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A combined finite and infinite element approach for modeling spherically symmetric transient subsurface flow

Wenjun Dong [*,1], A.P.S. Selvadurai

Department of Civil Engineering and Applied Mechanics, McGill University 817 Sherbrooke Street West, Montreal, QC Canada H3A 2K6

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A B S T R A C T

This paper presents a finite element-infinite element coupling approach for modeling a spherically symmetric transient flow problem in a porous medium of infinite extent. A finite element model is used to examine the flow potential distribution in a truncated bounded region close to the spherical cavity. In order to give an appropriate artificial boundary condition at the truncated boundary, a transient infinite element, that is developed to describe transient flow in the exterior unbounded domain, is coupled with the finite element model. The coupling procedure of the finite and infinite elements at their interface is described by means of the boundary integro-differential equation rather than through a matrix approach. Consequently, a Neumann boundary condition can be applied at the truncated boundary to ensure the C^1-continuity of the solution at the truncated boundary. Numerical analyses indicate that the proposed finite element-infinite element coupling approach can generate a correct artificial truncated boundary condition to the finite element model for the unbounded flow transport problem.

1. Introduction

Flow and transport problems in porous media are important topics in the fields of geo-environmental engineering and geosciences. Due to the multi-physical interactions involved in such transport process, most problems of practical interest have to be examined by numerical approaches at this moment (Bear and Bachmat, 1992). Currently, many finite element computational packages, such as COMSOL Multi-physics, are developed and being used for interdisciplinary problems involved in geosciences, due to their flexibility and strength on discretizing complicated physical geometries and material regions. These finite element computational codes can provide the accurate solutions for various bounded domain problems. However, subsurface fluid flow can be considered as the unbounded domain problem, in the sense that the flow potential at significant distances (or theoretically at infinity) should be taken into consideration during the modeling, since the physical nature of the problem in the remote region usually has influences on the behaviour of the solution in the near-field. For the finite element method (FEM) to give accurate solutions to such problems that involve unbounded domains, appropriate artificial boundary conditions should be applied at truncated boundaries (Zhao and Valliappan, 1993; Kaljevic et al., 1992; Khalili et al., 1999; Xia and Zhang, 2006).

A common engineering approach when using FEM for unbounded flow problems is to truncate the far-field boundary at a location remote from the near-field region of interest. But the discretization of the large domain usually requires a great amount of computational resources. On the other hand, for most engineering applications, a relatively small region close to the structure is of interest and the state in the remote region is usually assumed to be as simple as can be described by the

* Corresponding author. Tel.: +1 415 288 2744.
E-mail addresses: wenjun.dong@hotmail.com, wjnd@gerwick.com (W. Dong).
[1] Currently at Ben C. Gerwick, Inc., 20 California Street, Suite 400, San Francisco, CA 94111, USA.

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analytical solution. In these cases, if a boundary integral equation can be developed for the problem in the exterior domain, the boundary element method (BEM) can be coupled with the FEM to give the appropriate boundary condition at the truncated boundary (Zienkiewicz et al., 1977; Brebbia and Dominguez, 1992). However, finding the fundamental solution for the boundary integral equation is often difficult, especially for nonlinear problems and for anisotropic media. Furthermore, the programming effort of coupling boundary elements to finite elements at the truncated boundary limits the applicability of BEM to existing finite element software.

Another approach for extending the FEM to unbounded domain problems is to use an infinite element (Wood, 1976; Bettess and Zienkiewicz, 1977; Zienkiewicz et al., 1983; Selvadurai and Karpurapu, 1989; Bettess, 1992; Khalili et al., 1997), which utilizes a shape function to describe the basic far-field characteristic of the problem in the exterior region. Such an infinite element shape function can be obtained by using a mapping to transform the global infinite region into a local finite domain, in which the standard polynomial interpolation is employed (Bettess, 1977; Damjanic and Owen, 1984; Zienkiewicz et al., 1985; Simoni and Schrefler, 1987). The inverse mapping shows that the dependent variable (for steady state three-dimensional potential problems in particular) in the exterior domain should be expressed in a polynomial of $1/r$, where r represents the distance from a specific point (pole) in the interior domain. This is supported by the fact that the potential problem, as well as many others, has solutions of this type in the remote domain. For these problems, therefore, the infinite element with $1/r^n$-type decaying shape function can generate solutions with any degree of accuracy by choosing a high enough order of polynomial (Lynn and Hadid, 1981). Recently, the infinite element decay shape function has been constructed using the analytical solution of the problem; such transient infinite elements can be used to give rigorous solutions to time-dependent unbounded domain potential problems (Zhao and Valliappan, 1993, 1994; Khalili et al., 1999). However, the conventional finite element-infinite element coupling procedure at the truncated boundary is presented at a matrix level, and this requires programming and recoding efforts that may not be applicable to the existing commercial finite element software. From this point of view, the Dirichlet-to-Neumann (D-t-N) operator (Johnson and Nedelec, 1980) and absorbing boundary condition (Engquist and Majda, 1977), which are widely used for acoustic and wave propagation problems in the unbounded domain, have merit in that the appropriate truncated boundary condition can be expressed in terms of the local differential operator (Ihlenburg, 1998).

