SELVADURAI, A. P. S. (1979). *Géotechnique* 29, No. 2, 195–202

The displacement of a rigid circular foundation anchored to an isotropic elastic half-space

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This Paper examines the axisymmetric problem of a rigid circular foundation, resting on an isotropic elastic half-space which is subjected, simultaneously, to an external load and an internal anchor load. The anchor load consists of constant, linear or parabolic distributions of Mindlin forces of finite length, located along the axis of symmetry. A Mindlin force is defined as a concentrated force which acts at an interior point of the half-space along the axis of symmetry. The solution for the rigid displacement experienced by the circular plate is obtained in an exact closed form. This particular problem is of interest in connection with the study of rock anchors and in the examination of in situ tests such as the cable method of in situ testing.

Cet article examine en axisymétric le cas d'une fondation circulaire rigide reposant sur un semi-espace élastique isotrope, soumise simultanément à un effort externe et à un effort d'ancrage interne. L'effort d'ancrage se compose des forces de Mindlin de longueur finie, situées le long de l'axe de symétrie et dont la répartition est constante, linéaire ou parabolique. Une force Mindlin se définit comme étant une force concentrée agissant en un point interne du semi-espace le long de l'axe de symétrie. On traite le déplacement rigide de la plaque circulaire grâce à l'emploi d'une forme exacte fermée. Ce problème spécifique présente un intérêt pour l'étude des ancrages en terrain rocheux et l'analyse d'essais *in situ* tels que l'essai utilisant la méthode par câble.

The axisymmetric problem of a rigid circular foundation resting in smooth contact with an isotropic elastic half-space was first considered by Boussinesq (1885). The same problem was subsequently investigated by Harding and Sneddon (1945) who employed a Hankel transform technique for the solution of the elasticity problem. The results developed for this particular interaction problem have found extensive application in the geotechnical study of settlement of foundations and in the evaluation of in situ tests such as plate load tests (Terzaghi, 1943; Stagg and Zienkiewicz, 1968; Jaeger, 1972; Poulos and Davis, 1974; Selvadurai, 1979).

In the classical treatment of the interaction problem related to the rigid circular foundation it is assumed that the elastic half-space is subjected to axisymmetric loads that are applied at the surface of the plate. This Paper is concerned with the analysis of the settlement of a rigid circular foundation resting in smooth contact with an isotropic elastic half-space and subjected simultaneously to axisymmetric external and internal loads. The particular axisymmetric internal loads correspond to constant, linear or parabolic distributions of Mindlin (1936) forces, which are of finite length, located at a finite depth below the free surface of the half-space (Fig. 1). The above problem is of some interest with regard to the examination of tests such as the 'cable method of in situ testing', utilized in the determination of in situ properties of soil and rock media, or in the assessment of the mechanical behaviour of an anchor bolt (Stagg and Zienkiewicz, 1968; Jaeger, 1972). Here, the distributions of Mindlin-type forces represent, approximately, the influence of the anchor region. The effect of the anchor rod placement and other frictional effects are neglected in the ensuing analysis.

The solution to the problem in the title of this Paper can be approached by making use of the complex potential function formulation developed by Green (1949). Using such a technique, a normal contact stress distribution beneath the rigid foundation is sought, such that

Discussion on this Paper closes 1 September, 1979. For further details see inside back cover.
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the normal surface displacement field resulting from these contact stresses when combined with the normal surface displacement due to the internal anchor load gives a constant displacement within the foundation region $r \leqslant a$, where a is the radius of the rigid foundation. The solution for the plate displacement due to a concentrated Mindlin force is first developed; this result is integrated to generate the results for the distributed internal loading. (Since only internal load distributions are considered it implies that the anchor region is flexible.) In the formulation of this interaction problem it is assumed that the contact between the rigid foundation and the half-space is smooth. Accordingly, for the solutions developed to be physically admissible, the location of the anchor region should be such that no tensile stresses are generated at the interface. It is found that as the anchor region migrates to the boundary of the half-space, tensile stresses do tend to develop at the interface.

The analytical formulation of the axisymmetric interaction between a circular foundation and an internal load considered here yields exact closed-form solutions for the settlement of the rigid foundation. Numerical results presented illustrate the manner in which this settlement is influenced by the depth of location and the length and the load distribution in the anchor region.

