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The Interaction between a Uniformly Loaded Circular Plate and an Isotropic Elastic Halfspace: A Variational Approach

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ABSTRACT

The axially symmetric flexural interaction of a uniformly loaded circular plate resting in smooth contact with an isotropic elastic halfspace is examined by using an energy method. In this development the deflected shape of the plate is represented in the form of a power series expansion which satisfies the kinematic constraints of the plate deformation. The flexural behavior of the plate is described by the classical Poisson-Kirchhoff thin plate theory. Using the energy formulation, analytical solutions are obtained for the maximum deflection, the relative deflection, and the maximum flexural moment in the circular plate. The results derived from the energy method are compared with equivalent results derived from numerical techniques. The solution based on the energy method yields accurate results for a wide range of relative rigidities of practical interest.

I. INTRODUCTION

The flexural behavior of finite plates resting on deformable elastic media is of interest to several branches of engineering. Solutions to such problems are of particular importance in the analysis and design of structural foundations resting on soil and rock media [1]. The classical problem relating to the axisymmetric flexure of a circular plate resting on an isotropic elastic halfspace was examined by Borowicka [2] by using a power series expansion technique. This analytical procedure was subsequently modified by Ishkova [3] and Brown [4] who incorporated the effect of a singular term in the approximation of the contact stress distribution. Other numerical methods, such as discretization procedures, finite difference techniques, and finite element methods, have also found useful application in the treatment of the circular plate problem (see, e.g., Refs. 5–9). A complete account of the axisymmetric interaction problem related to the elastic halfspace is given by Selvadurai [1].

The present paper examines the application of an energy method for the analysis of axisymmetric flexure of a uniformly loaded circular plate resting in smooth contact with an isotropic elastic halfspace. The analysis assumes that the deflected shape of the circular plate can be represented in the form of a power series in terms of the radial coordinate r. This deflected shape is specified to within a set of undetermined constants and satisfies the kinematic constraints of the axisymmetric deformation. The assumption of continuous contact between the elastic plate and the isotropic elastic halfspace ensures that the deflected shape of the plate is compatible with the surface deflections of the halfspace within the plate region. The energy method of analysis requires the development of the total potential energy functional for the plate–elastic halfspace system which consists of (1) the strain energy of the halfspace region, (2) the strain energy of the circular plate, and (3) the work component of the external loads. The strain energy of the isotropic elastic halfspace can be developed by computing the work component of the surface tractions which compose the contact stresses at the plate–elastic medium interface. The contact stresses associated with the imposed displacement field can be determined by making use of the integral equation methods developed for mixed boundary value problems in classical theory of elasticity (see, e.g., Refs. 10–12). The strain energy of the plate region is composed of the flexural and membrane energies corresponding to the prescribed deflected shape. The total potential energy functional thus developed is defined in terms of the constants characterizing the deflected shape of the plate. We may, however, eliminate *two* of these constants by invoking the Kirchhoff boundary conditions applicable to the free edge of the plate. The remaining constants are uniquely determined from the linearly independent algebraic equations generated from the minimization of the total potential energy functional.

The general procedure outlined above is used to analyze the flexural interaction of a uniformly loaded circular foundation, with a free edge, resting on an isotropic elastic halfspace. The deflected shape of the plate is represented by an even order polynomial in r up to the sixth-order. This particular deflected shape is assumed to represent, approximately, the flexural behavior of a moderately flexible circular foundation. Using the energy method, analytical solutions are derived for the central deflection, the differential deflection, and the central flexural moment. Also, Boussinesq's [13] solution for the problem of a rigid circular plate resting on an isotropic elastic halfspace occurs as a limiting case of the energy solution (i.e., the relative rigidity R, defined by Eq. (20), tends to infinity). Similarly as $R \to 0$, the energy solutions for the central deflection and the differential deflection compare favorably with the corresponding solutions developed for the problem related to a uniformly loaded circular area. A comparison with an existing solution [4] indicates that the accuracy of the energy approximation is greatly improved at the range of relative rigidities 0 to ∞. Numerical results presented in this paper illustrate the manner in which the central deflection, the differential deflection, and the central moment of the uniformly loaded circular plate resting in frictionless contact with an isotropic elastic halfspace are influenced by the relative rigidity of the plate–elastic medium system.

II. ANALYSIS

We examine the axisymmetric interaction of a thin elastic plate resting in smooth contact with an isotropic elastic halfspace. The thin plate, of thickness h and radius a, is subjected to a uniform load of stress intensity p_0 over its entire surface area (Fig. 1). Due to the axisymmetric interaction, displacements are induced at the plate–elastic halfspace interface. We assume that there is no loss of contact at the interface. As such, the surface displacements of the contact region $r < a$ in the z-direction also represent the deflected shape of the circular plate, $w(r)$. An expression for the total potential energy functional appropriate to the elastic plate–elastic halfspace system can be developed by making use of the function $w(r)$.

