{\frac{\nu_1}{\nu_2}}\frac{(1+k_2)B(\xi)}{(1+k_1)\xi^2}; \quad A_2(\xi) = \frac{B(\xi)}{\xi^2}$$
$$F(\xi) = \left(1 - \frac{\psi}{\xi}\right); \quad \psi = \frac{aE_f V_f\sqrt{\nu_1\nu_2}(k_1-k_2)}{E_m l\Omega^*} \tag{16}$$

$$\mu^* = \frac{c_{44}\Omega^*}{2a^4\nu_2\sqrt{\nu_1}(1+k_1)}$$
$$\Omega^* = \sqrt{\nu_1}(1+k_1)\left\{\frac{k_2c_{33}-\nu_2c_{13}}{c_{44}}\right\} - \sqrt{\nu_2}(1+k_2)\left\{\frac{k_1c_{33}-\nu_1c_{13}}{c_{44}}\right\}$$

Using a finite Fourier cosine transform Eq. (15) can be identically satisfied (Sneddon [25]), the system of dual integral equations (14) and (15) can be reduced to a single *Fredholm integral equation of the second kind*, for a single unknown function $\Phi^*(t)$ that is related to $B(\xi)$: i.e.,

$$\Phi^*(t) - \frac{\psi}{\pi}\int_1^\infty K(t,\tau)\Phi^*(\tau)d\tau = g(t) \tag{17}$$

where the kernel function $K(t, \tau)$ is given by

$$K(t,\tau) = 2\int_0^\infty \frac{\cos(\xi t)\cos(\xi\tau)}{\xi}d\xi \tag{18}$$

$$g(t) = \frac{\zeta_1\sqrt{\nu_1}\eta}{(\nu_1t^2+\eta^2)} - \frac{\zeta_2\sqrt{\nu_2}\eta}{(\nu_2t^2+\eta^2)} \tag{19}$$

and $\eta = c/a$. The complete mathematical analysis of the bridged external circular crack is now reduced to the solution of the Fredholm integral equation (17). In the ensuing discussions, however, we shall restrict attention to an examination of the influence of the elastic bridging action on the stress intensity factor at the crack boundary.

3. Limiting cases

Prior to considering the numerical solution of the Fredholm integral equation of the second-kind for $\Phi^*(t)$, it is instructive to examine the stress intensity factor resulting from certain limiting cases of the bridging action. The stress intensity factor for the opening mode of the bridged external circular crack is defined by

$$K_I = \underset{r=a^-}{Lim}\{2(a-r)\}^{1/2}\sigma_{zz}(r,0) \tag{20}$$

From the results derived previously, it can be shown that

$$[K_I]^{trans.\ isotr.}_{bridged\ flaw} = \frac{P}{2\pi^2a^{3/2}}\{4\Phi^*(1)\} \tag{21}$$

In the limiting case when the elasticity of the bridged fibres (E_f) reduces to zero, the parameter $\psi = 0$ and the expression (21) yields the stress intensity factor for the *di-pole body force loading of an external circular flaw in a transversely isotropic elastic medium*; i.e.,

$$[K_I]^{trans.\ isotr.}_{unbridged\ flaw} = \frac{P}{2\pi^2a^{3/2}}\left(\frac{2(c_{13}+c_{44})\nu_1\nu_2}{c_{33}c_{44}(\nu_1\nu_2)} + \frac{(k_1c_{33}-\nu_1c_{13})\eta}{\sqrt{\nu_1}(\nu_1+\eta^2)} - \frac{(k_2c_{33}-\nu_2c_{13})\eta}{\sqrt{\nu_2}(\nu_2+\eta^2)}\right) \tag{22}$$

In the limit of material isotropy, $\nu_1, \nu_2 \to 1$ and $c_{11} = c_{33} = \lambda + 2\mu$, where λ, μ are the classical Lamé constants. For the limiting case of an isotropic material with no bridging action (Kassir and Sih [26])

$$[K_I]^{isotropic}_{unbridged\ flaw} = \frac{P}{2\pi^2a^{3/2}}\left[\frac{(1-2\nu)+(3-2\nu)\eta^2}{(1-\nu)(1+\eta^2)^2}\right] \tag{23}$$

where ν is Poisson's ratio for the isotropic material.

Finally, in the limiting case where the fibre-reinforced composite is composed of *inextensible fibres*, $\psi \to \infty$, and the integral equation yields a trivial result $\Phi^*(t) = 0$. Consequently,

$$[K_I]^{inextensible\ fibres}_{bridged\ flaw} \to 0 \tag{24}$$

The variation of $\sigma_{zz}(r, 0)$ in the bridged region can be calculated by determining the solution the integral equation along with the integral expression for the stress. While this can be done, attention in this paper has been directed to the evaluation of the Mode I stress intensity factor at the bridged crack tip and the influence of elasticity of the bridging fibres in moderating the SIF. It is sufficient to note that the stress singularity persists for all finite values of E_f; in the case when $E_f \to 0$, the stress state reverts to that of an external crack in an infinite solid loaded by a di-pole of forces. In the case when $E_f \to \infty$, when the singularity is suppressed and the stress state corresponds to that applicable for an infinite space acted on by a doublet of Kelvin forces in an intact infinite space.