GOVERNING EQUATIONS

A comprehensive account of the complex potential function approach together with its application to crack and indentation problems in classical elasticity is given by Green and Zerna (1968). Briefly, the class of problems in which the shearing stresses vanish at all points in a plane, $z = 0$, can be reduced to classical problems in potential theory. The displacement and stress components for this class of problem can be uniquely represented in terms of a single potential function $\Phi(r, \theta, z)$, where (r, θ, z) represents the cylindrical polar coordinate system. The particular displacement and stress components of interest to the interaction problem are u_z, σ_{zz} and σ_{rz}. For axisymmetric problems, these can be represented in the forms

$$\left.\begin{aligned} 2Gu_z(r, z) &= z\frac{\partial^2\Phi}{\partial z^2} - 2(1-\nu)\frac{\partial\Phi}{\partial z} \\ \sigma_{zz} &= z\frac{\partial^3\Phi}{\partial z^3} - \frac{\partial^2\Phi}{\partial z^2}; \quad \sigma_{rz} = z\frac{\partial^3\Phi}{\partial r\partial z^2} \end{aligned}\right\} \qquad (1)$$

where G and ν are the linear elastic shear modulus and Poisson's ratio, respectively.

When the contact between the rigid foundation and the elastic half-space ($z > 0$) is frictionless it is sufficient that all the stresses and displacements derived from $\Phi(r, z)$ should decay as $z \rightarrow \infty$ and that no shear stresses should act on the bounding plane $z = 0$. The third boundary condition is of a mixed type where

$$\left.\begin{aligned} u_z(r, 0) &= -\frac{(1-\nu)}{G}\frac{\partial\Phi}{\partial z} = u^*(r) \quad &\text{on } r < a \\ \sigma_{zz}(r, 0) &= -\frac{\partial^2\Phi}{\partial z^2} = 0 \quad &\text{on } r > a \end{aligned}\right\} \qquad (2)$$

Following Green and Zerna (1968) we consider the representation

$$-\frac{(1-\nu)}{G}\frac{\partial\Phi}{\partial z} = \frac{1}{2}\int_{-a}^{a}\frac{g(t)dt}{[r^2+(z+it)^2]^{\frac{1}{2}}} \qquad (3)$$

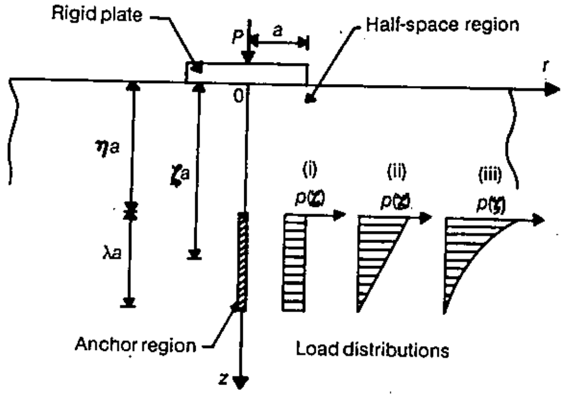

Fig. 1. The rigid plate–anchor system. Anchor load distribution $p(\zeta)$. **(i)** $p(\zeta)=P/\lambda a$, **(ii)** $p(\zeta)=2P/\lambda^2 a$ $[(\lambda+\eta)-\zeta]$, **(iii)** $p(\zeta)=3P/\lambda^3 a[(\lambda+\eta)-\zeta]^2$, **where** $P=$ **total load**

which satisfies $\nabla^2\Phi(r, z)=0$ and the regularity conditions at infinity. (In equation (3) $i^2=-1$.) Then from (3) we have

$$\left.\begin{aligned} -\frac{(1-\nu)}{G}\frac{\partial\Phi}{\partial z} &= \int_0^r \frac{g(t)\,dt}{(r^2-t^2)^{\frac{1}{2}}} \quad \text{on } z=0;\ 0<r<a \\ \frac{\partial^2\Phi}{\partial z^2} &= 0 \qquad\qquad\qquad \text{on } z=0;\ r>a \end{aligned}\right\} \quad . \ . \ . \ . \ (4)$$

and the mixed boundary conditions in equation (2) reduce to the single integral equation

$$u^*(r)=\int_0^r \frac{g(t)\,dt}{(r^2-t^2)^{\frac{1}{2}}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (5)$$

in terms of the unknown function $g(t)$. The Abel integral, equation (5), can be inverted to complete the solution. Assuming that $u^*(r)$ is continuously differentiable in the region $0\leqslant r\leqslant a$, the solution of (5) is given by

$$g(t)=\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{ru^*(r)\,dr}{(t^2-r^2)^{\frac{1}{2}}} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (6)$$