When considering the small deflection Poisson-Kirchhoff thin plate theory, the elastic strain energy of the circular plate subjected to the axisymmetric

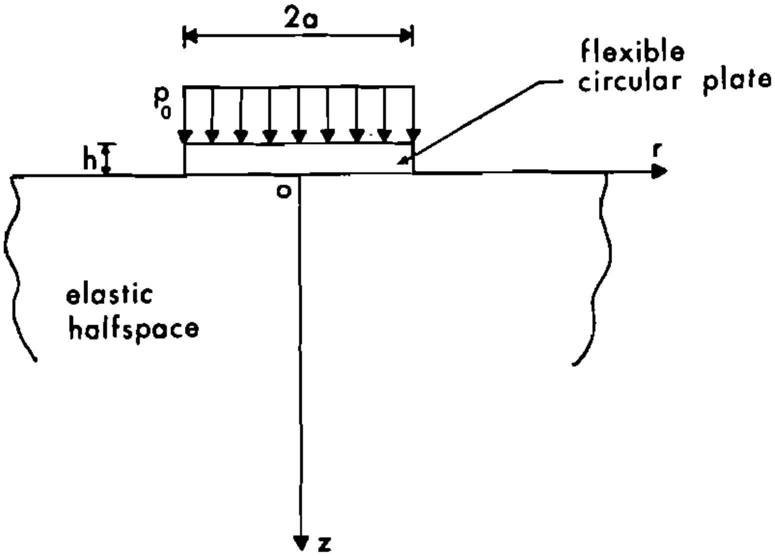

Fig. 1 Uniformly loaded circular plate on an isotropic elastic halfspace.

deflection $w(r)$ is composed of only the flexural energy of the plate U_F, given by

$$U_F = \frac{D}{2}\int\int_S \left[\{\nabla^2 w(r)\}^2 - \frac{2(1-\nu_b)}{r}\frac{dw(r)}{dr}\frac{d^2w(r)}{dr^2}\right] r\,dr\,d\theta \tag{1}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}; \qquad D = \frac{E_b h^3}{12(1-\nu_b^2)}$$

and E_b and ν_b are, respectively, the elastic modulus and Poisson's ratio for the plate material, and S corresponds to the plate region.

The second component of the total potential energy functional corresponds to the elastic strain energy U_E of the isotropic elastic halfspace which is subjected to the displacement field $w(r)$ in the region $r < a$. The elastic strain energy can be developed by computing the work component of the surface tractions which compose the interface contact stresses. Since the interface is assumed to be smooth, only the normal surface tractions contribute to the strain energy. These normal tractions can be uniquely determined by making use of the integral equation methods developed by Sneddon [10, 11], Green [14], and Green and Zerna [15] for the analysis of mixed boundary value problems in classical elasticity theory. We consider the problem of an isotropic elastic halfspace which is subjected to the axisymmetric displacement field

$$u_z = w(r) \text{ for } z = 0, \qquad 0 \leqslant r \leqslant a \tag{2}$$

where u_z is the component of the displacement vector in the z-direction. The surface of the halfspace is subjected to the traction boundary conditions

$$\begin{aligned} \sigma_{zz} &= 0 \text{ on } z = 0; \qquad a \leqslant r < \infty \\ \sigma_{rz} &= 0 \text{ on } z = 0; \qquad 0 \leqslant r < \infty \end{aligned} \tag{3}$$

where σ_{zz} and σ_{rz} are the normal and shear stress components of the Cauchy stress tensor σ_{ij} referred to the cylindrical polar coordinate system (r, θ, z). By employing the integral equation formulation it can be shown that the contact stress at the interface is given by

$$\sigma_{zz}(r, 0) = \frac{G_s}{(1 - \nu_s)r} \frac{d}{dr} \int_a^r \frac{tg(t)\, dt}{\sqrt{t^2 - r^2}} \qquad (\text{for } 0 \leqslant r \leqslant a) \tag{4}$$

where

$$g(t) = \frac{2}{\pi} \frac{d}{dt} \int_0^t \frac{rw(r)\, dr}{\sqrt{t^2 - r^2}} \qquad (\text{for } 0 \leqslant t \leqslant a) \tag{5}$$

and G_s and ν_s are, respectively, the linear elastic shear modulus and Poisson's ratio of the elastic medium. From the above results, the elastic strain energy of the isotropic elastic halfspace is given by

$$U_E = \frac{G_s a^3}{\pi(1 - \nu_s)} \iint_S \frac{w(r)}{a} \left[\frac{d}{dr} \int_a^r \frac{t}{\sqrt{t^2 - r^2}} \left\{ \frac{d}{dt} \int_0^t \frac{rw(r)\, dr}{a^2\sqrt{t^2 - r^2}} \right\} dt \right] dr\, d\theta \tag{6}$$