4. Numerical results

For any arbitrary value of fibre elasticity, the stress intensity factor has to be evaluated by a numerical solution of the Fredholm integral equation of the second-kind given by (17). (To the author's knowledge, there is no known analytical solution of this integral equation.) The various numerical procedures that can be adopted for the solution of the Fredholm integral equations of the second kind are well documented by Baker [27], Delves and Mohamed [28] and Atkinson [29]. Here, we employ a discretization method that reduces the Fredholm integral equation to a matrix equation, where a *Gaussian quadrature technique* is used for the numerical evaluation of the governing integrals. The Fredholm integral equation (17) can be reduced to the form

$$\Phi^*(t) - \frac{\psi}{\pi}\sum_{j=1}^n \omega_j K(t,\tau_j)\Phi^*(\tau_j) = g(t) \tag{25}$$

where $1 \leqslant t < \infty$, and ω_j are the Gaussian weights, etc. The functional equation (25) can be solved by setting $t = \tau_j$ ($j = 1, \ldots, n$) and the resulting system of equations can be written in the matrix form

$$\mathbf{\Phi}^* = \mathbf{Q}^{-1}\mathbf{G}; \quad \mathbf{Q} = \left[\mathbf{I} - \frac{\psi}{\pi}\boldsymbol{\omega}\mathbf{K}\right] \tag{26}$$

where $\mathbf{I}$ is the identity matrix, $\boldsymbol{\omega}$ is the vector of Gaussian weights, etc. The accuracy of the numerical procedure can be established by

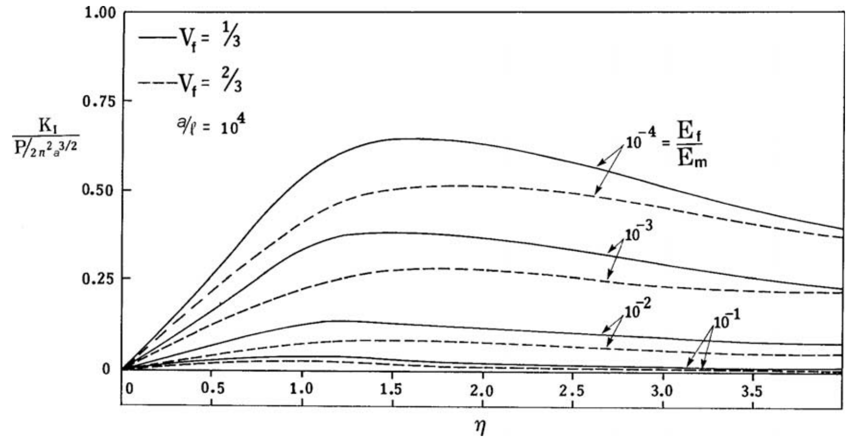

Fig. 2. The stress intensity factor for the bridged external crack [$(a/l) = 10^4$].

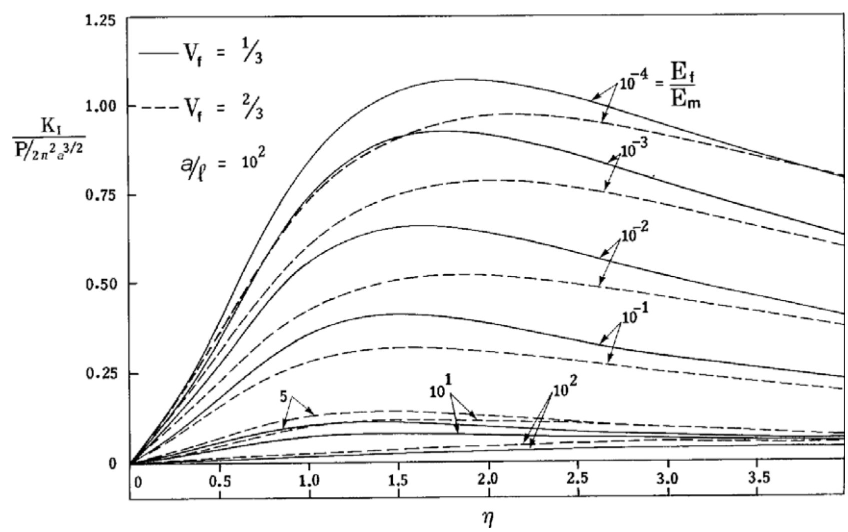

Fig. 3. The stress intensity factor for the bridged external crack [$(a/l) = 10^2$].

evaluating a conditioning number $E = \|\mathbf{Q}\| \left\|\mathbf{Q}^{-1}\right\|$ and through comparison with known exact solutions for Fredholm integral equations of the second kind.

