On the Mode I stress intensity factor for an external circular crack with fibre bridging

A.P.S. Selvadurai

Department of Civil Engineering and Applied Mechanics, McGill University, Montréal, QC, Canada H3A 2K6

1. Introduction

The more general applications of fibre reinforcement invariably involve multidirectional reinforcement to address directional variability in the loading of fibre-reinforced materials. Unidirectional reinforcement usually results from a highly specific functional requirement in the use of fibre reinforcement, where the dominant loading direction can be identified a priori. In spite of this limitation, the study of unidirectionally reinforced materials provides useful insight into micro-mechanical features that can influence the load transfer between the fibres and the matrix. In its fabricated condition, a fibre-reinforced material is expected to be defect free. The notion of a defect free fibre-reinforced material is largely a matter of definition, since even perfect fibre reinforcement can exhibit micro-mechanical defects during curing processes and under certain loading and environmental conditions resulting from localized loads, extreme temperatures and impact. The integrity of a fibre-reinforced material can therefore be influenced by the development of features such as fibre breakage, fibre pullout, matrix fracture, fibre-matrix interface delamination, matrix damage, matrix void growth, etc. The influence of damage and defects on the structural integrity of fibre-reinforced materials was discussed several decades ago by a number of researchers including Beaumont and Harris [1], Aveston and Kelly [2], Bowling and Groves [3] and Backlund [4]. In particular, flaw- or crack bridging in unidirectional fibre-reinforced composites was discussed by Kelly [5], Aveston and Kelly [6] and Sih [7]. The initial investigations dealing with the modelling of flaw bridging in composites were presented by Selvadurai [8,9], followed by the work of Stang [10], Rose [11], McCartney [12], Budiansky et al. [13], Budiansky and Amazigo [14] and Selvadurai [15] who investigated various aspects of the elastostatic problem of bridging-induced behaviour of flaws in unidirectional fibre-reinforced materials including frictional load transfer across single fibres bridging cracks. Recently, Movchan and Willis [16,17] have examined the crack tip effects in bridged cracks using asymptotic methods for the analysis of the associated integral equations. When there is complete continuity in the load transfer between the unidirectional fibres and the matrix, the composite will exhibit either orthotropy or transverse isotropy in its mechanical behaviour. The equivalent elasticity properties of the unidirectionally reinforced composite can be determined by considering the theories of effective elastic properties developed in the literature. Unidirectionally reinforced brittle elastic composites can exhibit various forms of structural damage and fracture depending on the types of loading contributing to such phenomena. Examples can include impact or dynamic loadings, thermal loadings, localized inhomogeneities introduced due to fabrication effects, etc. With unidirectionally reinforced brittle matrices, damage and fracture is usually restricted to either the brittle matrix or the interface between the fibre and the matrix. Interface failure can be minimized by the use of bond enhancing agents, and, as a result, the fracture and damage is largely restricted to the brittle matrix. A model of a fractured unidirectionally reinforced composite region can be visualized as a fractured matrix through which the fibres can exhibit continuity. The resulting defect in the composite is referred to as a bridged flaw, where the intact fibres exert a constraint on the mechanics of the flaw. Bridged-flaws can vary
depending on the mode of initiation of matrix cracking: In this paper, we examine the influence of flaw bridging in a unidirectionally reinforced composite by considering an axisymmetric problem where an external crack in a composite is loaded by a doublet of concentrated forces \( P \) located at a finite distance from the bridged external crack (Fig. 1). It is, however, important to emphasize the fact that analytical treatments of this nature are intended to examine localized flaws that can exist in regions of unidirectionally reinforced solids. If the assumption of axial symmetry is removed then other conditions have to be stipulated to ensure that the net stresses in the bridging fibres are always tensile. The absence of such a guarantee will make the problem inordinately complicated requiring an iterative incremental technique to identify regions where the bridging actions are present. Non-axisymmetric problems are better handled via computational approaches, which have issues of their own.

