The Analytical Method in Geomechanics[1]

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This article presents an overview of the application of analytical methods in the theories of elasticity, poroelasticity, flow, and transport in porous media and plasticity to the solution of boundary value problems and initial boundary value problems of interest to geomechanics. The paper demonstrates the role of the analytical method in geomechanics in providing useful results that have practical importance, pedagogic value, and serve as benchmarking tools for calibrating computational methodologies that are ultimately used for solving more complex practical problems in geomechanics. There are 315 references cited in this article. [DOI: 10.1115/1.2730845]

The development of classical soil mechanics and geomechanics owes a great deal to the availability and utilization of theoretical results in elasticity, plasticity, poroelasticity, and flow and transport in porous media. As the discipline evolves to encompass either new areas of application or the use of new theories of geomaterial behavior, the natural tendency is to resort to computational treatments of the complex problems. This is inevitable, but at the same time opportunities do exist for the development of analytical approaches to the solution of problems in geomechanics; these would be of considerable help to geotechnical engineers, particularly in preliminary assessments of the performance of problems in geotechnical engineering.

1 Introduction

The conventional definition of the term "analysis" refers to the resolution or separation of a problem or a task into its elements. In this sense all engineering endeavors are examples of analysis. In the context of problem solving in engineering, the term *analytical solutions* takes on a different meaning, one that specifically refers to the use of advanced mathematical procedures for the solution of problems. This raises the question: Is any solution scheme that uses mathematical procedures also an analytical solution? Therefore, are computational schemes that use mathematical techniques also providing analytical solutions? In the strict definition of the term analysis, they are indeed examples of analytical solutions, but conventional use of the term "an analytical solution" in past and recent times has become synonymous with solution schemes that rely on the exhaustive application of mathematical procedures for the development of a solution to a problem in engineering. It is the preoccupation with the latter that distinguishes a very elegant computational procedure from a purely analytical solution. Soil mechanics and geomechanics are excellent examples of disciplines that have extensively used analytical solutions to full advantage, not only to provide the foundations of the subject but also to develop a set of usable solutions that, to this day, continues to benefit many areas of application in geotechnical engineering. The historical treatises by Coulomb [1], Rankine [2], the volumes written after the coining of the term *soil mechanics*, by Terzaghi [3,4], Krynine [5], Taylor [6], Florin [7], Tschebotarioff [8], Caquot and Kerisel [9], and the more recent treatises by many authors including Sokolovskii [10], Leonards [11], Nadai [12], Scott [13,14], Harr [15], Suklje [16], Tsytovich [17], and Bell [18], illustrate the range and depth of the use of analytical methods for the solution of conventional problems in geomechanics and foundation engineering. They deal with diverse areas including stress distributions in soils, settlement analysis, stress states around embedded structures, stability and failure of soils, flow in porous media, and consolidation and creep of soils.

The analytical method has always played an important role not only as a component of the educational enterprise in geomechanics but also as a tool for the development of concise results of practical value for preliminary design calculations [19–25]. This latter aspect is particularly important to geotechnical engineering since, in most instances, preliminary designs are carried out with only a limited knowledge of the range of values associated with geotechnical material parameters.

This review aims to outline some seminal classical treatments and recent developments in the application of the analytical method, in particular to the study of problems of interest to geomechanics. The research, largely the purview of geomechanics half a century ago, now extends to a number of other disciplines, including mathematics, physics, materials science, earth physics, geophysics, particulate media, solid mechanics, biomechanics, chemical engineering science, etc. The literature covering these areas is extensive and it is difficult to adequately review and document all the available analytical developments within the limits of the present paper. Therefore this review will focus on a limited number of topics of interest to the general theme of the analytical approach and present a discussion of problems that may be of potential interest to the geomechanics community. Due to limitations of space, attention will be restricted to isothermal quasi-static problems. There is a wealth of research dealing with the dynamics, thermo-mechanics, and hydro-thermo-mechanics of the types of problems discussed in this paper. Discussions of other important methods that revolve around the semi-analytical techniques also require a fuller treatment, which cannot be achieved within the context of the present article. These are available in the leading journals and symposia devoted to geomechanics, solid mechanics, computational mechanics, materials science, applied mathematics, and applied mechanics.

2 Elasticity and Geomechanics

In their recent volume, Davis and Selvadurai [26] refer to elasticity as the "*glue*" that holds the governing equations. This is not an understatement, particularly when one looks at the typical civil engineering curriculum at the undergraduate level, which includes a large collection of subjects in mechanics of solids, structural mechanics, geomechanics, and advanced stress analysis that rely on principles that are deeply rooted in the *classical theory of elasticity*. The influences of nonlinear approaches are certainly being introduced into curricula, but not at the expense of the exclusion of the classical approaches. On occasions, particularly in the context of geomechanics, the linear theory has been referred to as a *children's model*; this is perhaps through ignorance and a lack of appreciation of the history of the subject, its content, and its impact on the engineering sciences: The *children* who were instru-

[1]Dedicated to Professor A. J. M. Spencer FRS, on the occasion of his 75th Birthday.

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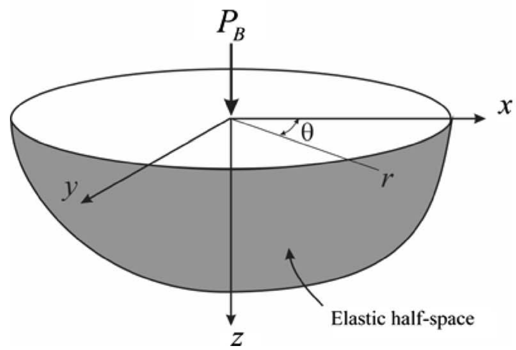

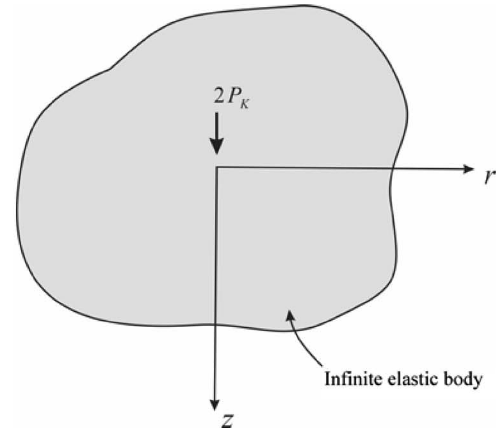

Fig. 1 Boussinesq's and Kelvin's problems

mental in developing the theory of elasticity include Euler, James Bernoulli, Hooke, Young, Lagrange, Poisson, Navier, Cauchy, Lamé, Clapeyron, Saint-Venant, Kirchhoff, Mohr, Green, Kelvin, Maxwell, Stokes, Rayleigh, Boussinesq, Hertz, Michell, Morera, Betti, Beltrami, Somigliana, Cerruti, Castigliano, Neuber, Papkovich, Galerkin, Airy, Lamb, Love, Filon, Southwell, Timoshenko, Inglis, Mushkhelishvili, Mindlin, Fichera and many others (see e.g., Todhunter and Pearson [27,28], Timoshenko [29], Goodier [30], Truesdell [31,32], Teodorescu [33], Volterra and Gaines [34], Gurtin [35], Szabo [36], and Selvadurai [37,38], Meleshko and Selvadurai [39]). Linear elasticity is regarded as one of the more successful theories of mathematical physics. Quite apart from its utility, the study of elasticity should be viewed as part of the classical education process, intended to develop the role of problem formulation, modelling, and analysis in order to bring problem solving to a successful conclusion.

2.1 Boussinesq's Problem. Every geotechnical engineer, both practitioner and student, has had the occasion to use Boussinesq's classical problem dealing with the action of a normal force at the surface of a traction-free isotropic elastic halfspace region (Fig. 1). Determination of the state of stress in an isotropic elastic halfspace, subjected to a concentrated force acting normal to a traction-free surface, was first considered by Boussinesq [40]. This problem can be solved via several approaches: the first consists of reducing the problem to a boundary value problem in potential theory. When the surface of the halfspace is subjected to only normal tractions, the elasticity problem is reduced to that of finding a single harmonic function with all the characteristic features of a single layer potential distributed over the plane region with intensity proportional to the applied normal tractions. The solution to the concentrated force problem is thus recovered as a special case of the general normal loading. The second approach, commences with the solution for the point force acting at the interior of an infinite space developed by Lord Kelvin [41] and utilizes a distribution of combinations of dipoles, which are equivalent to a distribution of centers of compression along an axis, to eliminate the shear tractions occurring on the plane normal to the line of action of the Kelvin force, thereby recovering Boussinesq's solution. A third approach involves the application of integral transform techniques to the solution of a governing partial differential equation (e.g., for Love's strain function) which can then be used to explicitly satisfy the traction boundary conditions applicable directly to Boussinesq's problem [42–44]. These procedures are well documented in classical treatises and papers by Michell [45], Love [46], Westergaard [47], Sokolnikoff [48], Lur'e [49], and Timoshenko and Goodier [50]. Alternative approaches to and interpretations of the Boussinesq, Flamant, and Cerruti problems are also presented by Podio-Guidugli [51].

While these approaches represent remarkably insightful procedures for obtaining a solution to Boussinesq's problem, there is a more direct approach. Selvadurai [52] recognized the advantages of formulating the problem in spherical coordinates and utilizes the properties of Lamé's strain potential $\varphi(R,\Theta)$ and Love's strain potential $\Phi(R,\Theta)$ that, respectively, satisfy

$$\nabla^2\varphi(R,\Theta)=0; \quad \nabla^2\nabla^2\Phi(R,\Theta)=0 \tag{1}$$

where (R,Θ) are the spherical coordinates and

$$\nabla^2=\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}+\frac{\cot\Theta}{R^2}\frac{\partial}{\partial\Theta}+\frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2} \tag{2}$$

These functions can then be used directly to determine the stress and displacement fields. For example, from the Lamé's strain potential

$$2Gu_R=\varphi_{,R}; \quad 2Gu_\Theta=R^{-1}\varphi_{,\Theta}; \quad \sigma_{RR}=\varphi_{,RR}; \quad \sigma_{R\Theta}=(R^{-1}\varphi)_{,R\Theta}; \quad \text{etc.} \tag{3}$$