In this paper, a finite element-infinite element model is presented for a *transient subsurface flow problem* in a spherically symmetric porous region of infinite extent. The transient nature of such subsurface flow problems is due to the compressibility of the system, and it should therefore be investigated using the piezo-conduction equation. A finite element model is created using the scientific finite element computational software package COMSOL Multiphysics (previously referred to as FEMLAB) to simulate the distribution of the flow potential in a truncated finite domain, and an infinite element is developed to take into consideration the regularity condition of the flow potential in the far field. The coupling procedure of the infinite element to finite elements at the truncated boundary is described by means of an *integro-differential* equation rather than through a matrix approach. Based on these integro-differential equations, either a Dirichlet or a Neumann boundary condition can be imposed at the truncated boundary to couple the infinite element to the finite element model, using the boundary PDE weak form provided in the COMSOL Multiphysics code. The proposed finite element-infinite element models are verified with the analytical solution, and are then used to investigate the transient development of the flow potential distribution at the truncated boundary corresponding to the time-dependent variation of the central cavity boundary condition.

2. The initial boundary value problem

2.1. Governing equations and conditions

The compressibility of either the skeleton of a porous medium and/or compressibility of the pore fluid will result in the development of transient effects in Darcy flow through a porous medium. A complete treatment of transient flow should, however, take into consideration the poroelastic nature of the fluid-saturated medium (Selvadurai, 1996; Lewis and Schrefler, 1998). In a simplified treatment of fluid flow problem that involves both the compressibility of the porous skeleton and the pore fluid, the partial differential equation (PDE) governing the flow potential can be reduced to the classical piezo-conduction equation (Selvadurai, 2002)

$$D\nabla^2\phi = \frac{\partial\phi}{\partial t} \tag{1}$$

where ϕ represents the flow potential, ∇^2 is the Laplace operator and D is the diffusion coefficient of the flow potential, which is defined by

$$D = \frac{k}{\gamma_w(n^* C_f + C_s)} \tag{2}$$

In (2), γ_w is the unit weight of the pore fluid, C_f is the compressibility of the pore fluid, C_s is the compressibility of the soil skeleton, k and n^* are the hydraulic conductivity and porosity of the porous medium, respectively. If the pore fluid and the soil skeleton are considered to be incompressible, the diffusion coefficient D becomes infinity and Eq. (1) reduces to Laplace's equation.

If the flow is restricted to the spherical symmetry, then the flow potential only varies along the spherical radial coordinate R, the piezo-conduction Eq. (1) reduces to the following one-dimensional diffusion equation

$$\frac{D}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right) = \frac{\partial\phi}{\partial t} \tag{3}$$

Such spherically symmetric flow problems can be encountered during deep geological disposal of chemical wastes in pressurized boreholes with enlarged cavities (Selvadurai, 2006) (see Fig. 1). In this case, the flow potential is also governed by the condition applied at the cavity boundary

$$\phi(a,t) = \phi_0 \mathrm{H}(t) \tag{4}$$

as well as the regularity condition applied at infinity:

$$\phi(\infty, t) \to 0 \tag{5}$$

Here a represents the radius of the central spherical cavity, and $\mathrm{H}(t)$ is the Heaviside step function. It is usually assumed that the flow potential is equal to the datum head before the application of the cavity boundary condition, implying the following initial condition for the flow potential

$$\phi(R,0) = 0; \quad R \in [a, \infty) \tag{6}$$

2.2. Weak forms

The numerical solution of the above subsurface fluid flow problem in a bounded region close to the central cavity can be obtained using a finite element model, which is usually based on the weak form of the governing equation. The weak form of the piezo-conduction Eq. (1) can be derived from its weighted residual integral equation

$$\int_\Omega w\left(\frac{\partial \phi}{\partial t} - D(\nabla^2 \phi)\right) \mathrm{d}\Omega = 0 \tag{7}$$

where w represents the arbitrary test function. For the spherically symmetric flow over a semi-infinite region Ω ($a \leqslant R < \infty$; $0 \leqslant \theta \leqslant 2\pi$; $0 \leqslant \varphi \leqslant \pi$), the infinitesimal volumetric element $\mathrm{d}\Omega = R^2 \sin\varphi \,\mathrm{d}R\,\mathrm{d}\theta\,\mathrm{d}\varphi$ (where θ and φ are the azimuth angle and the zenith angle, respectively), and consequently, the weighted residual integral Eq. (7) can be reduced to

$$4\pi\left[\int_a^\infty w\frac{\partial \phi}{\partial t} R^2\,\mathrm{d}R - \int_a^\infty wD\frac{\partial}{\partial R}\left(R^2 \frac{\partial \phi}{\partial R}\right)\mathrm{d}R\right] = 0 \tag{8}$$