The contact stress distribution at the interface of the circular foundation can be expressed in terms of $g(t)$ in the form

$$\sigma_{zz}=\frac{G}{(1-\nu)}\frac{1}{r}\frac{\partial}{\partial r}\int_r^a \frac{tg(t)\,dt}{(t^2-r^2)^{\frac{1}{2}}} \quad . \ . \ . \ . \ . \ . \ . \ . \ (7)$$

Using (7) it can be shown that the total force exerted by the rigid circular foundation is given

$$P=\frac{2\pi G}{(1-\nu)}\int_0^a g(t)\,dt \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (8)$$

In summary, once $u^*(r)$ is specified the load–displacement relationship for the circular foundation can be obtained by making use of the results in (6) to (8).

THE ANCHORED FOUNDATION PROBLEM

We first consider the problem of the indentation of the half-space by an external load P

and an internal concentrated Mindlin force P_0 acting at a distance c from the boundary of the half-space. It is assumed that under the combined action of P and P_0 the rigid foundation experiences a displacement w_0 and that no separation occurs at the interface region $r \leqslant a$. The prescribed displacement function corresponding to $u^*(r)$ is given by

$$u^*(r) = w_0 + \frac{P_0(1-\nu)}{2\pi G}\left[\frac{1}{(r^2+c^2)^{\frac{1}{2}}} + \frac{c^2}{2(1-\nu)(r^2+c^2)^{\frac{3}{2}}}\right] \quad (9)$$

Using (9) in (6) we obtain

$$g(t) = \frac{2}{\pi}\left\{w_0 + \frac{P_0 c}{4\pi G}\left[\frac{(3-2\nu)}{(t^2+c^2)} - \frac{2t^2}{(t^2+c^2)^2}\right]\right\} \quad (10)$$

The rigid displacement of the circular foundation due to the combined action of P and P_0 can be obtained by evaluating the expression (8) for the total load; we have

$$w_0 = \frac{P(1-\nu)}{4aG}\left\{1 - \frac{P_0}{P}\left[\frac{2}{\pi}\tan^{-1}\left(\frac{a}{c}\right) + \frac{ac}{\pi(1-\nu)(a^2+c^2)}\right]\right\} \quad (11)$$

It may be noted that as $c \to \infty$ the result (11) reduces to that of the classical Boussinesq problem. Similarly as $c \to 0$ and $P = P_0$, the rigid foundation is subjected to a doublet of forces; as such $w_0 = 0$. The stress distribution beneath the foundation is similarly given by

$$\sigma_{zz}(r, 0) = \frac{-P}{2\pi a[a^2-r^2]^{\frac{1}{2}}}\left[1 + \frac{P_0}{P}H(r, c)\right] \quad (12)$$

where

$$H(r, c) = -\left[\frac{2}{\pi}\tan^{-1}\left(\frac{a}{c}\right) + \frac{ac}{\pi(1-\nu)(a^2+c^2)}\right] + \frac{ac}{\pi(1-\nu)(r^2+c^2)^2}$$
$$\times\left\{(1-2\nu)(r^2+c^2) + c^2 + \frac{c^2(2a^2+c^2-r^2)}{(a^2+c^2)}\right.$$
$$\left. + [(1-2\nu)(r^2+c^2)+3c^2]\sqrt{\left(\frac{a^2-r^2}{r^2+c^2}\right)}\tan^{-1}\sqrt{\left(\frac{a^2-r^2}{r^2+c^2}\right)}\right\} \quad (13)$$

The solution for the displacement of the rigid circular foundation under the action of distributed line loads acting along the axis of symmetry can be generated from the integration of the result in equation (11) within appropriate limits. Three particular forms of internal load distributions are examined (Fig. 1)

(a) the anchoring load P_0 $(= P)$ is distributed along a finite length with a constant load intensity
(b) the anchor load varies linearly along a finite length
(c) the anchor load varies parabolically along a finite length.