Similar expressions can be obtained from $\Phi(R,\Theta)$. We can start with Kelvin's problem for a concentrated force of magnitude P_K acting at the interior of the elastic infinite space, since, in all of these concentrated force problems, there is no natural length scale in the problem. The requirement of solutions for $\varphi(R,\Theta)$ and $\Phi(R,\Theta)$ is that, through appropriate differentiations, the functions must yield expressions that are of the order $1/R^2$. Furthermore, they must also be applicable to an infinite space and give rise to stresses and displacements that decay to zero. For the solution of Kelvin's problem, the natural reaction is to select a spherically symmetric exterior solution for $\varphi(R,\Theta)$, which has a form C/R, where C is a constant. It can be shown that while the regularity condition is satisfied, the dimensional requirement is violated by the exterior solution. Considering Laplace's operator defined by Eq. (2) it is evident that the function CR is biharmonic. Alternatively, if C/R is harmonic then $CR(=R^2C/R)$ is biharmonic. Using the spherically symmetric Love's strain potential $\Phi(R)=CR$ will satisfy both requirements. The transition from Kelvin's solution to Boussinesq's solution involves only satisfying a traction-free constraint on the plane surface $z=0$ or $\Theta=\pi/2$. Kelvin's solution gives the following stresses on the plane $z=0$

$$\sigma_{\Theta\Theta}(R,\pi/2)=0; \quad \sigma_{R\Theta}(R,\pi/2)=P_K R^{-2}(1-\nu)(1-2\nu)\sin\Theta \tag{4}$$

where ν is Poisson's ratio. If the material was incompressible, then the problem is solved. We need to find solutions of either $\varphi(R,\Theta)$ or $\Phi(R,\Theta)$ that will enable us to satisfy the traction-free surface constraint applicable to Boussinesq's problem. We have already used the exterior solution to Lamé's strain potential to construct Kelvin's solution; therefore we seek a solution of $\varphi(R,\Theta)$ such that when the function is differentiated twice with respect to R, the resulting solution should be of the order $1/R^2$ with the form

$$\varphi(R,\Theta) = A\ln[Rf(\Theta)] \tag{5}$$

Substituting this in the first equation of Eq. (1) we obtain an ordinary differential equation (ODE)

$$\frac{d}{d\Theta}\left\{\frac{\sin\Theta}{f}\frac{df}{d\Theta}\right\} + \sin\Theta = 0 \tag{6}$$

The natural tendency is to carry out the differentiation, which will give rise to a nonlinear ODE; if one resists the temptation, and performs successive integrations, Eq. (6) yields

$$f(\Theta) = \exp\left\{\int_0^\Theta \left(\frac{\cos\zeta - 1}{\sin\zeta}\right)d\zeta\right\} = (1+\cos\Theta) \tag{7}$$

This result, in conjunction with Eq. (5) and the appropriate expressions for the stresses in terms of Love's strain function, gives

$$\sigma_{R\Theta}(R,\Theta) = \frac{A\sin\Theta}{[R^2(1+\cos\Theta)]} \tag{8}$$

The constant A can now be adjusted to eliminate the shear traction on the plane $\Theta=\pi/2$. The resulting solution for Boussinesq's problem is of the form

$$2G[u_R;u_\Theta] = \frac{P_B}{2\pi R}\left[\{4(1-\nu)\cos\Theta - (1-2\nu)\}; \left\{-(3-4\nu) + \frac{(1-2\nu)}{(1+\cos\Theta)}\right\}\right] \tag{9}$$

$$\sigma_{RR} = \frac{P_B}{2\pi R^2}[1-2\nu-2(2-\nu)\cos\Theta] \tag{10}$$

$$[\sigma_{\Theta\Theta};\sigma_{\Psi\Psi};\sigma_{R\Theta}] = \frac{P_B(1-2\nu)}{2\pi R^2(1+\cos\Theta)} \times [\{\cos^2\Theta\};\{\cos\Theta-\sin^2\Theta\};\{\sin\Theta\cos\Theta\}] \tag{11}$$

In Eq. (11), $\sigma_{\Psi\Psi}$ is the stress component in the azimuthal direction Ψ. Does this type of approach work for other categories of Boussinesq-type problems associated with material anisotropy and inhomogeneity? This same approach can be applied to determine the solution to Cerruti's problem for a halfspace that is subjected to a concentrated force acting tangential to the traction-free surface [53]. The approach is certainly applicable to the case of a halfspace that is transversely isotropic, although the algebra involved will be unwieldy and amenable only to the solution via symbolic manipulation methods. The formulation in terms of the spherical coordinates will work for halfspaces that are inhomogeneous, provided the form of the inhomogeneity lends itself to manageable governing equations. An example of such an application is due to Calladine and Greenwood [54], who examined Boussinesq's problem related to a Gibson soil, which refers to an incompressible medium with a linearly varying linear elastic shear modulus. With more general forms of inhomogeneities, the analytical approach outlined here becomes restrictive.

2.2 Elastic Nonhomogeneity. Elastic nonhomogeneity has always been a topic of interest to theoreticians devising solutions that could be applied to the study of load transfer during contact between elastic bodies. The nonhomogeneity was generally attributed to an alteration of the elastic properties of the body due to mechanical working and, on occasion, to chemical action. An extensive record of the literature on the subject of elastic nonhomogeneity is given in the works of Olszak [55], Goodman [56], Gibson [19], de Pater and Kalker [57], Selvadurai [22,37,58], Selvadurai et al. [59], Gladwell [60], and Aleynikov [61]. The most popular approach for considering elastic nonhomogeneity was to consider axial variations of the shear modulus of the form

$$G(z) = G_s\exp(\xi z) \tag{12}$$

where G_s is the surface shear modulus and ξ is a non-negative parameter, with the understanding that the halfspace region of interest occupied $r\in(0,\infty)$ and $z\in(0,\infty)$. The obvious reasons for the choice of the variation Eq. (12) is prompted by the considerable simplifications that arise in the governing equations through consideration of the exponential variation, which is filtered out from the governing equations only to leave the constant ξ as a remnant of the influence of the nonhomogeneity. The resurgence in the application of the concepts of elastic nonhomogeneity to problems in geomechanics is largely due to the paper by Gibson [62], who examined the traction boundary value problem for an incompressible isotropic elastic halfspace region, the shear modulus of which varied linearly with depth. The justification for the choice of the linear variation was prompted as a result of experimental evidence related to measurement of elastic properties of London clay reported by Skempton and Henkel [63] (see also, Ward et al. [64], Burland and Lord [65], Butler [66], and Hobbs [67]). One of the major observations of Gibson's analysis was that when the surface shear modulus approached zero, the surface displacement of the halfspace exhibited a discontinuous profile reminiscent of the *Winkler foundation* consisting of a set of independent spring elements. Consider the variation in the shear modulus with depth, defined by

$$G(z) = G_s + mz \tag{13}$$

and m is a constant. If the surface S of a halfspace, with $G_s\equiv 0$, is subjected to a normal surface traction over the region S_L, with the unloaded region defined by S_U, i.e.

$$\sigma_{zz}(x,y,0) = \begin{cases} f(x,y); & (x,y)\in S_L \\ 0; & (x,y)\in S_U \end{cases} \tag{14}$$

then the corresponding discontinuous axial surface displacement can be obtained from the result

$$u_z(x,y,0) = \frac{1}{2m}\begin{cases} f(x,y); & (x,y)\in S_L \\ 0; & (x,y)\in S_U \end{cases} \tag{15}$$

This is a remarkable analytical result, which could be revealed only through the complete mathematical formulation and analysis of the problem. It provided, once and for all, a clear continuum representation for the *Winkler medium*, which consists of a collection independent linearly deformable spring elements [22,68]. The resulting nonhomogeneous elastic continuum is now referred to as the "*Gibson Soil*." A computational solution to this problem, even fully accommodating for a variational principle to account for incompressible elastic behavior and infinite elements to account for the spatial extent, would have given only a hint of the form of the result but would not have given rise to the proof of the result Eq. (15). The elastic nonhomogeneity of the Gibson type has been extended to other types of elastic materials including transversely isotropic media, and several problems of interest to geomechanics are discussed by Gibson and co-workers (Gibson et al. [69], Brown and Gibson [70], Awojobi and Gibson [71], Awojobi [72], Gibson [19], Gibson and Kalsi [73], and Gibson and Sills [74]). One observation made in these studies is that the special discontinuous surface displacement is a consequence of the semi-infinite nature of the nonhomogeneous medium and the zero value of the surface shear modulus. The arguments also extend to orthotropic nonhomogeneous media. The arguments do not extend to either compressible media or incompressible media of finite depth [74]. The action of line, point, circular, and strip loads acting on nonhomogeneous Gibson-type halfspace regions are discussed by

Booker et al. [75,76]. Anchor and foundation problems related to nonhomogeneous media were examined by Rowe and Booker [77,78]. The problem related to the torsion of foundations and anchor plates embedded in Gibson-type media are given by Rajapakse and Selvadurai [79,80].

Holl [81], Popov [82] and Rostovtsev [83] were some of the earliest researchers to consider the class of problems where the elastic nonhomogeneity varies as a power law function of the axial coordinate (i.e., $E(z)=m_E z^{\alpha}$, where m_E and α are constants). The solutions obtained by Holl [81] are more specialized in the sense that the value of Poisson's ratio is restricted to $\nu=1/(\alpha+2)$, which, in the case of a Gibson soil, is limited to a specific Poisson's ratio, $\nu=1/3$. Booker et al. [75,76] developed explicit solutions to line loads, flexible and rigid circular loaded areas, and rectangular areas of flexible loads for arbitrary valued Poisson's ratio. These authors also confirm the validity of the Gibson soil and its equivalence to the Winkler foundation. A more general form of the power law variation was considered by Plevako [84]; i.e., $E(z)=E_0(1+kz)^b$, where E_0, k, and b are constants, and further references to studies in this area are given by Singh et al. [85]. The work of Stark and Booker [86,87] follows these studies and outlines computational procedures and gives extensive results for problems involving loading of halfspace regions with power law nonhomogeneities by irregular-shaped loads. In this sense, these works represent an efficient use of a fundamental solution. Rajapakse [88] presented the solution for interior loading of a Gibson soil, and Yue et al. [89] outline procedures for determining the stress state in a stratified soil using a transfer matrix approach. Similar investigations are given by Bufler [90]. Reasonably complete records of the literature in this area are given by Gladwell [60] and Selvadurai [58,91].

The assumption of either the exponential or the linear variation in $G(z)$ has to contend with unbounded values of the linear elastic shear modulus as $z\rightarrow\infty$. This is a limitation of the analytical developments, since the linear elastic shear modulus in geomaterials is rarely unbounded. An alternative is to consider a variation that gives bounded axial variations of $G(z)$ for $z\in(0,\infty)$. An example of such a variation was given by Selvadurai et al. [59], in which

$$G(z)=G_{\infty}+(G_s-G_{\infty})\exp(-\bar{\zeta}z/a) \tag{16}$$

where G_{∞} is the finite value of the linear elastic shear modulus as $z\rightarrow\infty$ and $\bar{\zeta}$ is a non-negative constant. Selvadurai et al. [59] use this variation of $G(z)$ to examine the Reissner–Sagoci problem, which relates to the torsional rotation of a rigid circular disk that is bonded to the surface of the halfspace region. Since the state of deformation induced by the torsional indentation of an elastic medium with axial variation of the linear elastic shear modulus is one that involves only shear stresses, the problem formulation does not require consideration of any variations in Poisson's ratio. Selvadurai [58] examined the axisymmetric indentation of a halfspace where the shear modulus varies according to Eq. (16) and Poisson's ratio is constant. The halfspace indented by a rigid circular foundation with a smooth flat base is shown in Fig. 2.