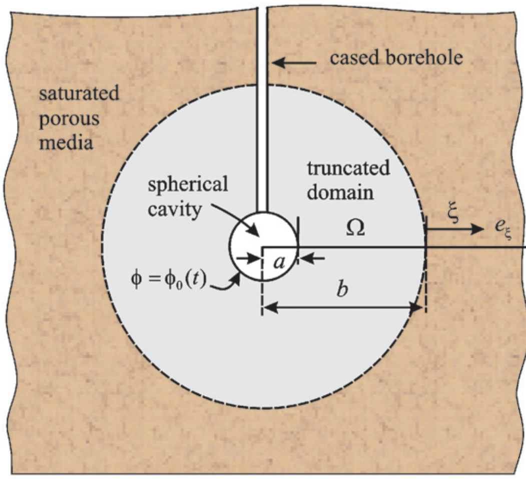

Fig. 1. A schematic drawing of finite computational domain Ω and an infinite element e_ξ for flow transport problem for deep geological disposal of chemical wastes.

The integral Eq. (8) should be satisfied by any test function w, and therefore it can be assumed that the test function w has a homogeneous boundary condition at $R = a$ and satisfies the regularity condition (5) when $R \to \infty$, i.e.

$$w(a) = 0; \; w(\infty) \to 0 \tag{9}$$

Then, applying Green's formula to integrate (8) by parts and considering the regularity conditions (5) and (9) of the flow potential and test function, we obtain a weak form of the one-dimensional diffusion Eq. (3) as follows:

$$\int_a^\infty w\frac{\partial \phi}{\partial t} R^2\,\mathrm{d}R + \int_a^\infty D\frac{\mathrm{d}w}{\mathrm{d}R}\frac{\partial \phi}{R} R^2\,\mathrm{d}R = 0 \tag{10}$$

The advantage of the weak form is that it requires the lower-order derivative of the dependent variable for the solution of the original governing equation.

3. The infinite element for the exterior domain

From the viewpoint of finite element modeling, the computations are carried out in a bounded domain that is discretized into polynomial elements for numerical calculations of the integrals involved in the weak form (10). However, it should be noted that the weak form (10) is derived not only from the governing piezo-conduction equation but also the regularity condition of the flow potential at infinity. This means that if the FEM is used to model the flow problem described by the weak form (10) in a truncated domain, say $a \leqslant R \leqslant b$, an appropriate artificial boundary condition should be applied on the truncated boundary $R = b$ in order to consider the regularity condition of the flow potential at infinity. Due to this consideration, special procedures should be coupled into the finite element model to take into account the influence of the flow potential in the exterior unbounded region on the one at the interior computational domain of interest. Among the approaches developed in the literature, the infinite element can be considered as a complementary element to the finite element modeling. A schematic of the finite truncated domain and an infinite element for the flow transport problem along the spherical coordinate R described in the previous section is shown in Fig. 1. The flow potential in the finite domain Ω is of interest and is determined by the finite element model, while an infinite element is used to describe the flow potential distribution in the exterior unbounded region. At the truncated boundary, an appropriate artificial condition should be applied to couple the finite and infinite elements.

The key issue when constructing an infinite element of the transient flow transport problem in an unbounded region is to choose an appropriate so-called hydraulic potential distribution function that can be used as the shape function to describe the flow potential distribution in the infinite element. A general form of the shape function for the transient infinite element can be derived from the analytical solution of a representative problem (Zhao and Valliappan, 1993; Khalili et al., 1999). The analytical solution of the diffusion Eq. (4) can be obtained in an exact closed form by applying the Laplace transforms (see e.g. Carslaw and Jaeger, 1986). The solution for the initial boundary value problem defined by (3)–(6) is given by

$$\phi(R,t) = \frac{a\phi_0}{R}\,\mathrm{erfc}\left(\frac{R-a}{2\sqrt{Dt}}\right) \tag{11}$$

where erfc(x) is the complimentary error function defined by

$$\mathrm{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-\zeta^2}\,\mathrm{d}\zeta \tag{12}$$