The procedure could no doubt be extended to include other variations in the distribution of the anchor load.

Constant load intensity

The displacement of the rigid foundation due to the combined action of the external load P and the anchor load uniformly distributed over the length λa is given by

$$[w_0]_{\text{constant}} = \frac{P(1-\nu)}{4aG}\left\{1 - \frac{2}{\pi\lambda}\left[I_1(\lambda, \eta) + \frac{1}{2(1-\nu)}I_2(\lambda, \eta)\right]\right\} \quad (14a)$$

where ηa is the depth of location of the anchor load (Fig. 1) and

$$I_1(\lambda, \eta) = (\lambda+\eta)\tan^{-1}\left(\frac{1}{\lambda+\eta}\right) - \eta\tan^{-1}\left(\frac{1}{\eta}\right) + \frac{1}{2}\ln\left[\frac{1+(\lambda+\eta)^2}{1+\eta^2}\right],$$

$$I_2(\lambda, \eta) = \frac{1}{2}\ln\left[\frac{1+(\lambda+\eta)^2}{1+\eta^2}\right] \qquad (14b)$$

Linearly varying anchor load

When the anchor load is linearly distributed over its length the displacement of the rigid circular foundation is given by

$$[w_0]_{\text{linear}} = \frac{P(1-\nu)}{4aG}\left(1 - \frac{4}{\pi\lambda^2}\left\{(\lambda+\eta)I_1(\lambda, \eta) - I_3(\lambda, \eta) + \frac{1}{2(1-\nu)}[(\lambda+\eta)I_2(\lambda, \eta) - I_4(\lambda, \eta)]\right\}\right) \qquad (15a)$$

where

$$I_3(\lambda, \eta) = \frac{1}{2}\left\{\lambda + [1+(\lambda+\eta)^2]\tan^{-1}\left(\frac{1}{\lambda+\eta}\right) - (1+\eta^2)\tan^{-1}\left(\frac{1}{\eta}\right)\right\},$$

$$I_4(\lambda, \eta) = \lambda - \tan^{-1}(\lambda+\eta) + \tan^{-1}(\eta) \qquad (15b)$$

Parabolically varying anchor load

When the total anchor load exhibits a parabolic distribution over its length, the displacement of the rigid circular foundation is given by

$$[w_0]_{\text{parabolic}} = \frac{P(1-\nu)}{4aG}\left(1 - \frac{6}{\pi\lambda^3}\left\{(\lambda+\eta)^2 I_1(\lambda, \eta) - 2(\lambda+\eta)I_3(\lambda, \eta) + I_5(\lambda, \eta) + \frac{1}{2(1-\nu)}[(\lambda+\eta)^2 I_2(\lambda, \eta) - 2(\lambda+\eta)I_4(\lambda, \eta) + I_6(\lambda, \eta)]\right\}\right) \qquad (16a)$$

where

$$I_5(\lambda, \eta) = \frac{(\lambda+\eta)^3}{3}\tan^{-1}\left(\frac{1}{\lambda+\eta}\right) - \frac{\eta^3}{3}\tan^{-1}\left(\frac{1}{\eta}\right) + \frac{\lambda}{6}(\lambda+2\eta) - \frac{1}{6}\ln\left[\frac{1+(\lambda+\eta)^2}{1+\eta^2}\right] \qquad (16b)$$

and

$$I_6(\lambda, \eta) = \frac{\lambda}{2}(\lambda+2\eta) - \frac{1}{2}\ln\left[\frac{1+(\lambda+\eta)^2}{1+\eta^2}\right]$$

THE CONTACT STRESS DISTRIBUTION

The contact stress distribution developed at the circular foundation–elastic half-space interface due to the action of a combination of Mindlin forces can also be developed by integrating the stress distribution due to the single concentrated force (equation (12)) within the appropriate limits. In general,

$$[\sigma_{zz}(r, 0)] = \frac{-P}{2\pi a[a^2-r^2]^{\frac{1}{2}}}\left[1 + \frac{1}{P}\int_{\eta a}^{(\eta+\lambda)a} a p(\zeta) H(r, \zeta)\, d\zeta\right] \qquad (17)$$