2. Governing equations

When complete continuity exists between the reinforcing fibres and the defect free matrix, the elastic behaviour of the unidirectionally reinforced composite can be modelled by appeal to the theory of transversely isotropic elastic solids and the effective elasticity properties can be determined by the theory of composite materials developed in the classical studies by Hashin and Rosen [18], Hill [19], Broutman and Krock [20], Hale [21] and Christensen [22]. We consider the mechanics of a transversely isotropic elastic material where the axis of elastic symmetry coincides with the fibre direction. As has been shown by Elliott [23] and Shield [24], the displacement and stress fields in the resulting transversely isotropic elastic material can be expressed in terms of two functions \( \phi_i(r, z) \), \( i = 1, 2 \), which are solutions of

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) \right) \phi_i(r, z) = 0
\]

where \( z = z/\sqrt{v_i} \), and \( v_i \) are roots of the equation

\[
c_{11}c_{44}v_i^2 + \left[ c_{13}(2c_{44} + c_{11}) - c_{11}c_{33} \right] v_i + c_{33}c_{44} = 0
\]

and \( c_i \) are the elastic constants of the transversely isotropic elastic material. In general,

\[
c_{ij} = c_g(E_t, E_m, v_f, v_m, V_f, V_m, C)
\]

where the material parameters of the composite depend on the constituent properties, the volume fractions and a contiguity factor, \( C \). Explicit expressions for \( c_g \) are given, for example, in [18,22] (see Appendix A). The displacement and stress fields relevant to the problems considered here are given by

\[
\begin{align*}
\begin{bmatrix}
u_1(r, z) \\
\sigma_{zz}(r, z) \\
\sigma_{rz}(r, z)
\end{bmatrix} &= \sum_{i=1}^{2} \left( \frac{\partial}{\partial r} k_i \phi_i ight) \\
&= \frac{\sigma_{zz}(r, z) - \nu_1 \sigma_{rr}(r, z)}{C_0} \sum_{i=1}^{2} \left( \frac{c_{ij}}{C_0} \frac{\partial \phi_i}{\partial r} \right)
\end{align*}
\]

where

\[
k_i = \frac{(c_{11}v_i - c_{44})}{(c_{11} + c_{44})}
\]

2.1. The loading of the intact fibre-reinforced solid by a doublet of forces

Prior to examining the bridged external circular crack problem we consider the axisymmetric loading of the intact fibre-reinforced solid by symmetrically placed body forces. Forces of magnitude \( P \) act at the locations \( z = \pm c \) along the \( z \)-axis. These are directed in the positive and negative \( z \)-directions, respectively, such that a state of tension exists in the plane of symmetry \( z = 0 \). The state of stress \( \sigma_{ij} \) due to this doublet of forces is such that

\[
\sigma_{ij}(r, 0) = 0
\]

and

\[
\sigma_{ij}(r, 0) = p_0 \left[ \frac{c_i r^2 c_1}{(r^2 + c_1^2)^2} - \frac{c_1 c_2 r^2}{(r^2 + c_2^2)^2} \right] = p'(r)
\]

where

\[
p_0 = \frac{P}{\pi a^2}; \quad c_i = \frac{C}{\sqrt{v_i}}
\]

and

\[
\zeta_i = \frac{(c_{11} + c_{44})(k_i c_{33} - v_i c_{13})v_i v_2}{2c_{33}c_{44}(v_1 - v_2)v_1}
\]

2.2. The boundary value problem for the external bridged crack

We now consider the axisymmetric problem of an external circular crack of radius \( a \) located in a unidirectionally reinforced composite. The external crack is bridged by fibres of effective length \( 2l \) that are present due to the bridging action. Unlike the classical formulations of crack problems, the cracked region is assumed to have a finite “effective length” \( 2l \) of exposed fibres. This “effective length” can be enhanced by the debonding of fibres resulting from the “load transfer” associated with matrix cracking and the ensuing lateral contraction of exposed regions due to a Poisson’s ratio effect.