The analysis of the mixed boundary value problem can be reduced to a Fredholm integral equation of the second kind for a single unknown function $\phi(\alpha)$

$$\phi(\alpha)+\frac{2}{\pi}\int_0^{\alpha}\phi(s)\,ds\int_0^{\infty}K(\eta)\cos(\alpha\eta)\cos(\alpha s)d\eta=1;\qquad 0\leqslant\alpha\leqslant a \tag{17}$$

where $K(\eta)$ is a Kernel function that depends on the solution of a pair of simultaneous ODEs that contain the effects of the nonhomogeneity. These ODEs can be solved either analytically or numerically. The solution of the integral equation involves the reduction of the problem to a matrix equation for values of $\phi(\alpha)$ at discrete points. In this sense the *analytical method* also involves a reasonable amount of *computational effort*. The result of practical importance concerns the influence of the nonhomogeneity on the settlement (Δ) of the circular foundation of radius a under the action of the axial load P. This can be evaluated in the form

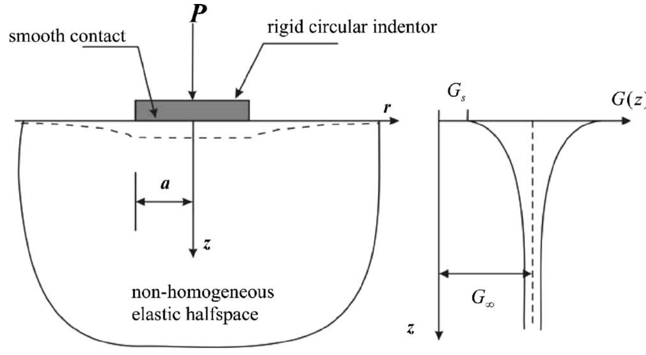

Fig. 2 Indentation of a nonhomogeneous elastic halfspace

$$\bar{P}=\frac{P(1-\nu)}{4\Delta G_{\infty}a}=\frac{2G_s(1-\upsilon)^2}{G_{\infty}(1-2\nu)}\int_0^a\phi(\alpha)d\alpha \tag{18}$$

and typical results are shown in Fig. 3.

In a recent paper, Vrettos [92] applied the variation of shear modulus proposed by Selvadurai et al. [59] to examine the associated Boussinesq's problem. In recent years, a number of authors [93,94] have extended the topic of elastic nonhomogeneity to include cross-anisotropic effects. While these are useful analytical efforts, the problems related to the experimental determination of at least five elastic parameters and their variation with depth will continue to be a formidable exercise in both laboratory and in situ investigations. The practical utility of analytical approaches that cannot fulfill the obligations concerning parameter identification are usually viewed with some skepticism.

A more basic question concerning the combination of anisotropy and nonhomogeneity in elasticity relates to a situation that is often encountered in the treatment of geomaterials formed through periodic deposition, such as varved clays and sedimentary rocks. Depending upon the choice of scale of the representative elemental volume, the material can display characteristics that can be described either by elastic nonhomogeneity or transverse isotropy. The periodic variation in the shear modulus of an isotropic elastic material was considered by Selvadurai and Lan [95], who examined the mixed boundary value problem associated with the surface indentation of a halfspace where the shear modulus varied according to

$$G(z)=G_1+G_2\cos(2\pi z/h) \tag{19}$$

where h is a length parameter in the problem, and the moduli G_1 and G_2 are chosen such that the thermodynamic constraint for a positive definite strain energy function is satisfied pointwise. Again, the analysis of the mixed boundary value problem for the indentation of the halfspace (Fig. 4) is reduced to a Fredholm integral equation of the second kind and a numerical procedure can be used to determine the load (P) versus settlement (Δ) for the rigid circular indentor. Other approaches to the formulation of elastostatic contact problems and embedded anchor problems dealing with isotropic and inhomogeneous elastic media are given by Selvadurai [22,37], Gladwell [60], and Aleynikov [61].

An entirely novel approach to the study of the axial elastic nonhomogeneity was proposed by Spencer and co-workers (Spencer [96], Rogers [97], and Abid Mian and Spencer [98]). Here, the formulation of the problem makes use of exact solutions to the laminate problem associated with composite materials. The exact closed form solution for the layer can be presented in terms of appropriate biharmonic and harmonic functions. The interlaminate

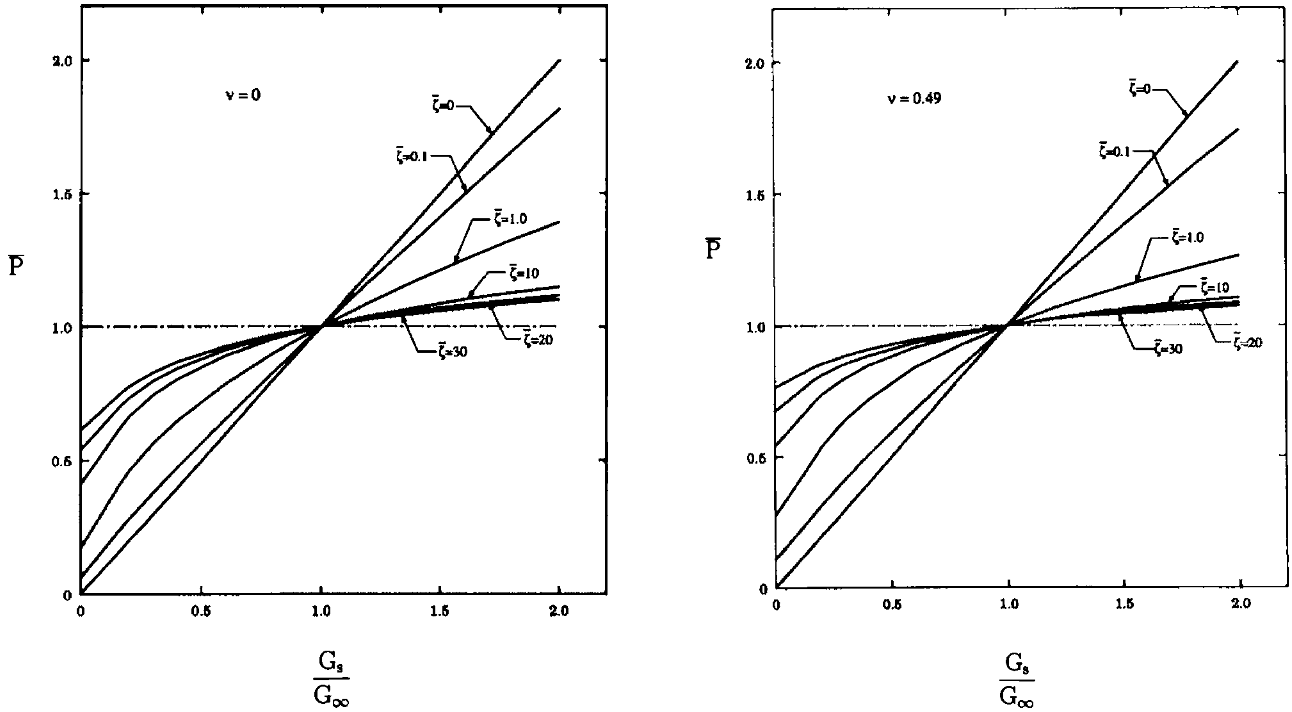

Fig. 3 Settlement of a rigid circular footing on a nonhomogeneous isotropic elastic halfspace. ($\bar{P}$ is the nondimensional load defined by Eq. (18); $\bar{\zeta}$ defines the variation in the inhomogeneity; the modular ratio G_s/G_∞ is defined by Eq. (16).)

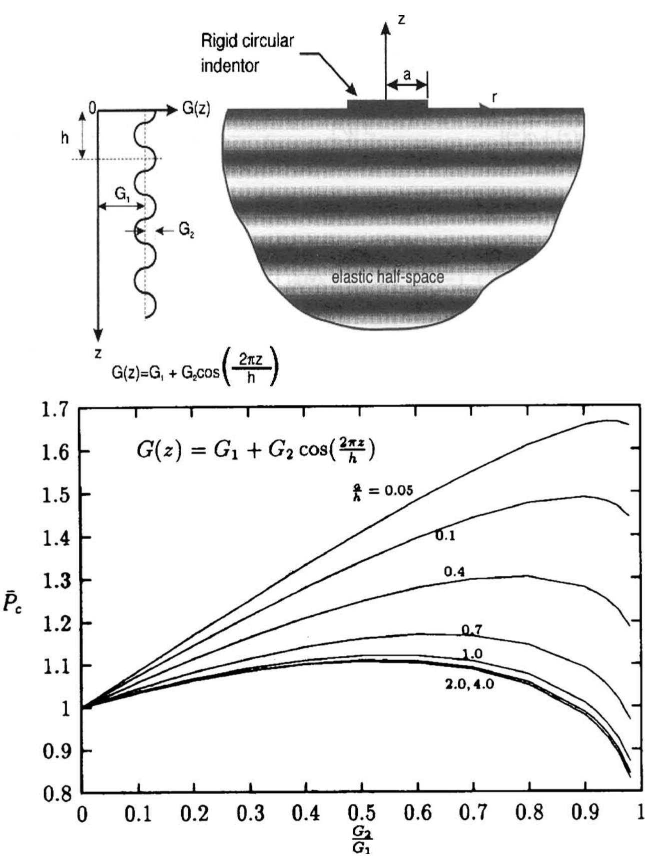

Fig. 4 Indentation of a halfspace with a harmonic elastic nonhomogeneity

continuity requirements can be explicitly satisfied to generate an assembly of regions that can model unbounded media. Spencer and Selvadurai [99] applied the procedure to develop solutions to certain anti-plane problems of interest to geomechanics, and they gave references to further studies that indicate the potential use of the procedure for the study of problems in material science and geophysics [100].

3 Poroelasticity and Geomechanics

Poroelasticity is a theory that considers the interaction between two phases composing a continuum. The origins of such studies date back to Lord Kelvin [101,102]. The study of the consolidation behavior of fluid-saturated geomaterials is generally attributed to Terzaghi [3], although the contributions of Fillunger [103] have also been recognized [104]. The theory of poroelasticity was originally developed by Biot [105,106], and later developed independently by Florin [7] and Mandel [107] (see also Zaretskii [108]). This theory represents one of the earliest rational continuum theories to account for the multiphase and three-dimensional nature of fluid-saturated geomaterials with an elastic porous skeleton. The assumptions of Hookean elastic behavior for the porous skeleton and Darcy flow for describing fluid flow through the pore space gives rise to a system of coupled partial differential equations (PDEs) for the skeletal displacements and the pore fluid pressure, that describe the conservation laws applicable to linear momentum and mass. For quasi-static problems, the governing PDEs have an elliptic-parabolic character and the well posedness of the initial boundary value problem, in a Hadamard sense, is assured through the availability of a uniqueness theorem [109]. Although the theory of poroelasticity was originally intended for the modeling of fluid-saturated geomaterials, the subject matter can be applied to the solution of a variety of other classes of materials including bone and other natural materials. The research in this area is quite extensive and no attempt will be made to give a comprehensive review. The articles and compilations by Scheidegger [110], Paria [111], Schiffman [112], and Detournay and Cheng [113] and the volumes by Desai and Christian [114], Coussy [115], Selvadurai [116], Cheng et al.