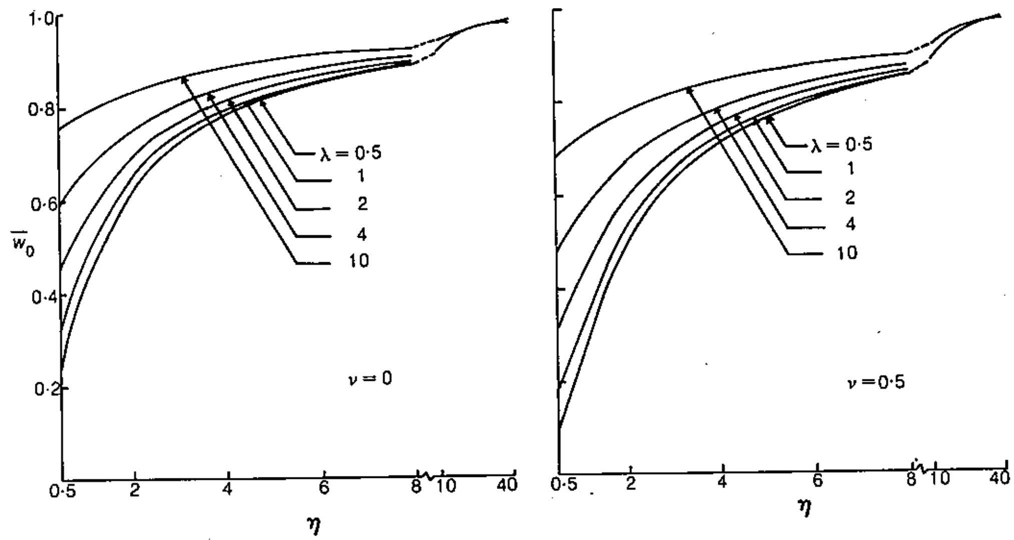

Fig. 2. The displacement of the rigid plate. Anchor load distribution constant $w_0 = [P(1-\nu)/4aG]\bar{w}_0$