The total energy of the uniform external load (p_0) applied to the plate is given by

$$U_P = -p_0 \iint_S w(r) r\, dr\, d\theta \tag{7}$$

By combining (1), (6), and (7) we obtain the following expression for the total potential energy functional:

$$\begin{aligned} U = \iint_S \Bigg[& \frac{Dr}{2} \left(\{\nabla^2 w(r)\}^2 - \frac{2(1 - \nu_b)}{r} \frac{dw(r)}{dr} \frac{d^2 w(r)}{dr^2} \right) \\ & + \frac{G_s a^3}{\pi(1 - \nu_s)} \left(\frac{w(r)}{a} \frac{d}{dr} \int_a^r \frac{t}{\sqrt{t^2 - r^2}} \left\{ \frac{d}{dt} \int_0^t \frac{rw(r)\, dr}{a^2\sqrt{t^2 - r^2}} \right\} dt \right) \\ & - rp_0 w(r) \Bigg] dr\, d\theta \end{aligned} \tag{8}$$

For the total potential energy functional to satisfy the principle of stationary potential energy, we require

$$\delta U = 0 \tag{9}$$

where δU is the variation in the total potential energy. Alternatively we note that when the prescribed deflected shape $w(r)$ satisfies the condition $\delta U = 0$ (for all $\delta w(r)$), it can be shown that $w(r)$ is the solution of the associated elasticity problem. In order to apply the principle of total potential energy to the circular foundation, we assume that the deflected shape $w(r)$ can be represented in the form

$$w(r) = a \sum_{i=0}^{n} C_i \phi_i(r) \tag{10}$$

where C_i are arbitrary constants and $\phi_i(r)$ are arbitrary functions which render the plate deflections kinematically admissible. Of the $n + 1$ arbitrary constants, two can be eliminated by invoking the Kirchhoff boundary conditions applicable for the free edge of the circular plate, i.e.,

$$\begin{aligned} M_r(a) &= -D\left[\frac{d^2 w(r)}{dr^2} + \frac{\nu_b}{r}\frac{dw(r)}{dr}\right]_{r=a} = 0 \\ Q_r(a) &= D\left[\frac{d}{dr}\{\nabla^2 w(r)\}\right]_{r=a} = 0 \end{aligned} \tag{11}$$

The total potential energy functional can now be represented in terms of $(n - 1)$ arbitrary constants. The principle of total potential energy requires that U be an extremum with respect to the kinematically admissible deflection field characterized by C_i (see, e.g., Refs. 16–18). Hence

$$\frac{\partial U}{\partial C_i} = 0 \qquad (i = 0, 1, 2, \ldots, n - 1) \tag{12}$$

The above minimization procedure yields $(n - 1)$ linear equations for the undetermined constants C_i $(i = 0, 1, 2, \ldots, n - 1)$.

III. THE UNIFORMLY LOADED CIRCULAR PLATE RESTING ON AN ISOTROPIC ELASTIC HALFSPACE

To apply the formal procedure developed in the preceding section to the analysis of the circular plate problem, it is assumed that the deflected shape of the plate can be approximated by the power series

$$w(r) = a \sum_{i=0}^{3} C_{2i} \left(\frac{r}{a}\right)^{2i} \tag{13}$$

where C_{2i} are arbitrary constants. In (13), the particular choice of functions corresponding to $\phi_i(r)$ give a kinematically admissible deflection and finite flexural moments and shearing forces in the plate region $0 \leqslant r \leqslant a$. By invoking the free edge boundary conditions (11), this assumed expression for the plate deflection can be reduced to the form

$$w(r) = a\left[C_0 + C_2\left\{\frac{r^2}{a^2} + \lambda_1 \frac{r^4}{a^4} + \lambda_2 \frac{r^6}{a^6}\right\}\right] \tag{14}$$

where

$$\lambda_1 = \frac{-3(1 + \nu_b)}{4(2 + \nu_b)}; \qquad \lambda_2 = \frac{(1 + \nu_b)}{6(2 + \nu_b)}$$

The contact stress distribution corresponding to the imposed displacement field (14) can be determined by making use of the relationships (4) and (5); we have

$$\begin{aligned}\sigma_{zz}(r, 0) = \frac{G_s a}{\pi(1 - \nu_s)\sqrt{a^2 - r^2}}\bigg[2C_0 + C_2\bigg\{&-4 - \frac{16}{9}\lambda_1 - \frac{32}{25}\lambda_2 \\ &+ \frac{r^2}{a^2}\left(8 - \frac{64}{9}\lambda_1 - \frac{64}{25}\lambda_2\right) \\ &+ \frac{r^4}{a^4}\left(\frac{128}{9}\lambda_1 - \frac{256}{25}\lambda_2\right) + \frac{r^6}{a^6}\left(\frac{512}{25}\lambda_2\right)\bigg\}\bigg]\end{aligned} \tag{15}$$