[117], Thimus et al. [118], Lewis and Schrefler [119], de Boer [104,120], Auriault et al. [121], Ehlers and Bluhm [122], and Schrefler [123] contain well in excess of 2000 references to articles covering various aspects of poroelastic media, ranging from fundamental concepts, analytical solutions, computational developments, and extensions to include nonlinear phenomena associated with the porous skeleton. One important factor concerning the mechanics of the classical poroelasticity problem is that the response of the poroelastic medium at the start of the transient processes and at the completion of the transient processes can generally be gleaned from the associated solution for the elastic medium. This provides a useful check on the accuracy of the solutions. The main emphasis of a poroelastic analysis therefore is concentrated on the study of the transient processes. The analytical solution of three-dimensional problems in poroelasticity commences with the work of Biot [124] and Biot and Clingan [125] who studied the consolidation settlement under a rectangular load. Biot and Clingan [126] subsequently examined the plane strain problem of the bending settlement of a slab on a consolidating halfspace, under the action of a concentrated line load. The problem of the consolidation around pore pressure meters was first examined by de Josselin de Jong [127] using Biot's theory. An important and perhaps less well cited paper by Biot [128] is of more interest to analysts; it deals with the general solutions of the equations of poroelasticity through the use of Boussinesq–Papkovich-type stress functions similar to those used in classical elasticity theory. The Biot consolidation due to a loaded rigid sphere embedded in a poroelastic halfspace was examined by de Josselin de Jong [129] using the stress function technique. Gibson and McNamee [130] present a solution to the problem of the loading of a poroelastic halfspace region by a rectangular load. The Mandel–Cryer effect is one of the key solutions that establish the phenomenological differences between the elementary consolidation theory of Terzaghi [3] and that of Biot [106]. The mathematical analysis of the effect of pore pressure rise in a poroelastic material loaded under plane strain conditions was demonstrated by Mandel [131], and Cryer [132] established a similar result for a poroelastic sphere subjected to an external radial stress field in the presence of boundary drainage. Experimental verification is also provided by Gibson et al. [133]. The presence of the Mandel–Cryer effect is documented by a number of investigators including Helm [134] and McKinley [135] who considered the problem of a radially deforming poroelastic cylinder. Gibson et al. [136] also examined the poroelastic sphere problem for the case of a material that exhibits large displacements; although the linear elastic constitutive equation used in this study has limitations, the results show that the pore pressure amplification effect persists in the poroelastic sphere problem. Paria [137] formulated the axisymmetric problem related to a consolidating halfspace and presented expressions for the axial stress. Axisymmetric, plane, and three-dimensional problems related to the consolidation of semi-infinite regions were presented by McNamee and Gibson [138,139] and Gibson and McNamee [140]. Freudenthal and Spillers [141] developed a solution to the axisymmetric problem for a fluid-saturated medium with a porous skeleton that exhibits both elastic and viscoelastic effects. The consolidation due to the application of a tangential load over a circular region of a poroelastic halfspace was presented by Schiffman and Fungaroli [142]. Parton [143] also considered the axisymmetric Biot consolidation problem for a halfspace region. Problems related to plane strain and axisymmetric consolidation of a clay layer with a smooth impervious base were presented by Gibson et al. [144]. Booker [145] considered the problem of consolidation of a finite layer subjected to surface loading, but utilized Laplace transformation properties obtained from general theorems, which makes computational aspects more efficient. Schneider and Bowen [146] examined the traction boundary value problem associated with the loading of the surface of a poroelastic medium, the constitutive behavior of which is obtained from the theory of binary mixtures proposed by Bowen and Weise [147]. The response of this model differs in aspects from that of the Biot [105,106] theory; the application of mixture theories for obtaining the constitutive equations of fluid saturated media is discussed by a number of authors, including Mills [148], Atkin and Craine [149], Bowen [150], and Ehlers [151]. The consolidation problem for a finite layer has also been investigated by Booker and Small [152,153]. The consolidation of a cross-anisotropic soil was examined by Booker and Randolph [154] who also present consolidation curves for halfspace regions with free-draining surfaces, which are subjected to circular and rectangular uniform loads. The application of integral transform techniques for the consolidation of a layered medium with the linear variation of elasticity properties is also given by Harnpattanapanich and Vardoulakis [155].

The classes of problems that deal with fundamental solutions related to poroelastic media are important in connection with applications to traction boundary value problems as well as for the solution of problems in poroelasticity theory using boundary integral equation techniques [156,157,113]. Other poroelasticity solutions related to line dislocations and pressurized cylindrical and spherical cavities are given by Rice and Cleary [158]. In a companion paper, Cleary [159] presented solutions for the action of a point force and a point fluid source in a poroelastic medium of infinite extent. Rudnicki [160–162] has developed solutions for fluid mass sources and point forces in poroelastic media. Dislocation solutions for elastic media have always been of interest in connection with the study of earthquake mechanisms in geologic media [163]. Nur and Booker [164], Rudnicki [162,165], and Pan [166] have presented solutions for dislocation problems associated with poroelastic media. The solutions presented by Apirathvorakij and Karasudhi [167] and Niumpradit and Karasudhi [168] relate to circular patch loads acting in the axial and horizontal directions at the interior of a poroelastic halfspace. Puswewala and Rajapakse [169] developed fundamental solutions for axisymmetric axial loads, radial loads, and pore pressure sources that are located at the interior of a poroelastic halfspace and arranged in a ring, while Rajapakse and Senjuntichai [170] extended these results to include poroelastic media saturated with a compressible pore fluid. Of related interest are the studies by Smith and Booker [171] and Jiang and Rajapakse [172] that deal with the fundamental solutions for thermo-poroelastic materials. In the absence of thermal effects, the analogous results for classical poroelasticity can be recovered. Green's functions for layered poroelastic halfspaces are presented by Pan [166] and the fundamental solutions for poroelastic media composed of transversely isotropic elastic materials are given by Taguchi and Kurashige [173]. A recent review of coupled deformation–diffusion effects in the mechanics of faulting and failure of geomaterials is given by Rudnicki [174]. Cavity expansion problems are also of interest in poroelastic modeling, particularly in the study of pore pressure generation around driven piles and in connection with hollow cylinder testing of saturated geomaterials. The work of Randolph and Wroth [175] presents an analytical solution to the poroelasticity problem related to cavity expansion. Detournay and Cheng [113] present a series of solutions applicable to various limiting cases associated with poroelastic media; recently Jourine et al. [176] examined the hollow cylinder problem related to the material testing of fluid-saturated rocks, developing an analytical result that ultimately involves a Laplace transform inversion for the generation of numerical results. The one-dimensional poroelastic/plastic consolidation of a fluid-saturated elastic–plastic material that obeys either a Mohr–Coulomb or a Drucker–Prager failure criterion and either an associated or nonasssociated flow rule was considered by Pariseau [177]. The analytical solution utilizes a formulation that gives rise to a Stefan problem for the time-dependent pore fluid pressure distribution.

3.1 Poroelastic Contact Problems. Considering the mixed nature of the boundary conditions for contact problems, their solutions in either elasticity or poroelasticity are generally nonroutine. The presence of the time variable in the poroelasticity theory makes the analytical approach all the more involved since recourse must be made to formulate the mixed boundary value problem, usually in the Laplace transform domain, followed by a numerical solution of the resulting integral equations and a numerical inversion to transform the solution to the time domain. Rigorous analyses of contact problems in poroelasticity are few, and are almost always restricted to problems with an axisymmetric geometry, which allows the application of Fourier, Hankel, and Laplace transforms, depending upon the nature of the loadings and the boundary conditions governing the displacements, tractions, and pore fluid pressures. Since the PDEs governing poroelastic behavior are linear, unless there are moving boundaries involved, the problem is linear and for this reason, attention is usually restricted to loadings applied on the contacting body that are constant with time, thereby developing a fundamental result for the initial boundary value problem, which can be superposed to generate results applicable to other time-dependent variations. Schimmerl [178] examined the problem of the eccentric loading of a rigid plate resting on a poroelastic halfspace, invoking the approximations proposed by Heinrich and Desoyer [179] with regard to the time-independent nature of the contact pressure distribution beneath the rigid plate. This is a useful engineering approximation that satisfies the displacement boundary conditions only in an approximate manner. The complete analysis of several poroelastic contact problems has been discussed in the literature. Briefly, for deformations of a poroelastic medium referred to a (r,θ,z) coordinate system, the PDEs governing the displacements $\mathbf{u}(=(u_r,u_\theta,u_z))$ and pore fluid pressure p for both a mechanically and hydraulically isotropic porous skeleton and an incompressible pore fluid take the forms

$$G\nabla^2\mathbf{u} + (\lambda + G)\,\nabla\,(\nabla\,.\,\mathbf{u}) = \nabla p \tag{20}$$

$$\frac{k}{\gamma_w}\nabla^2 p = \frac{\partial}{\partial t}(\nabla\,.\,\mathbf{u}) \tag{21}$$

where ∇ is the gradient operator; ∇^2 is Laplace's operator; λ and G are Lamé's constants for the porous elastic skeleton; k is the hydraulic conductivity; and γ_w is the unit weight of water.

We can of course proceed to formulate initial boundary value problems by treating the displacement components and the pore pressure as the dependent variables [180,181]. Experience with elasticity suggests that the use of special displacement and stress functions and representations results in a reduction in the number of dependent variables. The harmonic Lamé's potential, the biharmonic Love's potential, the Neuber–Papkovich functions, Maxwell–Morera stress functions, Airy's stress function, Stokes' stream function, etc., are typical examples. There is, of course, a requirement imposed on such representations; the resulting PDEs for the stress or displacement functions should be in a canonical form, preferably identifiable with any of the classical PDEs in mathematics. There are very few specific avenues for identifying such representations (see e.g., Truesdell [182], Gurtin [35], Gladwell [60], and Selvadurai [37,38]). The displacement function representations by Biot [128] and its equivalents given by McNamee and Gibson [138] and Schiffman and Fungaroli [142] can be used to perform the reduction. For completely asymmetric problems, expressed in (r,θ,z) coordinates, only *three stress functions* are necessary and sufficient, although there does not appear to be a formal proof of completeness of the representation. The displacements and the pore water pressure can be represented in terms of three independent functions $\varphi(r,\theta,z,t)$, $\psi(r,\theta,z,t)$, and $\chi(r,\theta,z,t)$ such that

$$u_r = \frac{\partial\phi}{\partial r} + z\frac{\partial\chi}{\partial r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} + \frac{z}{r}\frac{\partial\chi}{\partial\theta} - \frac{\partial\psi}{\partial r}$$

$$u_z(r,\theta,z,t) = \frac{\partial\phi}{\partial z} + z\frac{\partial\chi}{\partial z} - \chi \tag{22}$$

$$p = -2G\frac{\partial\chi}{\partial z} - 2G\eta\nabla^2\varphi \tag{23}$$

provided the functions satisfy the PDEs

$$\nabla^4\varphi = \frac{1}{c}\frac{\partial}{\partial t}\nabla^2\varphi;\quad \nabla^2\psi = 0;\quad \nabla^2\chi = 0;\quad c = \frac{2G\eta k}{\gamma_w} \tag{24}$$