Using the analytical solution (11), the flow potential at the interface between the finite computational domain and infinite element, i.e. $R=b$, can be expressed as follows:

$$\phi(b,t) = \frac{a\phi_0}{b}\,\mathrm{erfc}\left(\frac{\lambda}{\sqrt{t}}\right) \tag{13}$$

where $\lambda = ((b-a)/(2\sqrt{D}))$. Therefore, the flow potential at any point within the infinite element, say $b+\xi$, can be expressed as

$$\phi(b+\xi,t) = \phi(b,t)F(\xi,t) \tag{14}$$

where $\xi \geqslant 0$. In (14), $F(\xi,t)$ is referred to as a mass transfer function of the infinite element, which takes the following form:

$$F(\xi,t) = \frac{1}{\mathrm{erfc}(\lambda/\sqrt{t})}\,\frac{b}{b+\xi}\,\mathrm{erfc}\left(\frac{b+\xi-a}{2\sqrt{Dt}}\right) \tag{15}$$

It is evident from the Eq. (14) that the mass transfer function can be considered as the *shape* function in the infinite element, i.e. $N_\xi = F(\xi,t)$. In the infinite element, such a shape function has the following derivative:

$$\begin{aligned}\frac{\partial N_\xi}{\partial \xi} &= \frac{\partial F(\xi,t)}{\partial \xi}\\ &= \frac{-b}{\mathrm{erfc}(\lambda/\sqrt{t})}\left[\frac{1}{(b+\xi)^2}\,\mathrm{erfc}\left(\frac{b+\xi-a}{2\sqrt{Dt}}\right) + \frac{1}{b+\xi}\,\frac{1}{\sqrt{\pi Dt}}\right.\\ &\quad\left.\times\exp\left(-\left(\frac{b+\xi-a}{2\sqrt{Dt}}\right)^2\right)\right]\end{aligned} \tag{16}$$

Substituting the shape function in (15) and its derivative in (16) to the weak form (10) of the diffusion Eq. (3), we obtain the following the integro-differential equation for the flow potential ϕ_{b^+} at the left hand side of the infinite element, i.e. $R=b^+$,

$$K_{b^+}\phi_{b^+} + M_{b^+}\frac{\partial \phi_{b^+}}{\partial t} = 0 \tag{17}$$

where

$$\begin{aligned}K_{b^+} &= \int_0^\infty D\frac{\partial N_\xi}{\partial \xi}\frac{\partial N_\xi}{\partial \xi}(b+\xi)^2\,\mathrm{d}\xi\\ &= \frac{b^2}{\mathrm{erfc}(\lambda/\sqrt{t})^2}\\ &\quad\times\int_0^\infty D\left[\frac{1}{b+\xi}\,\mathrm{erfc}\left(\frac{b+\xi-a}{2\sqrt{Dt}}\right)\right.\\ &\quad\left.+\frac{1}{\sqrt{\pi Dt}}\exp\left(-\left(\frac{b+\xi-a}{2\sqrt{Dt}}\right)^2\right)\right]\mathrm{d}\xi\end{aligned} \tag{18a}$$

$$M_{b^+} = \int_0^\infty N_\xi N_\xi (b+\xi)^2\,\mathrm{d}\xi = \frac{b^2}{\mathrm{erfc}(\lambda/\sqrt{t})^2}\int_0^\infty \mathrm{erfc}\left(\frac{b+\xi-a}{2\sqrt{Dt}}\right)^2\mathrm{d}\xi \tag{18b}$$

The infinite integrals involved in (18a) and (18b) can be evaluated by the adaptive Simpson's quadrature (Gander and Gautschi, 2000).

4. Finite-infinite element coupling procedures

4.1. Finite element model

The finite element model for the spherically symmetric subsurface flow problem described in Section 2 can be easily constructed using the scientific finite element computational software package COMSOL Multiphysics. Several modes available in COMSOL Multiphysics can be used for this purpose; the convection-diffusion equation (CDE) mode was chosen to create the finite element model for the numerical simulation because of its flexibility on definition of equation coefficients and application of boundary conditions (COMSOL Multiphysics Modeling Guide, 2005a). It is implied from the weak form (10) that the spherically symmetric flow problem can be described by a one-dimensional diffusion equation with spatially varied mass and diffusion coefficients, e.g. R^2 and DR^2.

In addition to the governing PDE, two boundary conditions need to be applied at both ends of the truncated region for computational purposes. The cavity flow potential ϕ_0 can be used as a Dirichlet boundary condition on the left hand side of the finite computational domain $R=a$, i.e. $\phi(a,t)=\phi_0$. However, special attention should be paid to defining the boundary condition on the right hand side of the computational domain, i.e. $R=b$, the interface between the finite element and infinite element. As mentioned previously, the boundary condition at this point should include the influence of the flow potential in the exterior unbounded region on that in the finite computational domain. For this purpose, the infinite element described by the integro-differential Eq. (17) will be coupled to the finite element model that is created by COMSOL Multiphysics; such a finite element-infinite element coupling procedure can be implemented using a Neumann coupling boundary condition at $R=b$, leading to a C^1- continuity of the solution at the interface between the finite element and infinite element. In order to make the concept more clear, a C^0-continuous Dirichlet boundary coupling procedure will be introduced first in the following section.