Similarly, by making use of $w(r)$ as defined by (14), the total potential energy functional (8) reduces to the form

$$U = \frac{2G_s a^3}{(1 - \nu_s)}[C_0^2 + C_0 C_2 \chi_1 + C_2^2 \chi_2] + \pi D C_2^2 \chi_3 - \pi p_0 a^3 [C_0 + \chi_4 C_2] \tag{16}$$

where the constants χ_n $(n = 1, 2, 3, 4)$ are given by

$$\chi_1 = \left\{\xi_0 + \frac{2}{3}(1 + \xi_2) + \frac{8}{15}(\lambda_1 + \xi_4) + \frac{16}{35}(\lambda_2 + \xi_6)\right\}$$

$$\chi_2 = \left\{\frac{2}{3}\xi_0 + \frac{8}{15}(\lambda_1\xi_0 + \xi_2) + \frac{16}{35}(\lambda_2\xi_0 + \lambda_1\xi_2 + \xi_4) + \frac{128}{315}(\xi_2\lambda_2 + \xi_4\lambda_1 + \xi_6) + \frac{256}{693}(\xi_4\lambda_2 + \xi_6\lambda_1) + \frac{1024}{3003}(\xi_6\lambda_2)\right\}$$

$$\chi_3 = \left\{8 + 32\lambda_1 + \frac{144}{3}\lambda_2 + \frac{128}{3}\lambda_1^2 + \frac{1188}{5}\lambda_2^2 + 144\lambda_1\lambda_2 - (1 - \nu_b)(4 + 16\lambda_1 + 24\lambda_2 + 16\lambda_1^2 + 36\lambda_2^2 + 48\lambda_1\lambda_2)\right\}$$

$$\chi_4 = \frac{1}{2} + \frac{\lambda_1}{3} + \frac{\lambda_2}{4} \tag{17a}$$

and

$$\xi_0 = -2 - \frac{8}{9}\lambda_1 - \frac{16}{25}\lambda_2; \qquad \xi_4 = \frac{64}{9}\lambda_1 - \frac{128}{25}\lambda_2$$

$$\xi_2 = 4 - \frac{32}{9}\lambda_1 - \frac{32}{25}\lambda_2; \qquad \xi_6 = \frac{256}{25}\lambda_2 \tag{17b}$$

The constants C_0 and C_2 can be determined from the equations which are generated from the minimization conditions,

$$\frac{\partial U}{\partial C_0} = 0; \qquad \frac{\partial U}{\partial C_2} = 0 \tag{18}$$

The deflected shape of the uniformly loaded circular foundation corresponding to (13) is given by

$$w(r) = \frac{\pi a p_0(1 - \nu_s)}{2G_s\{\chi_1^2 - 4\chi_2 - 2R\chi_3\}}\left[\chi_1\chi_4 - 2\chi_2 - R\chi_3 + (\chi_1 - 2\chi_4)\left\{\frac{r^2}{a^2} + \lambda_1\frac{r^4}{a^4} + \lambda_2\frac{r^6}{a^6}\right\}\right] \tag{19}$$

where R is a relative rigidity parameter of the circular plate–elastic halfspace system defined by

$$R = \frac{\pi(1 - \nu_s^2)E_b}{6(1 - \nu_b^2)E_s}\left(\frac{h}{a}\right)^3 = \frac{\pi K_r}{6(1 - \nu_b^2)} \tag{20}$$

and K_r is a similar parameter defined by Brown [4]. The relative rigidity parameter R (or K_r) can be used to investigate the accuracy and the limits of applicability of the energy approximation for the plate deflection, (19).

A. Limiting Cases

We note that as $R \to \infty$, the circular foundation becomes infinitely rigid; as such, the displacement (w_0) of the uniformly loaded rigid circular plate resting in smooth contact with an isotropic elastic halfspace is given by

$$w_0 = \frac{\pi p_0 a(1 - \nu_s^2)}{2E_s} \tag{21}$$

Result (21) is in agreement with the classical solution obtained by Boussinesq [13] and Harding and Sneddon [19] for the rigid displacement of a circular punch resting on an isotropic elastic halfspace derived by considering results of potential theory and integral equation methods, respectively.

As $R \to 0$, the problem reduces to that of the axisymmetric loading of an isotropic elastic halfspace by a uniform flexible load. The two results of particular importance in geotechnical engineering are the maximum surface deflection $\{w(0)\}$ and the differential deflection $\{w(0) - w(a)\}$ within the uniformly loaded area. Considering the energy method, expression (19) yields

$$\{w(0)\}_{\text{energy}} = \frac{p_0 a(1 - \nu_s^2)}{E_s}\{2.09\} \tag{22a}$$

The exact result corresponding to the central surface displacement of the uniformly loaded area is given by (see, e.g., Ref. 20)

$$\{w(0)\}_{\text{exact}} = \frac{p_0 a(1 - \nu_s^2)}{E_s}\{2.00\} \tag{22b}$$

The energy estimate for the central deflection of the uniformly loaded circular area overpredicts the exact solution by approximately 4.5%.