These are recognizable PDEs in the engineering sciences [38,42]. Now we can focus attention on the application of Laplace, Fourier, and Hankel transforms to generate solutions to these PDEs with the proviso that the initial boundary value problem is posed in terms of the physical quantities associated with the description of the specific problem. Examples of the structure of the representations can be of the forms

$$\Lambda(r,\theta,z,t) = \sum_{n=0}^{\infty}\Lambda_n(r,z,t)\cos n\theta \tag{25}$$

where Λ can be identified with either φ, ψ, or χ. Upon application of Laplace transforms to remove the time variable, the integral representations for these functions can be written as

$$\bar{\varphi}_n(r,z,s) = \int_0^\infty [A_n(\alpha,s)e^{-\alpha z} + \hat{A}_n(\alpha,s)e^{\alpha z} + B_n(\alpha,s)e^{-\beta z} + \hat{B}_n(\alpha,s)e^{\beta z}]\alpha J_n(\alpha r)d\alpha \tag{26}$$

$$\bar{\chi}_n(r,z,s) = \int_0^\infty [C_n(\alpha,s)e^{-\alpha z} + \hat{C}_n(\alpha,s)e^{\alpha z}]\alpha J_n(\alpha r)d\alpha \tag{27}$$

$$\bar{\psi}_n(r,z,s) = \int_0^\infty [D_n(\alpha,s)e^{-\alpha z} + \hat{D}_n(\alpha,s)e^{\alpha z}]\alpha J_n(\alpha r)d\alpha \tag{28}$$

where $J_n(\alpha r)$ are nth-order Bessel functions of the first kind; s is the Laplace transform parameter; $A_n(\alpha,s), \hat{A}_n(\alpha,s), B_n(\alpha,s), B_n(\alpha,s),\ldots$, etc., are arbitrary functions; and

$$\beta^2 = \alpha^2 + (s/c) \tag{29}$$

As an example of the problem formulation, consider the axisymmetric indentation of the surface of a poroelastic halfspace by a circular rigid indentor. The formulation of the initial boundary value problem should take into consideration the nature of the displacement and pore water pressure boundary conditions in the contact region and the traction and pore fluid pressure boundary conditions in the region exterior to the indentor. These can be varied and should reflect the physical conditions associated with the contact. Consider, for example, the case of a completely permeable halfspace that is indented by a rigid, flat, smooth, circular indentor of radius a with a pervious contact. In addition, the rigid indentor is subjected to a central axial load P that is kept constant with time. The boundary conditions associated with the problem are

$$u_z(r,0,t) = w_0(t);\quad r\in[0,a) \tag{30}$$

$$\sigma_{rz}(r,0,t) = 0;\quad r\in[0,\infty);\quad \sigma_{zz}(r,0,t) = 0;\quad r\in[a,\infty) \tag{31}$$

$$p(r,0,t) = 0; \quad r \in [0,\infty) \tag{32}$$

A comment concerning the Dirichlet pore water pressure boundary condition Eq. (32) is in order. If we assume that a free-draining porous indentor in contact with the poroelastic halfspace, has a large area porosity in the contacting zone, then the pore pressure boundary condition Eq. (32) is a correct representation. If on the other hand, the indentor has a micromechanical contact that distinguishes areas of the contact region that are porous and those that are impervious, then the boundary condition Eq. (32) is only an approximation, since solid impervious regions of the contacting porous medium will inhibit dissipation of pore fluid pressures. The final displacements of the indentor will not be influenced by this distinction between void and solid areas of the contacting plane. The time for the consolidation process can however be influenced by the area porosity of the contacting region [183]. The displacement, stress, and pore pressure fields should satisfy the regularity conditions applicable to a three-dimensional poroelastic response. The equations will have a solution, which is proportional to $\bar{w}_0(s)$. This constant of proportionality is determined from the equilibrium condition

$$\int_0^a r\bar{\sigma}_{zz}(r,s)dr = P/(2\pi s) \tag{33}$$

In addition to the boundary and regularity conditions, initial conditions should be prescribed on the primary dependent variables governing the problem. Usually, this assumption relates to the fact that the analysis of the contact problem involves only displacement and stress states that are considered to be in excess of the in situ stress, displacement, and pore pressure distributions that are attributable to geostatic states. Consequently, the initial conditions assume that the additional displacements, stresses, and pore water pressures are zero at $t=0$, i.e.

$$u_i(r,\theta,z,0) = 0; \quad p(r,\theta,z,0) = 0 \tag{34}$$

The system of integral equations obtained from Eqs. (30)–(32) and (34) can be reduced to a pair of dual integral equations. Using an Abel-type finite Fourier transform, these in turn can be reduced to a single Fredholm integral equation of the second kind in terms of the Laplace transform variable s for an unknown function $\bar{\Omega}(\xi,s)$ (see e.g., Agbezuge and Deresiewicz [184], Szefer and Gaszyński [185], and Chiarella and Booker [186]), i.e.

$$\bar{\Omega}(\xi,s) = \frac{2}{\pi}\bar{w}_0(s) - \frac{2}{\pi}\int_0^\infty \bar{k}(\alpha,s)\cos\alpha\xi\left(\int_0^a \bar{\Omega}(r,s)\cos\alpha r\, dr\right)d\alpha \tag{35}$$

where

$$\bar{k}(\alpha,s) = \alpha\sqrt{c}[\eta(\alpha^2 c + s)^{1/2} + (\eta-1)\alpha\sqrt{c}]^{-1} \tag{36}$$

The poroelastic contact problem is effectively reduced to the solution of the Fredholm integral Eq. (35) and the Laplace transform inversion. A review of the literature on integral equations [187–191] indicates that the solution of the integral Eq. (35) cannot be obtained in explicit closed form. Therefore its numerical solution and the associated numerical inversion of the Laplace transforms must progress, giving due attention to the accuracy of the schemes.

The contact problem is made more complicated when full bonded contact conditions and the accompanying impervious boundary conditions are specified in the contact region. In this case the relevant boundary conditions take the forms

$$u_z(r,0,t) = w_0(t); \quad r \in [0,a); \quad u_r(r,0,t) = 0; \quad r \in [0,a) \tag{37}$$

$$\sigma_{rz}(r,0,t) = 0; \quad r \in [a,\infty); \quad \sigma_{zz}(r,0,t) = 0; \quad r \in [a,\infty) \tag{38}$$

$$p(r,0,t) = 0; \quad r \in [a,\infty); \quad \frac{\partial p(r,0,t)}{\partial z} = 0; \quad r \in [0,a) \tag{39}$$

The natural assumption would be to extend the method of analysis of the mixed boundary value problem described previously for the solution of the bonded contact between the indentor and the halfspace. Such a direct application results in an expedient, but a mathematically incorrect, solution, unless the correct form of the stress singularity at the boundary of the contact region is known. In the elasticity problem the rapid change of the boundary conditions on a surface, planar or otherwise, will give rise to a stress singularity which will have oscillatory characteristics, governed by the Poisson's ratio of the material. When $\nu=1/2$, the stress singularity will be regular and of the $1/\sqrt{r}$ type. For any other value of Poisson's ratio, the stress singularity is determined from a solution of a Hilbert problem and its order will be a function of the Poisson's ratio of the material [192–194,60]. At the start of the consolidation problem the behavior of the poroelastic medum will correspond to that of an elastic material with incompressible behavior. For all other $t>0$, $\nu<1/2$, with the result that the oscillatory singularity will persist. The full transient contact problem that incorporates the characteristics of the singularity is yet to be solved. It has, however, been shown [195–199] that although the order of the stress singularity is of importance to the mathematical formulation, it has only a marginal influence on the load–displacement relationship. Where the pore pressure boundary condition changes from a Dirichlet to a Neumann type, the potential problem suggests the existence of singularities that are nonoscillatory. The work of Atkinson and Craster [200] and Craster and Atkinson [201], which examines the quasi-static crack extension in two dimensions, points to the velocity-dependent stress singularities at the pervious–impervious demarcation point. The problem of the transient form of the stress singularity in the stress and displacement fields needs to be properly formulated and solved as an eigenvalue problem that can establish any transient effects associated with the order of the stress singularity. With these limitations in mind, the initial boundary value problems described by Eqs. (37)–(39) can be solved using the conventional approaches to generate useful results of practical value.

The earliest recorded solution of a mixed boundary value problem for a contact problem in classical poroelasticity is that of Szefer and Domski [202] who examined the plane contact problem for a poroelastic layer. These authors employed a finite difference technique both spatially and temporally to solve the initial boundary value problem. By virtue of the approximations involved in the finite difference approach, the influence of stress singularities is not accounted for in the study. Agbezuge and Deresiewicz [184] examined the frictionless indentation of a poroelastic halfspace by both permeable and impermeable spherical rigid indentors with frictionless traction boundary conditions at the contacting region and traction-free boundary conditions exterior to the contact zone. Since the indentor has a spherical shape, the extent of the contact region is defined by a time-dependent radius, which needs to be determined by considering the smooth transition of the contact stresses at the line of separation. These authors present a comprehensive study of the problem formulation and provide numerical results for the evolution of the contact region and the contact pressures within it with time. Deresiewicz [203] also examined the influence of the Poisson's ratio and the pore water pressure boundary conditions on the radius of the contact zone. Agbezuge and Deresiewicz [204] extended the analysis to include the axisymmetric indentation of a poroelastic halfspace by a rigid circular indentor with a flat base. The analysis also provides results for the time-dependent evolution of contact stresses beneath the footing as well as the results for the time-dependent

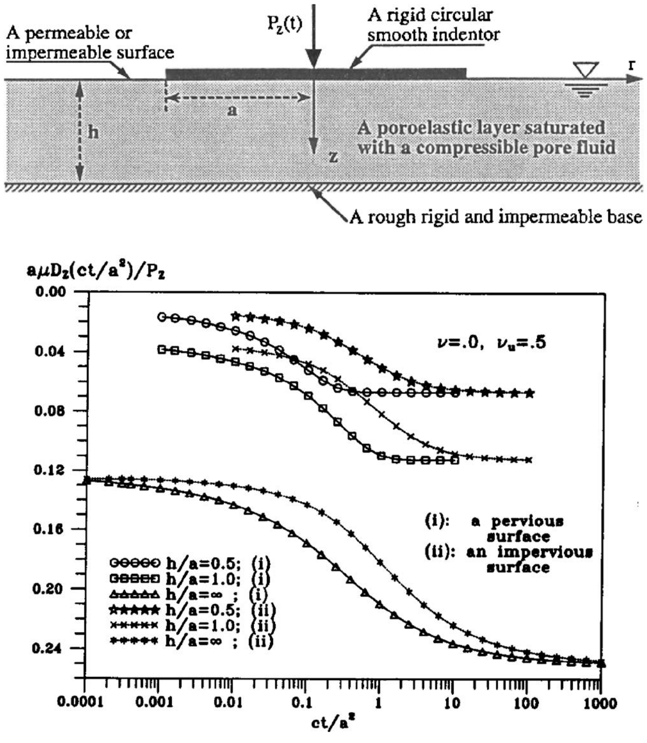

Fig. 5 Consolidation characteristics of a rigid foundation on a poroelastic layer