The unidirectionally reinforced composite containing the bridged external crack is loaded by a doublet of axial forces that are located at distances \( c \) from the intact region (Fig. 1). The analysis of the axisymmetric mixed boundary value problem governing this problem requires that

\[
u_2(r, 0) = 0; \quad r \in (0, a)
\]

\[
\sigma_{zz}(r, 0) = -P(r) + \frac{V_f E_t}{T} \nu_2(r, 0); \quad r \in (a, \infty)
\]

\[
\sigma_{rz}(r, 0) = 0; \quad r \in (0, \infty)
\]

where \( E_t \) and \( V_f \) are respectively the fibre elastic modulus and volume fraction and \( P(r) \) is the stress state in the plane of the intact solid due to the doublet of forces. As is evident from (11), in the present analysis the length of the bridging fibres \( l \) is assumed to be a constant over the externally cracked region. This assumption has been used by other authors (see e.g. [1–10]) and others in their
investigation of bridged cracks. If a computational approach is adopted this restriction can be replaced by a bridging region that can vary spatially. Owing to the symmetry of the bridged-crack problem about the plane \( z = 0 \), the problem can be formulated as a mixed boundary value problem related to a transversely isotropic halfspace region. The solutions of (1), applicable to the region \( z > 0 \), can be written in the integral form

\[
\varphi_i(r, z) = \frac{1}{a^2} \int_0^\infty \xi A_i(\xi) e^{-i\xi J_0(\xi r/a)} d\xi; \quad (i = 1, 2) \tag{13}
\]

where \( A_i(\xi) \) are arbitrary functions and \( \lambda_i = \xi/a \sqrt{\nu_i} \).

By making use of the Hankel integral solutions (3) and the relationships (4) and (5), it can be shown that the mixed boundary conditions (11) and (12) are equivalent to the system of dual integral equations

\[
\int_0^\infty \xi B(\xi) F(\xi J_0(\xi r/a)) d\xi = \frac{P'(r)}{2\pi r}, \quad a < r < \infty \tag{14}
\]

\[
\int_0^a B(\xi) J_0(\xi r/a) d\xi = 0, \quad 0 \leq r < a \tag{15}
\]

where

\[
A_1(\xi) = \frac{\sqrt{v_1(1 + k_2) B(\xi)}}{\sqrt{v_2(1 + k_1) \xi^2}}; \quad A_2(\xi) = \frac{B(\xi)}{\xi} \tag{16}
\]

\[
\mu' = \frac{\epsilon_{44} Q}{2a^2 \sqrt{v_1}(1 + k_1)} \tag{17}
\]

\[
\Omega' = \sqrt{v_1(1 + k_1)} \left\{ \frac{k_2 c_{13} - v_2 c_{13}}{c_{44}} \right\} - \sqrt{v_2(1 + k_2)} \left\{ \frac{k_1 c_{13} - v_1 c_{13}}{c_{44}} \right\} \tag{18}
\]

Using a finite Fourier cosine transform Eq. (15) can be identically satisfied (Sneddon [25]), the system of dual integral equations (14) and (15) can be reduced to a single Fredholm integral equation of the second kind, for a single unknown function \( \Phi'(t) \) that is related to \( B(\xi) \): i.e.,

\[
\Phi'(t) = \frac{\psi}{2\pi} \int_0^\infty K(t, \tau) \Phi'(\tau) d\tau = g(t) \tag{19}
\]

where the kernel function \( K(t, \tau) \) is given by

\[
K(t, \tau) = 2 \int_0^\infty \frac{\cos(\xi t) \cos(\xi \tau)}{\xi} d\xi \tag{20}
\]

\[
g(t) = \frac{\xi_1 \sqrt{v_1} \eta}{(v_1 t^2 + \eta^2)} - \frac{\xi_2 \sqrt{v_2} \eta}{(v_2 t^2 + \eta^2)} \tag{21}
\]

and \( \eta = c/a \). The complete mathematical analysis of the bridged external circular crack is now reduced to the solution of the Fredholm integral equation (17). In the ensuing discussions, however, we shall restrict attention to an examination of the influence of the elastic bridging action on the stress intensity factor at the crack boundary.