The result for the differential deflection obtained from the energy method is

$$\{w(0) - w(a)\}_{\text{energy}} = \frac{p_0 a(1 - \nu_s^2)}{E_s}\{0.728\} \tag{23a}$$

The corresponding exact solution for the uniformly loaded circular area is

$$\{w(0) - w(a)\}_{\text{exact}} = \frac{p_0 a(1 - \nu_s^2)}{E_s}\left\{\frac{2\pi - 4}{\pi}\right\} \tag{23b}$$

In this case the energy estimate for the differential deflection of the uniformly loaded area overpredicts the exact solution by approximately 0.3%.

B. Flexural Moments in the Circular Plate

The flexural moments induced in the uniformly loaded circular foundation due to its interaction with the isotropic elastic halfspace can, in principle, be calculated by making use of the expression for the foundation deflection (19) and the relationships

$$M_r = -D\left\{\frac{d^2w(r)}{dr^2} + \frac{\nu_b}{r}\frac{dw(r)}{dr}\right\}$$

$$M_\theta = -D\left\{\frac{1}{r}\frac{dw(r)}{dr} + \nu_b\frac{d^2w(r)}{dr^2}\right\} \tag{24}$$

Although the energy method provides an accurate estimate of the deflections of the plate, the accuracy with which $w(r)$ is able to predict the flexural moments in the foundation is, in general, considerably less (see, e.g., Refs. 21 and 22). Any inaccuracies that may be present in the energy estimate for $w(r)$ are greatly magnified in the computation of M_r and M_θ owing to the presence of derivatives of $w(r)$ up to the second order. It can be shown that the flexural moments in the circular foundation as determined from (19) and (24) are somewhat lower than those predicted by Brown [4] using the numerical method involving a power series technique. A more accurate estimate of the flexural moments in the circular foundation can be obtained by considering the flexural response of the circular plate under the combined action of the uniform external load and the contact stresses $\sigma_{zz}(r, 0)$. The maximum flexural moment $[M_r(0) = M_\theta(0)]$ at the center of the circular foundation can be computed by using the solutions developed for the flexure of a circular plate *simply supported along its boundary*.

(a) The maximum flexural moment at the center of an edge-supported plate due to the external load p_0 is given by

$$M_p = p_0 a^2 \frac{(3 + \nu_b)}{16} \tag{25a}$$

(b) The maximum flexural moment due to the contact stress $\sigma_{zz}(r, 0)$ acting on an edge-supported plate is given by

$$M_c = \int_0^a \frac{\zeta}{2}\sigma_{zz}(\zeta, 0)\left\{\frac{(1 - \nu_b)}{2}\frac{(a^2 - \zeta^2)}{a^2} - (1 + \nu_b)\ln\left(\frac{\zeta}{a}\right)\right\}d\zeta \tag{25b}$$

where $\sigma_{zz}(\zeta, 0)$ is defined by (15). The expression (25b) is, in part, composed of integrals which take the form

$$I_n = \int_0^1 \frac{x^n \ln x}{\sqrt{1 - x^2}}\, dx \qquad (n = 1, 3, 5, \ldots) \tag{26}$$

For the evaluation of (26), the following recurrence relationships have been developed:

$$nI_n = (n - 1)I_{n-2} + \frac{1, 2, 4, \ldots (n - 3)}{1, 3, 5, \ldots n} \qquad (n > 3) \tag{27}$$

and

$$I_1 = (\ln 2 - 1), \qquad 3I_3 = 2I_1 + \frac{1}{3}$$

Alternatively, I_n can be expressed in the form of the double series

$$I_n = -\left[\frac{1}{(n + 1)^2} + \sum_{i=1}^{\infty} \left\{\sum_{j=1}^{i} \left(\frac{2j - 1}{2j}\right)\right\} \frac{1}{(2i + n + 1)^2}\right] \tag{28}$$

Since the flexural moments derived from (25a) and (25b) are referred to the same total load, their combined result will eliminate the free edge support reaction. Using these results we obtain the following expression for the central flexural moment (M_0) in the uniformly loaded plate resting on an isotropic elastic halfspace:

$$M_0 = p_0 a^2 \left\{\frac{(3 + \nu_b)}{16} - \frac{(1 - \nu_b)}{4} m_1 + \frac{(1 + \nu_b)}{2} m_2\right\} \tag{29}$$

where

$$m_1 = \frac{c_0^*}{3} + c_2^* \left\{\frac{\xi_0}{3} + \frac{2}{15}\xi_2 + \frac{8}{105}\xi_4 + \frac{16}{315}\xi_6\right\}$$

$$m_2 = \left\{c_0^* + c_2^*\left(\xi_0 + \frac{2}{3}\xi_2 + \frac{8}{15}\xi_4 + \frac{16}{35}\xi_6\right)\right\} \ln 2$$
$$- \left\{c_0^* + c_2^*\left(\xi_0 + \frac{5}{9}\xi_2 + \frac{94}{225}\xi_4 + \frac{1276}{3675}\xi_6\right)\right\} \tag{30a}$$

and

$$\{c_0^*; c_2^*\} = \frac{1}{[\chi_1^2 - 4\chi_2 - 2R\chi_3]}\{(\chi_1\chi_4 - 2\chi_2 - R\chi_3); (\chi_1 - 2\chi_4)\} \tag{30b}$$