4.2. C^0-continuous Dirichlet boundary condition

Due to the fact that the truncated boundary $R=b$ is a connection point between the finite computational domain and the exterior unbounded region, the flow potential at $R=b$ should include the contributions of the flow potential from these two regions. As mentioned in the previous sections, the flow potential in the internal bounded region can be determined by the finite element model based on the governing diffusion Eq. (3); the flow potential in the exterior unbounded region can be described by an infinite element using the integro-differential Eq. (17). This implies that the flow potential at $R=b$ should be governed by the following coupling equation:

$$M_{b^+}\frac{\partial\phi_{b^+}}{\partial t}+K_{b^+}\phi_{b^+}+\left(R^2\frac{\partial\phi}{\partial t}-D\frac{\partial}{\partial R}(R^2\frac{\partial\phi}{\partial R})\right)_{R=b^-}=0 \tag{19}$$

where $\phi|_{R=b^-}$ represents the flow potential at the right hand side of the interior computational domain, i.e. $R=b^-$, and ϕ_{b^+} represents the flow potential at the left-hand side of the infinite element, i.e., $R=b^+$. The multiplication of R^2 to the diffusion equation included in the bracket in the left-hand side of (19) is due to the implementation of the spherically symmetrical governing Eq. (3) in terms of the weak form at the truncated boundary. Substituting the value of ϕ at $R=b^-$, obtained from the finite element model, into (19) and solving the resulting equation for ϕ_{b^+}, we obtain a flow potential ϕ_b at $R=b$, which includes the contribution of the flow potentials from both finite and infinite elements. Once this is determined, the flow potential ϕ_b can be used as a Dirichlet boundary condition at the truncated boundary in the finite element model for the determination of the flow potential distribution in the finite computational domain $a\leqslant R\leqslant b$ during the second part of the iteration loop, which will be terminated when the state is stabilized. Fig. 2 gives a schematic illustration of this finite element-infinite element coupling procedure at the boundary $R=b$, which includes the determination of the flow potential in two different geometric dimensions, e.g. both over a line and at a single point. As described previously, the flow potential distribution over the finite radial distance is modeled using the CDE mode. The flow potential on the truncated boundary at point $R=b$ determined by (19) can be modeled with the boundary weak form of the PDE mode embedded in COMSOL Multiphysics (COMSOL Multiphysics Modeling Guide, 2005b).

4.3. C^1-continuous Neumann boundary condition

Applying the Dirichlet condition at the truncated boundary $R=b$ only ensures C^0-continuity of the flow potential at this point. However, the fact that the point $R=b$ is a physically internal boundary within the entire physical domain implies that both the flow potential and its first spatial derivative in the normal direction should be continuous across this point, leading to a set of C^1-continuous conditions as follows:

$$\phi|_{R=b^-}=\phi|_{R=b^+}=\phi_{b^+} \tag{20}$$

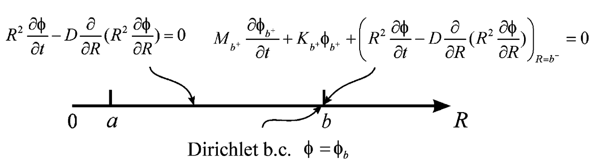

Fig. 2. A schematic illustration of finite-infinite element coupling procedure with a Dirichlet boundary condition at boundary $R=b$.

$$\left.\frac{\partial\phi}{\partial R}\right|_{R=b^-}=\left.\frac{\partial\phi}{\partial R}\right|_{R=b^+}=\frac{\partial\phi_{b^+}}{\partial R}=-\frac{a\phi_0}{b}\left[\frac{1}{b}\operatorname{erfc}\left(\frac{\lambda}{\sqrt{t}}\right)+\frac{1}{\sqrt{\pi Dt}}\exp\left(-\frac{\lambda^2}{t}\right)\right] \tag{21}$$

The expression on the right-hand side of Eq. (21) is obtained from the analytical solution given by (11). Substituting (20) to (19) gives the following integro-differential equation for the flow potential ϕ on the boundary $R=b$

$$(R^2+M_{b^+})\frac{\partial\phi}{\partial t}-D\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right)+K_{b^+}\phi=0 \tag{22}$$