3. Limiting cases

Prior to considering the numerical solution of the Fredholm integral equation of the second-kind for \( \Phi'(t) \), it is instructive to examine the stress intensity factor resulting from certain limiting cases of the bridging action. The stress intensity factor for the opening mode of the bridged external circular crack is defined by

\[
K_i = \lim_{r \to 0} \{ 2(a - r) \}^{1/2} \sigma_{22}(r, 0) \tag{22}
\]

From the results derived previously, it can be shown that

\[
|K_i|_{\text{bridged flow}} = \frac{P}{2\pi^2 a^2} \{ 4\Phi'(1) \} \tag{23}
\]

In the limiting case when the elasticity of the bridged fibres (\( E_2 \)) reduces to zero, the parameter \( \psi = 0 \) and the expression (21) yields the stress intensity factor for the di-pole body force loading of an external circular flaw in a transversely isotropic elastic medium; i.e.,

\[
|K_i|_{\text{unbridged flow}} = \frac{P}{2\pi^2 a^2} \left\{ \frac{2(c_{12} + c_{44}) v_1 v_2}{c_{33} c_{44}(v_1 v_2)} + \frac{(k_1 c_{13} - v_1 c_{13})}{\sqrt{v_1(v_1 + \eta^2)}} \right\} \tag{24}
\]

The stress singularity persists for all finite values of \( E_2 \). For any arbitrary value of fibre elasticity, the stress intensity factor at the bridged crack tip and the influence of elastic bridging action on the stress intensity factor for the bridged flaw can be expressed as

\[
|K_i|_{\text{unbridged flow}} = \frac{P}{2\pi^2 a^2} \left\{ 1 - 2\nu + \frac{(3 - 2\nu)(1 + \eta^2)}{(1 - \nu)(1 + \eta^2)} \right\} \tag{25}
\]

4. Numerical results

For any arbitrary value of fibre elasticity, the stress intensity factor has to be evaluated by a numerical solution of the Fredholm integral equation of the second-kind given by (17). (To the author's knowledge, there is no known analytical solution of this integral equation.) The various numerical procedures that can be adopted for the solution of the Fredholm integral equations of the second kind are well documented by Baker [27], Delves and Mohamed [28] and Atkinson [29]. Here, we employ a discretization method that reduces the Fredholm integral equation to a matrix equation, where a Gaussian quadrature technique is used for the numerical evaluation of the governing integrals. The Fredholm integral equation (17) can be reduced to the form

\[
\Phi'(t) - \frac{\psi}{2\pi} \sum_{i=1}^{n} \omega_i K(t, \tau_j) \Phi'(\tau_j) = g(t) \tag{26}
\]

where \( 1 \leq t < \infty \), and \( \omega_i \) are the Gaussian weights, etc. The functional equation (26) can be solved by setting \( t = \tau_j \) (\( j = 1, \ldots, n \)) and the resulting system of equations can be written in the matrix form

\[
\Phi' = Q^{-1} G; \quad Q = I - \frac{\psi}{2\pi} \alpha K \tag{27}
\]

where \( I \) is the identity matrix, \( \alpha \) is the vector of Gaussian weights, etc. The accuracy of the numerical procedure can be established by
evaluating a conditioning number \( E = \|Q\|\|Q^{-1}\| \) and through comparison with known exact solutions for Fredholm integral equations of the second kind.

The elastic constants \( c_{ij} \) of the transversely isotropic idealization are obtained by using the composite cylinder assemblage analogy given by Hashin and Rosen [18]. For purposes of illustration, the elastic properties for the matrix are assigned the following values: \( E_m = 30 \) GN/m\(^2\); \( \nu_m = 0.35 \). The modular ratio \( M = E_f/E_m \) is assigned values in the range \( 10^2 - 10^4 \); \( V_f = 1/3, 2/3 \) and \( V_f = 0.2 \). Other variables in the problem are the location of the body forces \( \eta = c/a \) and the flaw aspect ratio \( a/l \). Figs. 2 and 3 illustrate the manner in which the various geometric and material parameters influence the stress intensity factor at the boundary of the bridged crack. These results exhibit trends that have been identified previously in connection with the limiting conditions.