IV. CONCLUSIONS

The classical problem relating to the axisymmetric flexure of a uniformly loaded circular plate resting in smooth contact with an isotropic elastic halfspace is examined by using an energy method. The energy method proposed here utilizes an even order power series representation, up to the sixth order in r, to approximate the deflected shape of the flexible plate. This power series is specified to within a set of arbitrary constants. These constants are evaluated by making use of the free edge Kirchhoff boundary conditions and from the minimization of the total potential energy functional. Using such a technique, analytical results can be developed for the deflection of the circular

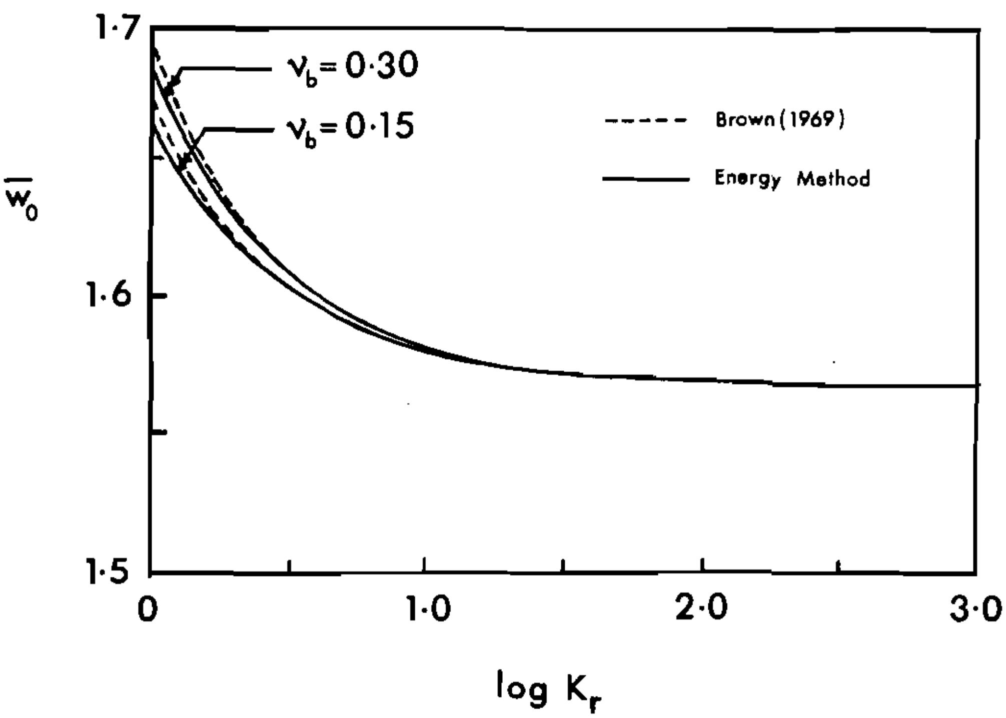

Fig. 2 The variation of central displacement of the uniformly loaded circular plate:

$$w(0) = \frac{ap_0(1 - \nu_s^2)\bar{w}_0}{E_s}$$

plate and the contact stresses at the plate–elastic medium interface. The latter result is employed in the calculation of the maximum flexural moment in the circular plate. The energy result for the plate deflection converges to the exact result when the relative rigidity of the plate–elastic medium system (K_r) tends to infinity. Also, when the relative rigidity K_r tends to zero, the energy result yields total and differential surface displacements for the loaded area of the halfspace which compare very accurately with existing exact solutions.

In Figs. 2 to 4, the results derived from the energy method for the central deflection, the differential deflection, and the central flexural moment in the circular plate are compared with equivalent results derived from the power series method [4]. The results of the energy method compare very accurately with the exact solutions for relative rigidities, K_r, in the range 0 to ∞. [The practical applicability of this range of relative rigidities can be illustrated in the following manner. Consider a uniformly loaded circular reinforced con-