Again, R^2 needs to be multiplied by the corresponding temporal and spatial derivative terms of the dependent variable ϕ in order to implement the spherically symmetric equation using the weak form at the truncated boundary. It is implied from (22) that two terms, represented by $M_{b^+}\partial\phi/\partial t$ and $K_{b^+}\phi$ on the left-hand side of (22), should be added to the original governing diffusion Eq. (3) at the truncated boundary $R=b$ due to the consideration of the contribution from the flow potential in the infinite element. These additions can be easily implemented in COMSOL using its embedded boundary weak form of the CDE mode. Besides, it is implied from (21) that a Neumann boundary condition should be applied at $R=b$ to ensure the C^1-continuity of the flow potential at this point. Fig. 3 gives a schematic illustration of the boundary governing equation and the Neumann boundary condition at $R=b$ of the finite element model in order to couple the infinite element for the exterior domain. It can be seen that compared with the finite and infinite element coupling procedure using C^0-continuous Dirichlet boundary condition described in the previous section, no iteration loop is necessary in the finite and infinite element coupling procedure using C^1-continuous Neumann boundary condition described in the this section since only one flow potential is involved in the linear computation.

5. Numerical computations

As a numerical example, we considered a spherically symmetric flow transport problem in an infinite porous region with a hydraulic conductivity of $k=0.03$ m/day and a porosity of $n^*=0.3$. The porous aquifer material and pore fluid were assumed to be compressible and have compressibilities of $C_s=1.0\times10^{-8}\,\mathrm{m^2/N}$ and $C_f=4.4\times10^{-10}\,\mathrm{m^2/N}$, respectively (Freeze and Cherry, 1979). The flow is initiated by a flow potential $\phi_0=100$ m applied in

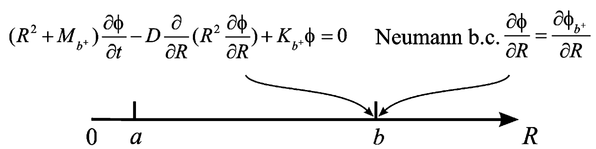

Fig. 3. A schematic illustration of boundary governing equation and Neumann boundary condition at boundary $R=b$ in finite-infinite element model.

the central spherical cavity with a radius of $a = 3\,\mathrm{m}$ and is subject to the regularity condition of the flow potential at the infinity of the porous region. The finite element model created with the CDE mode in COMSOL Multiphysics, as described in the previous section, was used to investigate the flow potential distribution in a truncated finite domain of $a \leqslant R \leqslant b$ with $b = 3\,\mathrm{m}$. Two finite element-infinite element coupling procedures described in Section 4 were used in the finite element model to incorporate the regularity condition of the flow potential at the infinity of the porous region, using either a C^0-continuous Dirichlet or a C^1-continuous Neumann boundary condition at the truncated boundary $R = b$. The quadratic element and default time-dependent solver embedded in COMSOL Multiphysics were used in the finite element modeling. The numerical results of time-dependent flow potential distributions over the finite computational domain $a \leqslant R \leqslant b$ within a 100-day period, obtained from two finite-infinite element models, are shown in Fig. 4(a) and (b), respectively. As a comparison, the corresponding analytical solution over the finite computational domain, obtained from Eq. (11), is shown in Fig. 4(c); it can be seen that the flow potential reaches an almost steady state after 10 days. Fig. 5 shows the comparisons of the numerical results with the analytical solution of the spatial distribution of flow potential over the finite computational domain at $t = 100$ days, as well as their temporal developments at the boundary $R = b$. It can be seen from Fig. 5 that both finite-infinite element models can generate numerical solutions that are close to the analytical solution; The relative numerical errors of two coupling procedures to the analytical solution at the truncated boundary, defined by $(\phi_{num} - \phi_{ana})/\phi_{ana}$, increases from zero at the beginning of the computation to maximum values of 12.4% and 5.6%, respectively, at 7th day when the flow potential starts to approach its final stable stage; and then they drop to 8.7% and 2.4%, respectively, at the end of computation of 100th day. As can be expected, the coupling approach with the Neumann boundary coupling procedure behaves better due to the satisfaction of the C^1-continuity of the solution at the truncated boundary. It can be also expected that these relative errors at the finite and infinite element interface can be decreased when the truncation boundary is located remote from the central cavity.

It should be noted from Fig. 5 that the flow potential at the truncated boundary $R = b$ is time-dependent due to the initial condition and the compressible property of the porous system. Such time-dependence of the flow potential distribution at the truncated boundary can also be induced by the variation in the cavity boundary condition. This connection between the flow potential distribution at the truncated boundary and the cavity boundary condition can be illustrated by the numerical computation of a flow transport with a time-decaying boundary condition applied in the cavity

$$\phi'_0 = \phi_0 \exp\left(-6.0\frac{k}{l}t\right) \tag{23}$$

The finite element-infinite element coupling approach with a C^1-continuous Neumann boundary condition at the truncated boundary is used in the ensuing computations. The corresponding numerical results of the flow potential distribution over the computational domain in a 100-day period are shown in Fig. 6, in which the time-decaying characteristic of the flow potential at the outer boundary $R = b$ can be observed.