5. Conclusions

The bridged-crack problem provides a useful mathematical model for examining the influences of matrix cracks that have fibre continuity. The results of the model can be used to assess the implications of fibre continuity in moderating fracture extension at the tip of the bridged crack. An elastic flaw-bridging model is proposed for the study of fibre continuity across a crack in a unidirectionally fibre-reinforced elastic composite. The modelling is applied to investigate the axial loading of a “bridged external crack” that is located in a unidirectional fibre-reinforced material. The bridging action due to fibre continuity indicates a displacement-dependent traction condition on the faces of the crack. For the problem to be properly posed, the loading of the external crack should give a finite axial load resultant over the entire region. The mathematical modelling of the “bridged-crack problem” can be reduced to the solution of a Fredholm integral equation of the second kind, which can be solved numerically, using quadrature techniques. It is shown that the continuity of the fibres over the faces of the crack leads to a reduction in the crack opening mode stress intensity factor \( K_I \) at the crack tip. As the elasticity of the bridging fibres increases this stress intensity factor decreases.
The model of the bridging region incorporated in this study assumes that the exposed length of the bridging fibres contributing to its elastic constraint is constant over the entire crack region. This assumption can be improved by adopting non-uniform variations of the bridging length over the faces of the crack. The resulting analysis can be better examined via computational stress analysis techniques.

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Appendix A

The constants \( C_i \) can be expressed in terms of the elastic constants \( E_i, G_i, \) and \( K_{23} \) (where the subscript 1 refers to the fibre-direction and the subscripts 2 and 3 refer to the transversely isotropic plane) in the form

\[
C_{11} = C_{22} + C_{33} \\
C_{33} = E_1 + 4(\nu_1^2)K_{23} \\
C_{12} = 2\nu_1 K_{23} \\
C_{44} = G_1
\]

The relationships between \( E_1, \nu_1, G_1, \) etc., and the properties of the fibre (subscript \( f \)) and the matrix (subscript \( m \)) constituents \( (E_f, E_m, \nu_f, \nu_m) \) and the respective volume fractions \( (V_f, V_m) \) are given below. The expression for \( G_{23} \) is equivalent to the upper bound for the assemblage presented by Hashin and Rosen [18]:

\[
K_{23} = \left\{ \frac{3(1+2\nu_m)V_f + 2\nu_m V_m}{\nu_m + V_f + 2V_m} \right\} (\lambda_m + G_m) \\
G_{23} = \frac{\nu_f E_f}{V_f E_f L_1 + V_m E_m L_2} \\
\nu_1 = \frac{V_f E_f L_1 + V_m E_m L_2}{V_f E_f L_1 + V_m E_m L_2} \\
G_1 = \frac{\eta (1 + \nu_f) + V_m}{\eta V_m + V_f + 1}; \quad E_1 = V_f E_f + V_m E_m
\]

where

\[
L_1 = 2V_f(1 - \nu_f^2) + V_m V_m (1 + \nu_m); \quad L_2 = 2V_f(1 - \nu_f^2) \\
L_3 = 2(1 - \nu_m^2)V_f + (1 + \nu_m)V_m \\
\zeta = \frac{\lambda_f + G_f}{\lambda_f + G_f}; \quad \alpha = \frac{\eta_f + \beta_m}{\eta_f - 1}; \quad \rho = \frac{\beta_m - \eta \beta_f}{1 + \eta \beta_f} \\
\eta = \frac{G_f}{G_m}; \quad V_m + V_f = 1 \\
G_i = \frac{E_i}{2(1 + \nu_i)}; \quad \lambda_i = \frac{\nu_i E_i}{(1 + \nu_i)(1 - 2\nu_i)}; \quad \beta_i = (3 - 4\nu_i)^{-1}; \quad (i = m,f)
\]