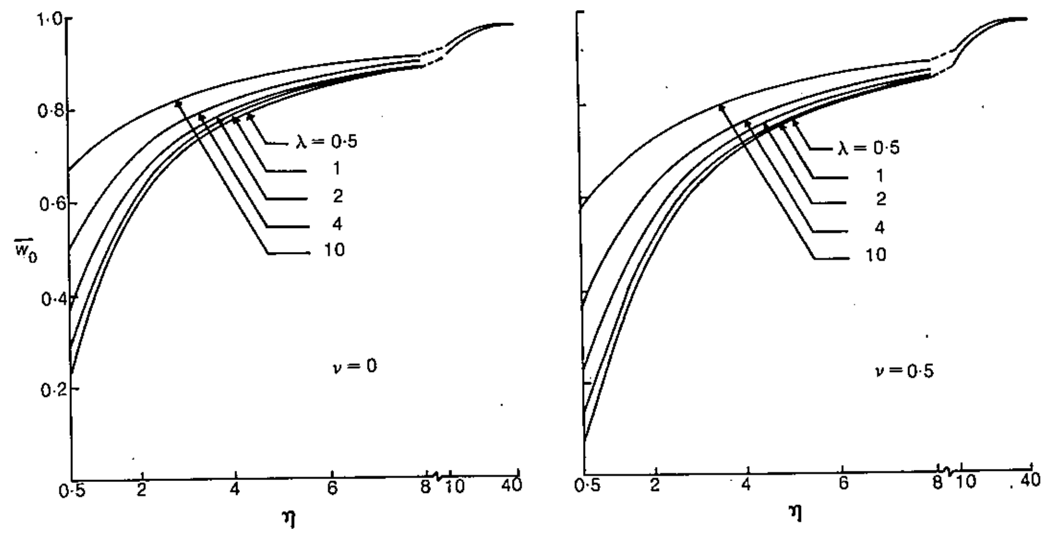

Fig. 3. The displacement of the rigid plate. Anchor load distribution linear

where various expressions for $p(\zeta)$ are given in Fig. 1. The integrals appearing in (17) do not appear to reduce to any exact closed forms such as (14) to (16). The integral can, however, be evaluated by adopting a numerical scheme based on Gaussian quadrature. Such an evaluation is necessary to establish the critical values of ηa for which tensile contact stresses would develop at the rigid foundation–elastic half-space interface $r \leqslant a$. Alternatively, the critical values of ηa can be established by assuming that the entire anchor load P acts as a concentrated load located at a depth ηa. Using such a technique it is found that tensile stresses develop at the interface at $\eta \approx 0{\cdot}6$, for $\nu = 0{\cdot}5$, and $\eta \approx 0{\cdot}4$ for $\nu = 0$. A numerical evaluation of the integral (17) for the three cases of internal anchor load distributions considered earlier gives the following bounds for η. No tensile stresses are developed at the interface for $\eta \geqslant 0{\cdot}52$ when $\nu = 0{\cdot}5$ and for $\eta \geqslant 0{\cdot}23$ when $\nu = 0$. (These results are valid for $\lambda \geqslant 0{\cdot}5$.)

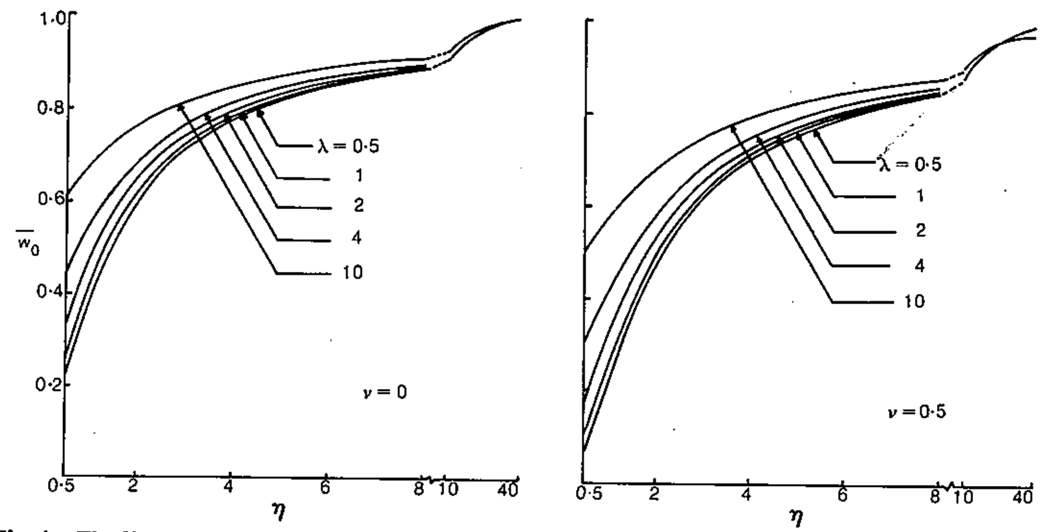

Fig. 4. The displacement of the rigid plate. Anchor load distribution parabolic

CONCLUSIONS

The axisymmetric interaction between a rigid circular foundation and a distribution of internal Mindlin forces is examined within the context of the classical theory of isotropic elasticity. This Paper presents closed-form analytical results for the rigid displacement experienced by the circular foundation under the combined action of an external load P and an internal load ($=P$) distributed uniformly, linearly or parabolically over a finite length. The numerical results presented in Figs 2 to 4 indicate that the length of the anchor region, its depth of location and the distribution of load within the anchor region have a significant influence on the resultant displacement experienced by the rigid circular foundation. The effects appear to be more pronounced when $\eta a < 8a$. When the depth of location of the anchor region ηa exceeds $50a$, neither the anchor load nor its distribution has any appreciable effect on the rigid settlement. This value of ηa would then set a realistic limit for the depth of location of an anchor region used for the purpose of providing the jacking load for a plate load test or for the cable method of in situ testing. Also, the numerical results presented here deal with the case where all the load on the footing is provided by the anchor. The theoretical developments, however, are also applicable to cases where an additional (positive or negative) external load is applied to the foundation. These results can be achieved by adding suitable proportions of a separate Boussinesq's solution. The techniques outlined in this Paper could be further extended to include frictional effects or complete bonding at the foundation interface or to examine the analogous problem related to a transversely isotropic elastic half-space.

ACKNOWLEDGEMENT

The work described in this Paper forms a part of the theoretical study of ground anchors which was supported in part by the National Research Council of Canada. The Author is indebted to the referee for his valuable comments.

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