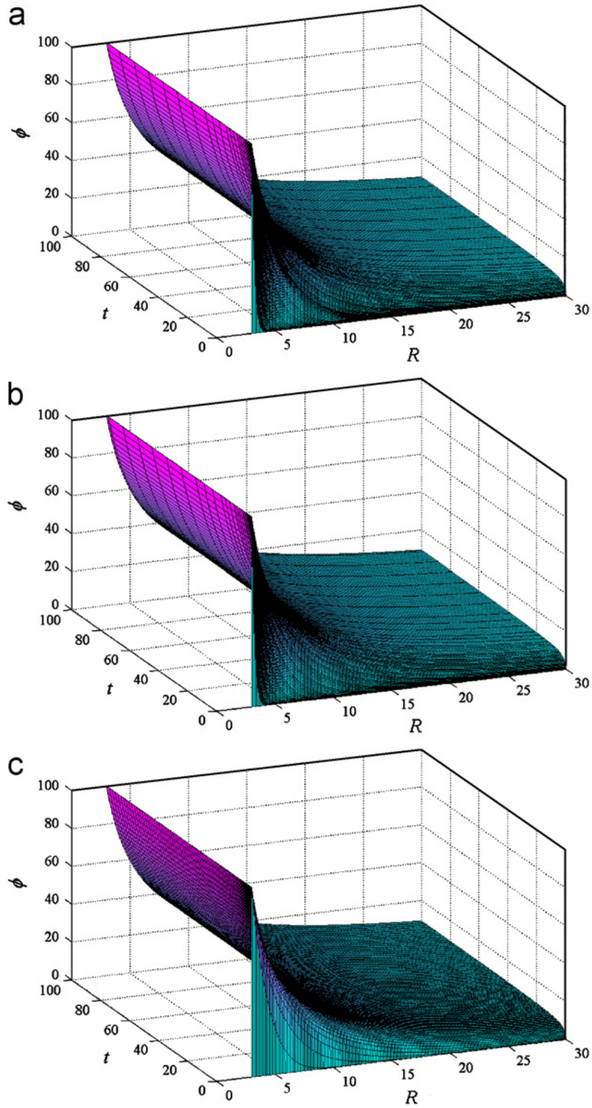

Fig. 4. Time-dependent distributions of flow potential over finite computational domain in a 100-day period, obtained from (a) finite-infinite element model with C^0-continuous Dirichlet coupling b.c., (b) finite-infinite element model with C^1-continuous Neumann coupling b.c., and (c) analytical solution.

Finally, the following cavity boundary condition that includes a discontinuous pulse is considered

$$\phi'_0 = \begin{cases} \phi_0, & 0 \leqslant t < 40 \\ 0, & 40 \leqslant t < 60 \\ \phi_0, & 60 \leqslant t \leqslant 100 \end{cases} \tag{24}$$

It can be seen from the numerical results shown in Fig. 7 that the step variation in the cavity boundary condition of

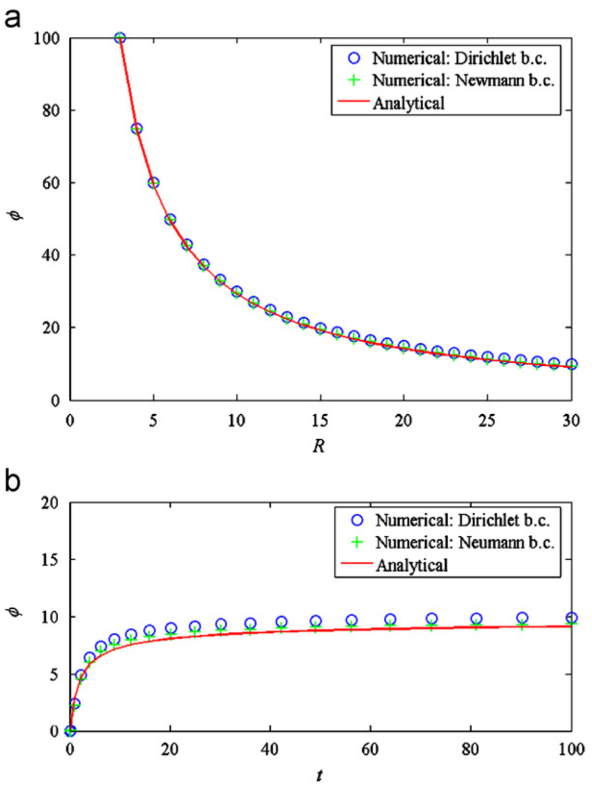

Fig. 5. Comparisons of numerical results with analytical solution of flow potential, (a) spatial distribution over finite computational domain at $t = 100$ days and (b) temporal development at boundary $R = b$.