The elastic constants c_{ij} of the transversely isotropic idealization are obtained by using the composite cylinder assemblage analogy given by Hashin and Rosen [18]. For purposes of illustration, the elastic properties for the matrix are assigned the following values: $E_m = 30\ \mathrm{GN/m^2}$; $v_m = 0.35$. The modular ratio $M(=E_f/E_m)$is assigned values in the range 10^2–10^{-4}; $V_f = 1/3, 2/3$ and $v_f = 0.2$. Other variables in the problem are the location of the body forces $\eta = c/a$ and the flaw aspect ratio a/l. Figs. 2 and 3 illustrate the manner in which the various geometric and material parameters influence the stress intensity factor at the boundary of the bridged crack. These results exhibit trends that have been identified previously in connection with the limiting conditions. Finally, the type of problem examined in the paper is better handled through analytical techniques and any computational results are bound to be less accurate, particularly the range of high modular ratio mis-match $(E_f/E_m) \in (10^2, 10^{-4})$ as demonstrated in the paper. The use of conventional finite element techniques without the development of a "bridged crack element" would give rise to unreliable computational estimates.

5. Conclusions

The bridged-crack problem provides a useful mathematical model for examining the influences of matrix cracks that have fibre continuity. The results of the model can be used to assess the implications of fibre continuity in moderating fracture extension at the tip of the bridged crack. An elastic flaw-bridging model is proposed for the study of fibre continuity across a crack in a unidirectionally fibre-reinforced elastic composite. The modelling is applied to investigate the axial loading of a "*bridged external crack*" that is located in a unidirectional fibre-reinforced material. The bridging action due to fibre continuity indicates a displacement-dependent traction condition on the faces of the crack. For the problem to be properly posed, the loading of the external crack should give a finite axial load resultant over the entire region. The mathematical modelling of the " *bridged-crack problem*" can be reduced to the solution of a Fredholm integral equation of the second kind, which can be solved numerically, using quadrature techniques. It is shown that the continuity of the fibres over the faces of the crack leads to a reduction in the crack opening mode stress intensity factor K_I at the crack tip. As the elasticity of the bridging fibres increases this stress intensity factor decreases.

The model of the bridging region incorporated in this study assumes that the *exposed length of the bridging fibres* contributing to its elastic constraint is *constant over the entire crack region*. This assumption can be improved by adopting non-uniform variations of the bridging length over the faces of the crack. The resulting analysis can be better examined via computational stress analysis techniques.

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Appendix A

The constants c_{ij} can be expressed in terms of the elastic constants $E_1, \nu_1^*, G_1, G_{23}$ and K_{23} (where the subscript 1 refers to the fibre-direction and the subscripts 2 and 3 refer to the transversely isotropic plane) in the form

$$\begin{aligned} c_{11} &= K_{23} + G_{23} \\ c_{33} &= E_1 + 4(\nu_1^*)^2 K_{23} \\ c_{13} &= 2\nu_1^* K_{23} \\ c_{12} &= K_{23} - G_{23} \\ c_{44} &= G_1 \end{aligned} \tag{A1}$$

The relationships between E_1, ν_1^*, G_1, etc., and the properties of the fibre (subscript f) and the matrix (subscript m) constituents (E_f, E_m, ν_f, ν_m) and the respective volume fractions (V_f, V_m) are given below. The expression for G_{23} is equivalent to the upper bound for the assemblage presented by Hashin and Rosen [18]:

$$\begin{aligned} K_{23} &= \left\{\frac{\zeta(1+2\nu_m V_f) + 2\nu_m V_m}{\zeta V_m + V_f + 2\nu_m}\right\}(\lambda_m + G_m) \\ G_{23} &= G_m\left\{\frac{(\alpha + \beta_m V_f)(1+\rho V_f^3) - 3V_f V_m^2 \beta_m^2}{(\alpha - V_f)(1+\rho V_f^3) - 3V_f V_m^2 \beta_m^2}\right\} \\ \nu_1^* &= \left\{\frac{V_f E_f L_1 + V_m E_m L_2 \nu_m}{V_f E_f L_3 + V_m E_m L_2}\right\} \\ G_1 &= G_m\left\{\frac{\eta(1+V_f) + V_m}{\eta V_m + V_f + 1}\right\}; \quad E_1 = V_f E_f + V_m E_m \end{aligned} \tag{A2}$$

where

$$\begin{aligned} & L_1 = 2\nu_f(1-\nu_m^2) + V_m \nu_m(1+\nu_m); \quad L_2 = 2V_f(1-\nu_f^2) \\ & L_3 = 2(1-\nu_m^2)V_f + (1+\nu_m)V_m \\ & \zeta = \frac{\lambda_f + G_f}{\lambda_m + G_f}; \quad \alpha = \frac{\eta + \beta_m}{\eta - 1}; \quad \rho = \frac{\beta_m - \eta\beta_f}{1 + \eta\beta_f} \\ & \eta = \frac{G_f}{G_m}; \quad V_m + V_f = 1 \\ & G_i = \frac{E_i}{2(1+\nu_i)}; \quad \lambda_i = \frac{\nu_i E_i}{(1+\nu_i)(1-2\nu_i)}; \quad \beta_i = (3-4\nu_i)^{-1}; \quad (i = m, f) \end{aligned} \tag{A3}$$

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