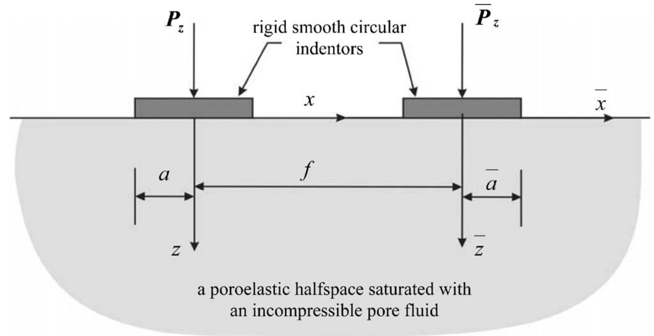

Fig. 6 Interacting rigid foundations on a poroelastic halfspace

variation of the settlement of the rigid indentor. The contact at the interface between the indentor and the poroelastic halfspace is assumed to be smooth. The frictionless indentation problem for the isotropic poroelastic halfspace was also examined by Chiarella and Booker [186], invoking a fully draining boundary condition at the entire surface of the halfspace region. These authors also present the procedures for the Laplace transform inversion and the solution of the resulting Fredholm–Volterra-type integral equation. Numerical results are provided for the time-dependent variation of the contact stresses and the degree of consolidation settlement of the rigid indentor. These authors provide a comparison of the complete analytical solution with the approximate result derived by making the approximation for the contact stresses suggested by Heinrich and Desoyer [179]. The axisymmetric indentor problem for a poroelastic halfspace region was examined by Gaszyński and Szefer [205] who present a formal development of the integral equations governing the problem. Similar formal developments are also presented by Gaszyński [206]. The problem of the axisymmetric contact between a viscoelastic consolidating halfspace and a rigid circular indentor with a flat base was considered by Szefer and Gaszyński [185]. In this study the pore water pressure boundary conditions are assumed to be of either the Dirichlet or Neumann type, over the entire surface of the halfspace. Standard procedures are adopted for the formulation of the governing integral equations and their reduction to Fredholm integral equations of the second kind. Numerical results are presented for the contact pressure distributions beneath the indentor and the time-dependent settlements of the indentor for a specific choice of a poro-visco-elastic material. The one-dimensional and axisymmetric traction boundary value problems related to the consolidation of a poroelastic medium with a Gibson-type linear variation in the shear modulus has been examined by Mahmoud and Deresiewicz [207,208]. The axisymmetric indentation of the surface of a poroelastic layer underlain by a rough impermeable base (Fig. 5) was examined by Selvadurai and Yue [209]. Here the entire surface is assumed to be frictionless, and homogeneous pore water pressure boundary conditions of the Dirichlet or Neumann type are prescribed over the entire surface of the layer. The associated boundary conditions take the forms

$$u_z(r,0,t) = w_0(t); \quad r \in [0,a) \tag{40}$$

$$\sigma_{rz}(r,0,t) = 0; \quad r \in [0,\infty); \quad \sigma_{zz}(r,0,t) = 0; \quad r \in [a,\infty) \tag{41}$$

$$u_z(r,h,t) = 0; \quad r \in [0,\infty); \quad u_r(r,h,t) = 0; \quad r \in [0,\infty) \tag{42}$$

with either

$$p(r,0,t) = 0; \quad r \in [0,\infty); \quad \text{or } \frac{\partial p(r,0,t)}{\partial z} = 0; \quad r \in [0,\infty) \tag{43}$$

The formal mathematical development of the contact problem follows the application of Laplace and Hankel transforms to reduce the problem to a single Fredholm integral equation of the second kind and to employ numerical procedures for the solution of the integral equation and for the Laplace transform inversion. The theoretical developments also take into consideration the influence of compressibility of the pore fluid. Some results that indicate the influence of the depth of the layer on the time-dependent consolidation settlement are shown in Fig. 5. In a series of papers that followed, these authors applied the techniques for the solution of a variety of problems of interest to geomechanics. Yue and Selvadurai [180,210] and Yue et al. [211] have examined the problems of both asymmetric and axisymmetric indentation of a poroelastic halfspace where the pore pressure boundary conditions at the surface of the halfspace can either be completely impervious or completely pervious over the entire surface of the halfspace or impervious over the indented region and pervious exterior to it. Again, the pore fluid is considered to be compressible. A very comprehensive treatment of the problem of a rigid disk inclusion embedded in a poroelastic infinite space was presented by Yue and Selvadurai [181]. In addition to considering the effects of a compressible pore fluid, these authors examine the consolidation response of the disk inclusion that is subjected to a set of generalized forces that induce axisymmetric and asymmetric deformations of the poroelastic medium. The disk inclusion itself has either completely impervious or fully pervious pore fluid pressure boundary conditions. The problem of the interaction of two circular rigid indentors of unequal radii resting in smooth contact with a poroelastic halfspace saturated with an incompressible pore fluid was examined by Lan and Selvadurai [212] (Fig. 6).

In this study, the entire surface of the poroelastic medium is assumed to be pervious and mixed boundary conditions are prescribed on the contact plane. The details of the solution will not be pursued here; the formulation of the single indentor problem is first achieved and the solution for the companion indentor is obtained through a coordinate transformation. The superposition of

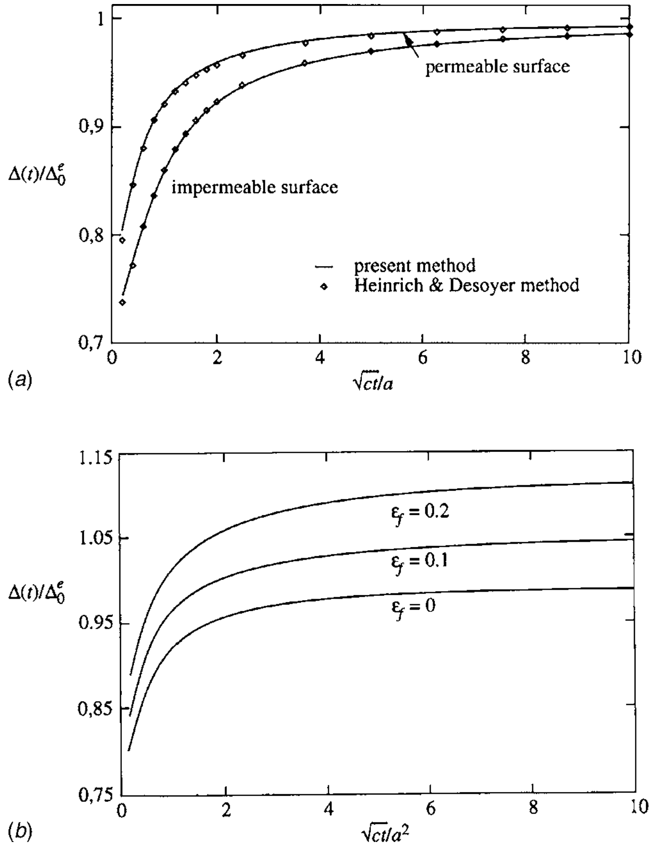

Fig. 7 Consolidation behavior of interacting identical foundations of radius *a*: (*a*) influence of surface drainage conditions; (*b*) interaction of symmetrically loaded foundations (a) ($\Delta(t)$ is the displacement at the point of application of the load; $\Delta_0^e = (1-\nu)P_z/4\,Ga$; $\varepsilon_f = f/a$; $c = 2Gk(1-\nu)/(1-2\nu)\gamma_w$)

the solutions satisfies all boundary conditions except the displacement boundary conditions at the indentor locations. The approach involves the numerical solution of the coupled Fredholm integral equations and the numerical Laplace transform inversion. Results of practical interest include the time-dependent interaction between: (i) the indentor and a halfspace with extreme pore pressure boundary conditions and (ii) the indentor and an externally placed concentrated force (Fig. 7). Comparisons are also provided for results obtained with the Heinrich–Desoyer [179] approximation that considers the contact pressure to be time-independent, at its distribution at $t=0$.

A highly successful application of the theory of poroelasticity and variations thereof deals with the problem of subsidence resulting from extraction of water and energy resources such as oil and natural gas from geologic formations. The literature in this area is too extensive to document the advances in detail. Comprehensive accounts are given by Gambolati [213], Geertsma [214], Saxena [215], Scott [216], Corapcioglu [217], and Lewis and Schrefler [119]. Analytical investigations in this area are due to Booker and Carter [218,219] and Carter and Booker [220] who have examined the settlement and consolidation around point sources and sinks. These studies have been extended by Tarn and Lu [221] to include anisotropic effects in the poroelastic response.

4 Porous Media Flow, Transport, and Geomechanics

Despite its potential importance to many civil and geological engineering applications, the measurement and interpretation of the fluid flow properties of geotechnical materials is far from routine. One of the key factors that influences the interpretation of the fluid transport characteristics for geomaterials is the choice of scale. This can range from geological crustal scales of 0.5–5 km, to borehole scales ranging from 30 m to 300 m, and laboratory scales that can range from 5 cm to 15 cm [222]. The bulk fluid flow characteristics of naturally occurring geologic media will be influenced by factors such as fissures, inclusions, and other inhomogeneities that become important at the various scales. Efforts are continually being made to develop methodologies for the accurate in situ determination of fluid flow properties of geomaterials through careful measurement and interpretation.

4.1 Intake Shape Factors. The rate of rise of the water level in a cased borehole is one of the more popular and practical procedures for determining the in situ hydraulic conductivity properties of geomaterials. Although the scientific origins of the method may date back to Kirkham [223] and Luthin and Kirkham [224], the importance of the procedure for geotechnical applications was brought about by Hvorslev [225] in an authoritative study of in situ investigations. The basic theory assumes the applicability of Darcy's law for describing the flow velocity in the porous medium. For a hydraulically isotropic medium that is nondeformable, the flow velocities are given by

$$\tilde{\mathbf{v}}(\mathbf{x}) = -\tilde{k}\,\nabla\Phi \tag{44}$$

where $\tilde{\mathbf{v}}$ are the velocities; $\mathbf{x}$ are the Cartesian coordinates; $\tilde{k}$ is the conventional Darcy hydraulic conductivity [226–231]; and Φ is the reduced Bernoulli potential. When steady incompressible flow exists in a homogeneous, nondeformable porous medium, Φ is harmonic: i.e.

$$\nabla^2\Phi = 0 \tag{45}$$

Although the flow into the well casing is unsteady, it is assumed that the local flow field can be described by Eq. (45), with suitable boundary conditions prescribed to solve a Dirichlet problem for the entry point geometry. This enables the calculation of the flux q, which can be represented in the form

$$q = F\tilde{k}\Phi_0 \tag{46}$$

where Φ_0 is the potential difference causing flow. The term F is referred to as the "intake shape factor," which has dimensions of length and depends solely on the geometrical characteristics of the entry region. Based on a result by Dachler [232] (although this can be gleaned from the results given by Legendre, Laplace, Green, Lamé and others and from standard texts in potential theory by Kellogg [233], MacMillan [234], Hobson [235], and Morse and Feshbach [236]), for the potential problem for a spheroid, Hvorslev [225] suggested the now familiar result for the intake shape factor for a cylindrical intake with diameter $\tilde{D}$ and length $\tilde{L}$

$$F = \frac{2\pi\tilde{L}}{\ln\left[\dfrac{\tilde{L}}{\tilde{D}} + \sqrt{1+\left(\dfrac{\tilde{L}}{\tilde{D}}\right)^2}\right]} \tag{47}$$