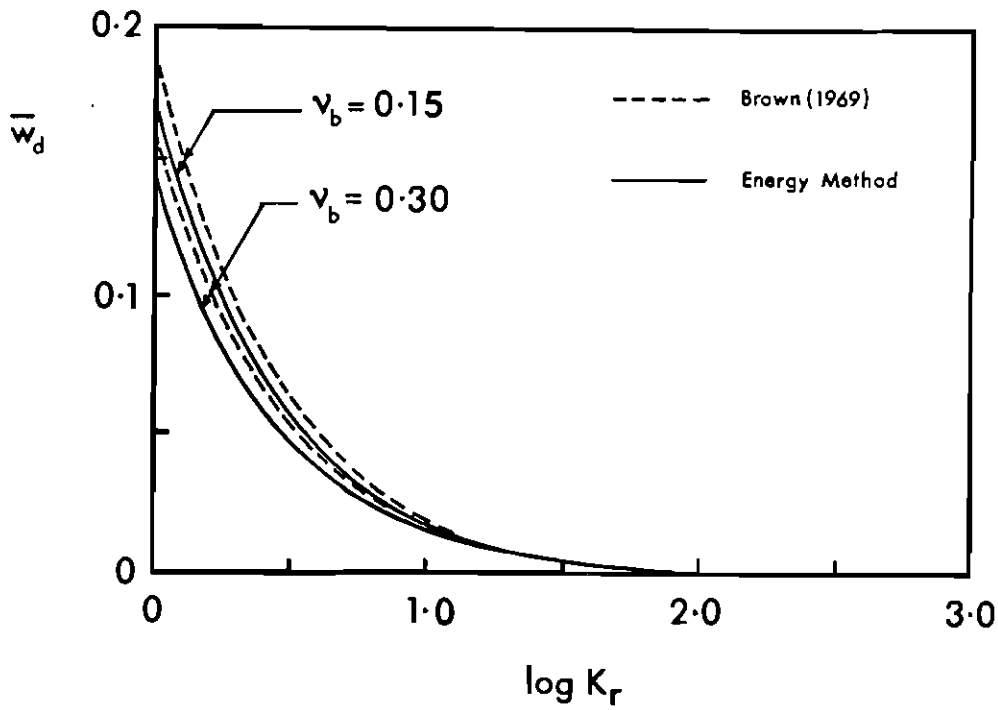

Fig. 3 . The variation of the differential deflection of the uniformly loaded circular plate:

$$w(0) - w(a) = \frac{ap_0(1 - \nu_s^2)\bar{w}_d}{E_s}$$

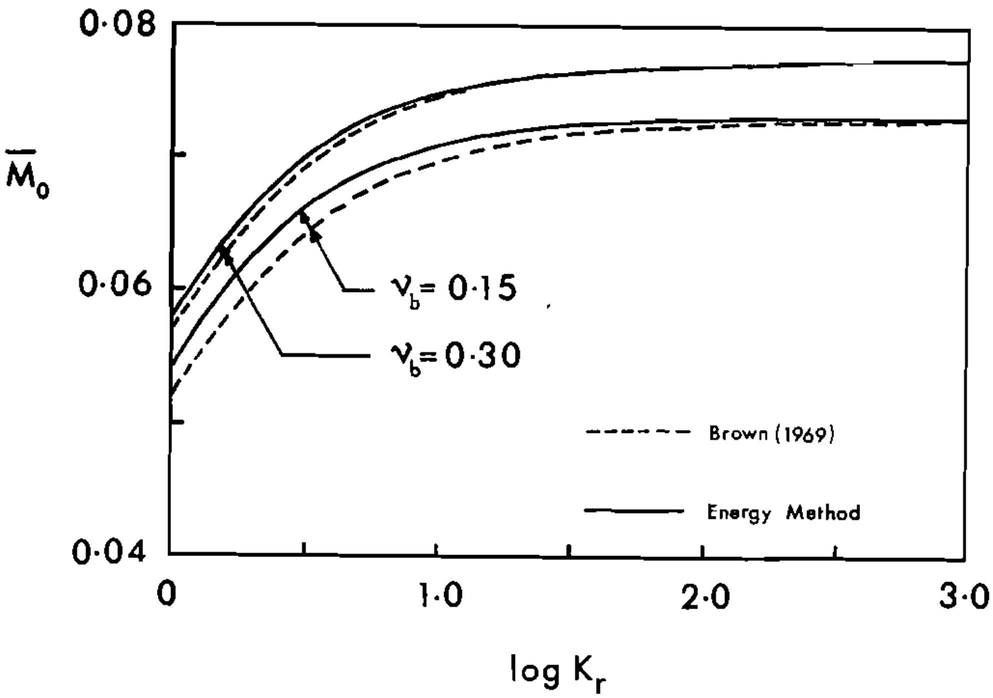

Fig. 4 The variation of the central flexural moment in the uniformly loaded circular plate: $M_r(0) = M_\theta(0) = p_0 a^2 \bar{M}_0$

crete raft (diameter 10 m; thickness 1 m; $E_b = 1.5 \times 10^7$ kN/m^2; $\nu_b = 1/3$) which is founded on a saturated cohesive soil medium ($\nu_s = 1/2$). In the case of a soft clayey soil with $E_s = 4 \times 10^3$ kN/m^2, we have $K_r = 22.5$. Similarly for a stiff clayey soil with $E_s = 8 \times 10^4$ kN/m^2, we have $K_r = 1.125$.]