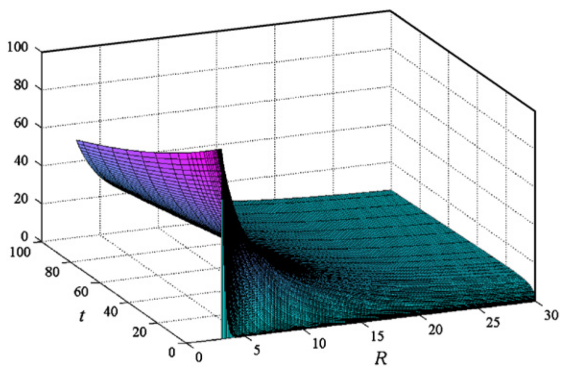

Fig. 6. Numerical results of flow potential distribution over finite computational domain in a 100-day period corresponding to time-decaying cavity boundary condition.

the flow potential has an apparent influence on the flow potential distribution at the truncated boundary of the finite computational domain. The corresponding numerical result of the temporal development of the flow potential at the truncated boundary during a 100-day period is shown in Fig. 8. It is implied from this computation that the flow potential distribution shown in Fig. 8 should be chosen as the appropriate boundary condition at the truncated boundary, so that the finite element model gives the correct solution, in a truncated

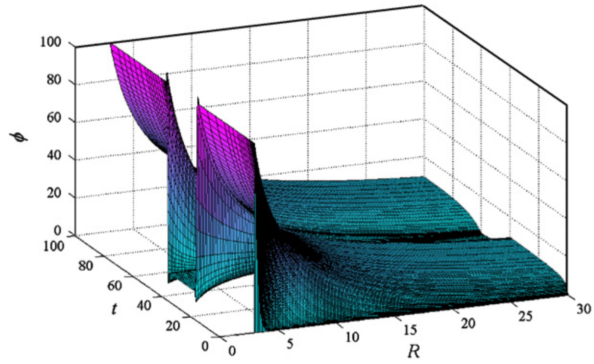

Fig. 7. Numerical results of flow potential distribution over finite computational domain in a 100-day period corresponding to discontinuous cavity boundary condition.

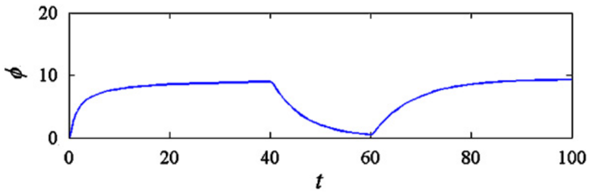

Fig. 8. Numerical result of flow potential distribution at boundary $R = b$ during a 100-day period corresponding to discontinuous cavity boundary condition.

domain, to the transport problem with the discontinuous cavity boundary condition, as given in (24). It can be concluded that using the finite element-infinite element coupling procedures presented in Section 4, the appropriate truncated boundary condition can be incorporated in the finite element model for the unbounded flow transport problem; and such finite element-infinite element coupling at the truncated boundary can be easily implemented using the boundary PDE weak form embedded in COMSOL Multiphysics without a great deal of programming effort.

6. Conclusions

Current concerns of subsurface transport problems in the field of geosciences usually include multi-physics couplings. More and more researchers and engineers are tending to rely on the existing well developed finite element computational package, e.g. COMSOL Multiphysics, for the solutions of the interdisciplinary problems rather than writing their own program code. For the FEM to give accurate solutions to the problem that involves unbounded domain, appropriate artificial boundary conditions should be applied at truncated boundaries, e.g. absorbing b.c. in modeling the acoustic propagation. These artificial truncated boundary conditions can be obtained by coupling infinite element to FEM. However, conventional finite and infinite element coupling procedures are usually implemented through the combination of the property matrix of the infinite element and the stiffness matrix of finite elements at the common nodes. Corresponding programming efforts usually limit its applications to a variety of engineering problems and especially to the well developed commercial finite element computational codes.

Due to this consideration, a general concept of an infinite element for transient flow problems is reviewed based on a flow transport example in a spherically symmetrical porous region. Then, a coupling procedure of finite and infinite elements is described by means of a boundary integro-differential equation instead of at the matrix level, making the coupling procedure clearer. This integro-differential equation finally has to be implemented in coding at the overall matrix formulation, but this task is done by COMSOL Multiphysics using its embedded PDE weak form formulation. In the proposed finite-infinite element coupling procedure, on the other hand, a Neumann boundary condition is also introduced at the truncated boundary to satisfy C^1-continuity of the solution at the interface between the bounded interior domain and unbounded exterior domain. Although the description is based on a spherically symmetrical flow transport problem, the proposed finite and infinite element coupling procedure is general and can be applied to the other unbounded domain problems as long as their shape function (that can be obtained from the analytical solution of the problem) for the exterior region is available.

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