An extensive account of the advances that took place from Hvorslev's original work to current developments is given in Refs. [237,238]. The basic problem concerning the influence of hydraulic anisotropy on the intake shape factor has been discussed by several authors, notably Childs and Collis-George [239], Childs [240,241], Childs et al. [242], Maasland [243], Maasland and Kirkham [244], and Philip [245,246]. Selvadurai [237] has presented an alternative procedure for obtaining an intake shape factor in the form of a spheroid located in a transversely isotropic porous medium. The idealization of the porous medium as a hydraulically transversely isotropic medium is not entirely unrealistic, since depositional effects invariably involve the gravity direction, which is perpendicular to the plane of transverse isotropy. Considering a system of axisymmetric cylindrical polar coordinates, (r, z) the PDE for the potential takes the form

$$k_{rr}\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right)+k_{zz}\frac{\partial^2\Phi}{\partial z^2}=0 \tag{48}$$

where k_{rr} and k_{zz} are the principal hydraulic conductivities. We can transform Eq. (48) to a Laplacian form and proceed to develop solutions to the harmonic problem, which can be used to determine the total flux at the boundary of the intake. The details of the analysis are given by Selvadurai [237] and the salient results of interest are summarized. The analysis given by Selvadurai [237] provides the expressions equivalent to Eq. (47), to take into account transverse isotropy of hydraulic conductivity. In presenting the results, it is necessary to distinguish between both the geometry of the intake region and the degree of transverse isotropy, defined in relation to whether $k_{rr} \geqslant k_{zz}$ or $k_{rr} \leqslant k_{zz}$. In summary, the complete range of the solutions for the intake shape factors, applicable to both *prolate* and *oblate* spheroidal cavities in hydraulically transversely isotropic porous media with either $k_{rr} \geqslant k_{zz}$ or $k_{rr} \leqslant k_{zz}$, can be stated in the form, with $\lambda = k_{rr}/k_{zz}$ and $\eta = b^*/a^*$

$$\left.\begin{matrix} F_{pp} \\ F_{op} \end{matrix}\right\} = \frac{4\pi\tilde{D}\sqrt{\lambda-\eta^2}}{\eta\sqrt{\lambda}\ln\left[\dfrac{\sqrt{\lambda}+\sqrt{\lambda-\eta^2}}{\sqrt{\lambda}-\sqrt{\lambda-\eta^2}}\right]}; \quad \lambda \geqslant 1; \quad \left\{\begin{matrix} \eta = \leqslant 1 \\ \eta \geqslant 1 \end{matrix}\right. \tag{49}$$

Similarly

$$\left.\begin{matrix} F_{oo} \\ F_{po} \end{matrix}\right\} = \frac{2\pi\tilde{D}\sqrt{\eta^2-\lambda}}{\eta\sqrt{\lambda}\cot^{-1}(\sqrt{\lambda}/\sqrt{\eta^2-\lambda})}; \quad \lambda \leqslant 1; \quad \left\{\begin{matrix} \eta \geqslant 1 \\ \eta \leqslant 1 \end{matrix}\right. \tag{50}$$

where a^* is the semi-major axis and b^* is the semi-minor axis; and to distinguish the respective cases we introduce the subscripts 'o' and 'p', referring, respectively, to the oblate and prolate geometries. The adoption of these results to the borehole casing cylindrical intake problem requires establishing a relationship between the two problems. Equivalence between the geometric aspect ratios of the cylindrical and spheroidal intakes is a poor choice, since it does not capture the physics of the flow problem. A most realistic correlation can be obtained by considering the equivalence of surface areas between the spheroidal intake and the cylindrical intake. Consider a cylindrical intake for a spheroidal cavity with aspect ratio η. If we set $\tilde{D} = 2b^*$, the equivalence in the entry point surface areas gives

$$\frac{\tilde{D}}{\tilde{L}} = \frac{2\eta\sqrt{1-\eta^2}}{[\eta\sqrt{1-\eta^2}+\sin^{-1}\{\sqrt{1-\eta^2}\}]} \tag{51}$$

For a specified intake shape geometry, the flow rate is determined using expressions (49) and (50) along with the flow rate equation of the form

$$q = F_{ij}k_{rr}\Phi_0; \quad (i, j = p, o) \tag{52}$$

From a practical perspective, the geometric aspect ratio of the intake can be altered to determine the different flow rates that are associated with the different intakes. These expressions for the flow rate, which incorporate the hydraulic transverse isotropy measure, can then be solved to determine the hydraulic transverse isotropy ratio k_h/k_v in a unique manner. Admittedly, this requires some knowledge of whether $k_h/k_v < 1$ or whether $k_h/k_v > 1$. Site investigations that involve core recovery from the stratified geological medium should indicate the plausible choice. Also, for most sedimentary geologic media the sequential deposition of relatively impervious layers will result in the condition where, invariably, $k_h/k_v > 1$. Application of fuzzy analysis to the estimation of the hydraulic anisotropy is also given by Hanss and Selvadurai [247] and Hanss [248].

4.2 Advective Transport in Porous Media. The topic of the migration of chemicals, pollutants, and other hazardous substances in fluid-saturated porous media has been in the forefront of the development of the discipline of environmental geomechanics. This is a very complex topic, with difficulties stemming largely from the interaction of a variety of processes influenced by geochemical, mechanical, and thermal effects and the inherent material variability in the geosphere [249–257]. The advances that can be made in terms of the development of analytically oriented results must, of necessity, make radical assumptions with respect to incorporating salient features that are perceived to be of interest to practical problems. Advective transport of a chemical species through the pore space of a porous medium due to hydraulic gradients present in the porous medium is one of the simplest idealizations of the transport process. The process can be made more realistic through the incorporation of other factors such as concentration gradient-dependent diffusion, hydrodynamic dispersion, natural attenuation, restricted sorption capabilities, alteration in the transport properties of the porous medium by the migrating species, moving boundaries, etc. The purely advective transport of the chemical species in an unchanging porous medium, however, represents the canonical problem. The mathematical aspects of the advective transport problem itself has applications to a variety of other disciplines including vehicular traffic flow, movement of waves in shallow water, meteorology and geostrophic processes in the atmosphere, movement of charged particles such as electrons, biological processes, mechanics of surging glaciers, migration of fine particulates in porous media, marine ecology, resin migration during injection molding, and in the study of heat exchangers. Consider advective–diffusive transport of the species with the concentration $C(\mathbf{x}, t)$, measured per *unit volume of the fluid*, due to an advective flow velocity $\mathbf{v}(\mathbf{x}, t)$ in the pore fluid. For a hydraulically isotropic nondeformable porous medium, the PDE governing the problem is

$$\frac{\partial C}{\partial t} + \nabla . (\mathbf{v}C) = \pm \xi C + D\nabla^2 C \tag{53}$$

where ξ is a generation/decay factor measured per unit time and D is a diffusion coefficient. If the pore fluid is incompressible and the porous medium is nondeformable, the flow can only be steady, and nonsteady flow fields can occur only due to time-dependent variations in the boundary potential. The pore space flow velocities $\mathbf{v}$, for a hydraulically isotropic medium, is given by the Dupuit–Forchheimer law

$$\mathbf{v} = -k\,\nabla\,\Phi \tag{54}$$

where k is related to the conventional Darcy measure of hydraulic conductivity $\tilde{k}$ given by Eq. (44) through the relationship $k = \tilde{k}/n^*$, where n^* is the porosity and $\Phi(\mathbf{x})$ is the reduced Bernoulli potential. The PDEs governing the advective–diffusive transport problem are therefore Eqs. (45) and (53) which represent a weakly coupled system, which is second-order elliptic for $\Phi(\mathbf{x})$ and can be first-order hyperbolic for $C(\mathbf{x}, t)$ for *advection-dominated* problems and second-order parabolic for *diffusion-dominated* problems. The weak coupling stems from the fact that the potential problem governing $\Phi(\mathbf{x})$ can be solved independently of the advective transport problem. The initial boundary value problem governing $C(\mathbf{x}, t)$ is subject to an initial condition and Dirichlet and Neumann-type boundary conditions specified on a limited number of *subsets* of $S_D (= S_{D1} \cup S_{D2} \cup \ldots\ldots \cup S_{Dn})$ and $S_N (= S_{N1} \cup S_{N2} \cup \ldots\ldots \cup S_{Nm})$, such that $S_D \cup S_N = S$ and $S_D \cap S_N = 0$. We also note that other mixed boundary conditions of the Robin type could be prescribed on a separate subset of S, but for the purpose of the discussion we shall restrict attention to the conventional Dirichlet and Neumann boundary conditions that can be readily identified in relation to certain physical attributes of the advective flow problem. We assume that the boundary value problem governing $\Phi(\mathbf{x})$ is subject to Dirichlet-type boundary conditions on S_D and Neumann-type boundary conditions on S_N. The uniqueness of solution to the potential problem is well established [38,258–260]. We further assume that certain surfaces on which Dirichlet and Neumann boundary conditions are prescribed for the concentration $C(\mathbf{x},t)$ also coincide with surfaces on which Dirichlet and Neumann boundary conditions are prescribed for the potential $\Phi(\mathbf{x})$. Under these conditions, proofs of uniqueness exist for both the advective and advective–diffusive transport problems [260,261]. It may also be noted that the above formulation is equally applicable to unconfined flows. The accurate computational modeling of the generalized advection–diffusion transport equation, especially in the presence of an advection-dominated term, with either a discontinuity or steep gradient of the dependent variable, has been a challenging problem in computational fluid dynamics. Higher-order methods (such as the central difference, Lax–Wendroff, and Beam–Warming techniques [262]) require the size of the domain discretization element to be small enough, such that the elemental Péclet number ($\mathrm{Pe}=\|\mathbf{v}\|h/D$, where $\|\mathbf{v}\|$ is a flow velocity norm within the element, h is a characteristic length of the element, and D is the diffusion coefficient) should not be greater than unity. When the elemental Péclet number is greater than unity the methods give rise to unrealistic numerical phenomena such as oscillations, negative concentrations, artificial diffusion, etc., at regions close to a leading edge with a discontinuous front. For this reason, in conventional higher-order methods for the convection-dominated problems, a finer mesh is invariably used throughout the region, since the velocity field is usually not known a priori. This places a great demand on computational resources, particularly in simulations involving three-dimensional problems. The first-order methods such as the Lax–Friedrich scheme, on the other hand, eliminate the oscillatory behavior at discontinuous fronts ($\mathrm{Pe}=\infty$), but give rise to numerical diffusion or numerical dispersion in the solution at discontinuous fronts. This feature is generally accepted for the purpose of engineering usage of the procedures, but from a computational point of view gives rise to strong reservations concerning the validity of the procedure for the solution of the purely advective transport problem. Furthermore, if physical diffusive phenomena are present in the transport problem, it becomes unclear as to whether the diffusive patterns observed in the solution are due to the physical process or an artifact of the numerical scheme. To date many approaches have been developed to examine the stabilized higher-order numerical schemes for the purely advective transport problem with sharp discontinuities in the concentration profile [263]. The guiding principle in these numerical methods involves the addition of numerical dispersion effects to the higher-order schemes, by means of procedures such as flux control, slope limiter methods, and total variation reducing properties, particularly in the vicinity of locations where the dependent variable is either discontinuous or exhibits a high spatial gradient. Evaluating the accuracy of the purely advective transport problem is therefore a necessary prerequisite to gain confidence in the application of the computational scheme for studying the advection–diffusion problem. The conventional approach for testing the accuracy of the advective transport problem involves the use of the classical solution to the one-dimensional initial boundary value problem defined by