The energy method described in this paper can be effectively employed to provide approximate solutions to interaction problems where exact or numerical solution is either mathematically complex or intractable. Interaction problems which involve external loads of nonuniform variation or of limited extent, edge boundary conditions of complete or partial fixity, etc., can be examined by using the general approach outlined in this paper. The accuracy of the energy method depends on the choice of functions that are used to represent the deflected shape of the plate. For example, the assumed form of the deflected shape (13) is clearly inappropriate for situations involving localized loading of highly flexible ($K_r \to 0$) circular plates. In this instance, higher order and logarithmic terms in r have to be incorporated in the assumed deflection functions to accurately model the flexural moments and deflections in the plate. Alternatively, the flexural response of a highly flexible locally loaded circular plate can be examined by analyzing the appropriate problem related to the infinite plate resting on an isotropic elastic halfspace (see, e.g., Refs. 1, 23, and 24).

REFERENCES

1. A. P. S. Selvadurai, *Elastic Analysis of Soil-Foundation Interaction* (Developments in Geotechnical Engineering Series), Elsevier, New York, 1979.
2. H. Borowicka, Influence of rigidity of a circular foundation slab on the distribution of pressure over the contact surface, in *Proceedings of the 1st International Conference on Soil Mechanics and Foundation Engineering*, Vol. 2, 1936, pp. 144–149.
3. A. G. Ishkova, Bending of a circular plate on the elastic halfspace under the action of a uniformly distributed axisymmetrical load, *Uch. Zap. Mosk. Gos. Univ.* 3(152): (1951) (in Russian).
4. P. T. Brown, Numerical analyses of uniformly loaded circular rafts on deep elastic foundations, *Geotechnique* **19**: 399–404 (1969).
5. L. Barden, Contact pressures under circular slabs, *Struct. Eng.* **43**: 153–154 (1965).
6. I. M. Smith, A finite element approach to elastic soil-structure interaction, *Can. Geotech. J.* **7**: 95–105 (1970).
7. J. A. Hooper, Elastic settlement of a circular raft in adhesive contact with a transversely isotropic medium, *Geotechnique* **25**: 691–711 (1975).
8. J. A. Hooper, Parabolic adhesive loading of a flexible raft foundation, *Geotechnique* **26**: 511–525 (1976).
9. A. K. Chakravorty and A. Ghosh, Finite difference solution for circular plates on elastic foundations, *Int. J. Numerical Methods Eng.* **9**: 73–84 (1975).
10. I. N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York, 1951.
11. I. N. Sneddon, The relation between load and penetration in the axisymmetric Boussinesq problem for a punch of arbitrary profile, *Int. J. Eng. Sci.* **3**: 47–57 (1965).
12. B. Noble and D. A. Spence, *Formulation of Two-Dimensional and Axisymmetric Problems for an Elastic Halfspace* (Technical Report No. 1089), Mathematics Research Center, University of Wisconsin, 1971.
13. J. Boussinesq, *Application des potentials a l'etude de l'equilibre et du movement des solides elastique*, Gauthier-Villars, Paris, 1885.
14. A. E. Green, On Boussinesq's problem and penny-shaped cracks, *Proc. Cambridge Philos. Soc.* **45**: 251–257 (1949).
15. A. E. Green and W. Zerna, *Theoretical Elasticity*, 2nd ed., Oxford University Press, London, 1968.
16. I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, 2nd ed., McGraw-Hill, New York, 1956.
17. S. G. Mikhlin, *Variational Methods in Mathematical Physics* (translated from Russian by T. Boddington), Pergamon, New York, 1964.
18. K. Washizu, *Variational Methods in Elasticity and Plasticity*, 2nd ed., Pergamon, Oxford, 1975.

19. J. W. Harding and I. N. Sneddon, The elastic stresses produced by the indentation of the plane surface of a semi-infinite elastic solid by a rigid punch, *Proc. Cambridge Philos. Soc.* **41**: 16–26 (1945).

20. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 3rd ed., McGraw-Hill, New York, 1970.

21. S. P. Timoshenko and S. Woinowsky-Krieger, *Theory of Plates and Shells* (Engineering Society Monographs), 2nd ed., McGraw-Hill, New York, 1959.

22. C. Dym and I. H. Shames, *Solid Mechanics: A Variational Approach* (Advanced Engineering Series), McGraw-Hill, New York, 1973.

23. K. S. Pister and R. A. Westmann, Bending of plates on an elastic foundation, *J. Appl. Mech.* (*Trans. A.S.M.E.*) **29**(Ser. E, 2): 369–374 (1962).

24. A. P. S. Selvadurai, The influence of a Mindlin force on the axisymmetric interaction between an infinite plate and an elastic halfspace, *Lett. Appl. Eng. Sci.* **5**: 379–388 (1977).

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