$$\frac{\partial C}{\partial t} + v\frac{\partial C}{\partial x} = -\xi C; \quad x \in (0,l); \quad t > 0 \tag{55}$$

subject to the boundary condition

$$C(0,t) = C_0 \mathsf{H}[t] \tag{56}$$

where ξ is a natural attenuation; $\mathsf{H}[t]$ is the Heaviside step function, and the initial condition

$$C(x,0) = 0 \tag{57}$$

While the solution to this initial boundary value problem is a standard result of the form

$$C(x,t) = C_0 \exp(-\xi x/v)\mathsf{H}[t-(x/v)] \tag{58}$$

this is not the most challenging of tests for the numerical procedures. A one-dimensional problem defined by Eqs. (55)–(58) gives rise to a constant flow velocity, whereas in most practical problems the velocities can be nonuniform and can vary both spatially and with time. Suitable analytical solutions can be developed for this purpose and elementary examples of these are given by Selvadurai [264]. For example, consider the problem of advective transport from a spherical cavity located in a porous medium of infinite extent, the boundary of which $(R=a)$ is maintained at a constant reduced Bernoulli potential φ_0 to induce steady Darcy flow. The boundary of the cavity of the initially species-free porous medium is subjected to a time-varying species concentration of the form

$$C(a,t) = C_0 \exp(-\zeta t) \tag{59}$$

where ζ is a constant. The time-dependent, spherically symmetric distribution of the concentration is given by

$$C(R,t) = C_0 \exp\{-\zeta t - [\xi - \zeta]\lambda(\rho)\}\mathsf{H}[t-\lambda(\rho)] \tag{60}$$

where

$$\lambda(\rho) = a^2(\rho^3-1)/3\varphi_0 k; \quad \rho = R/a \tag{61}$$

When *two-dimensional circular* flow takes place in a region $a \leq r < \infty$ and $0 \leq \theta < 2\pi$, where the plane $\theta=0$ is maintained at the potential φ_0, the velocity vector for potential flow reduces to

$$\mathbf{v}(r,\theta) = \frac{k\varphi_0}{2\pi r}\mathbf{i}_\theta \tag{62}$$

and the advective transport equation reduces to

$$\frac{\partial C}{\partial t} + \frac{k\varphi_0}{2\pi r^2}\frac{\partial C}{\partial \theta} = -\xi C \tag{63}$$

Assuming that the plane $\theta=0$ is subjected to the time-dependent concentration

$$C(r,0,t) = C_0 \mathsf{H}[t] \tag{64}$$

the spatial distribution of concentration of the species is given by

$$C(r,\theta,t) = C_0 \exp\{-\xi\tau(\eta)\}\mathsf{H}[t-\tau(\eta)] \tag{65}$$

where

$$\tau(\eta) = 2\pi a^2 \eta^2 \theta / k\varphi_0; \quad \eta = r/a \tag{66}$$

In recent papers Selvadurai [261,265] has extended these studies to include the advective transport of a species from a spheroidal cavity located in a fluid-saturated porous medium of infinite extent, the boundary of which is maintained at a constant potential φ_0. In this condition, the boundary of the cavity is subjected to a time-dependent concentration. For the case of an oblate spheroidal cavity, the potential problem is governed by

$$\nabla^2\varphi(\alpha,\beta) = h_o^2\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \tanh\alpha\frac{\partial}{\partial\alpha} + \cot\beta\frac{\partial}{\partial\beta}\right)\varphi(\alpha,\beta) = 0 \tag{67}$$

where $\alpha=\text{const}$ corresponds to oblate spheroids and $\beta=\text{const}$, corresponds to hyperboloids of one sheet. The potential problem gives rise to a velocity field

$$\mathbf{v}(\alpha,\beta) = \frac{k\varphi_0}{c_o \cosh\alpha \cot^{-1}(\sinh\alpha_0)\sqrt{\cosh^2\alpha - \sin^2\beta}}\mathbf{i}_\alpha \quad (68)$$

where α_0 corresponds to the spheroidal cavity surface and $c_o^2 = b_o^2 - a_o^2$, where a_o is the semi-minor axis, and b_o is the semi-major axis. The corresponding advective transport equation takes the form

$$\frac{\partial C}{\partial t} + \frac{k\varphi_0}{c_o^2 \cosh\alpha \cot^{-1}(\sinh\alpha_0)(\cosh^2\alpha - \sin^2\beta)}\frac{\partial C}{\partial\alpha} = -\xi C \quad (69)$$

This PDE can be solved, subject to the initial and boundary conditions

$$C(\alpha,\beta,0) = 0; \quad C(\alpha_0,\beta,t) = C_0\mathsf{H}[t] \quad (70)$$

respectively. The solution takes the form

$$\frac{C(\alpha,\beta,t)}{C_0} = \exp[-\xi\Omega_o(\alpha,\beta,\mu)]\mathsf{H}[t - \Omega_o(\alpha,\beta,\mu)] \quad (71)$$

where

$$\Omega_o(\alpha,\beta,\mu) = \frac{b_o^2(1-\mu^2)\cot^{-1}(\sinh\alpha_0)}{3k\varphi_0}[\sinh\alpha\cosh^2\alpha - \sinh\alpha_0\cosh^2\alpha_0 + (2 - 3\sin^2\beta)(\sinh\alpha - \sinh\alpha_0)] \quad (72)$$

The solution is exact and the time-dependent evolution of the chemical plume can be evaluated quite conveniently using this analytical result (Fig. 8).

In the context of its use in the calibration of computational methodologies, it is noted that the advective velocity is nonuniform over the porous region of infinite extent and furthermore, as $(a_o/b_o) \to 0$, the oblate spheroid flattens to acquire the shape of a penny-shaped crack. The advective flow velocity at the boundary of the crack is singular and the existence of the singularity is a severe test on the capabilities of the computational schemes. This analytical result therefore is of particular interest to establishing the computational efficiency of many schemes *developed for hyperbolic conservation laws*.

The types of problems that have been described in the preceding paragraphs assume that the advective flow velocity field in the porous medium is time independent. Time dependencies in the advective flow velocity can result from a number of processes including time-dependent variations in the boundary potentials, transient effects associated with *elastic-drive* type phenomena [266] arising from considerations of compressibilities of both the pore fluid and/or the porous skeleton, and poroelastic effects. Considering the one-dimensional advection–diffusion problem, the PDE governing the chemical migration in a one-dimensional column, similar to that used in a *falling head–type* experimental configuration (Fig. 9), is given by

$$\frac{\partial C}{\partial t} + v(t)\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} \quad (73)$$

where, $v(t)$ is the time-dependent velocity of the fluid in the pore space; D is the classical Fickian diffusion coefficient; x is the spatial variable; and t is time. In the particular instance when the head decreases exponentially with time, Eq. (73) reduces to

$$\frac{\partial C}{\partial t} + v_0\exp(-\lambda t)\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} \quad (74)$$

where $v_0 = kH_0/l$; $\lambda = k/l$; H_0 is the height of the water column at the start of the transport process; and l is the height of the porous column (Fig. 9). Therefore, for the advective velocities to be applicable l must necessarily be finite. The boundary and initial conditions applicable to the problem are, respectively

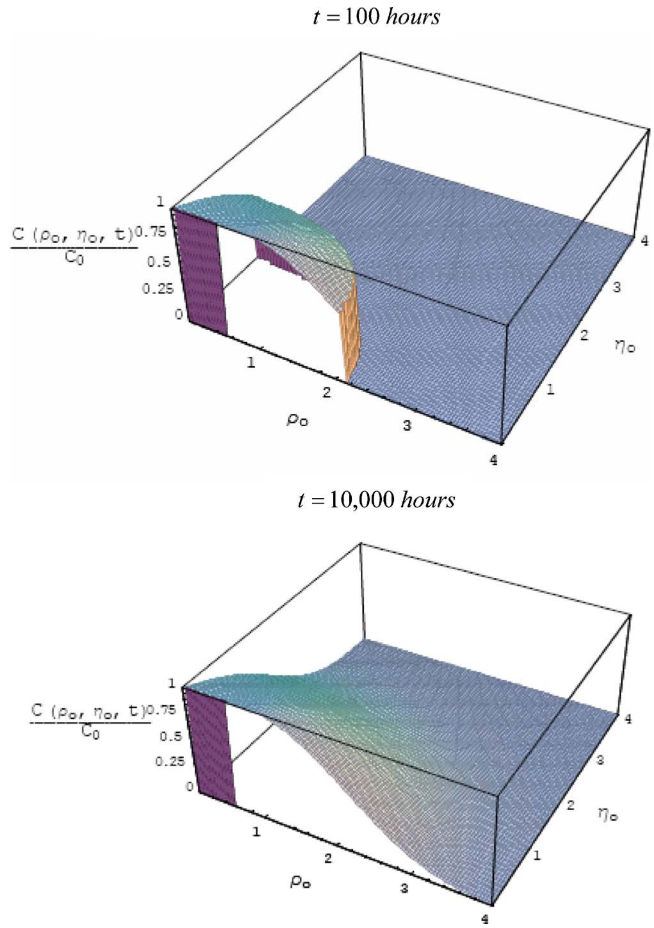

Fig. 8 Advective transport of a contaminant from a penny-shaped cavity in a porous medium with attenuation coefficient $\xi = 0.005$/day

$$C(0,t) = C_0\mathsf{H}[t]; \quad C(x,0) = 0 \quad (75)$$

No conditions can be prescribed at the boundary $x = l$ until the concentration reaches the location. Furthermore, the rate at which the contaminant is maintained at the location $x = l$ must be specified in order to solve the initial boundary value problem. A simplification that is often invoked for the analysis of the advective–

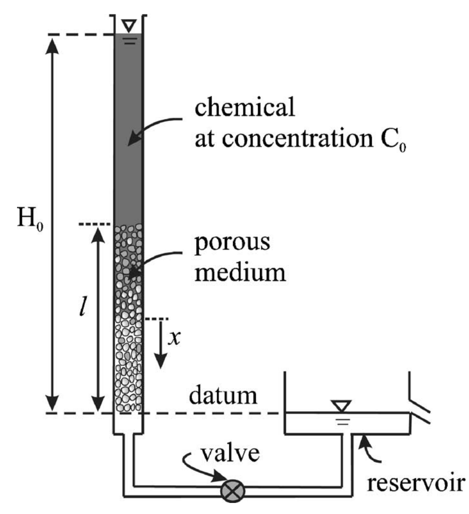

Fig. 9 The column experiment

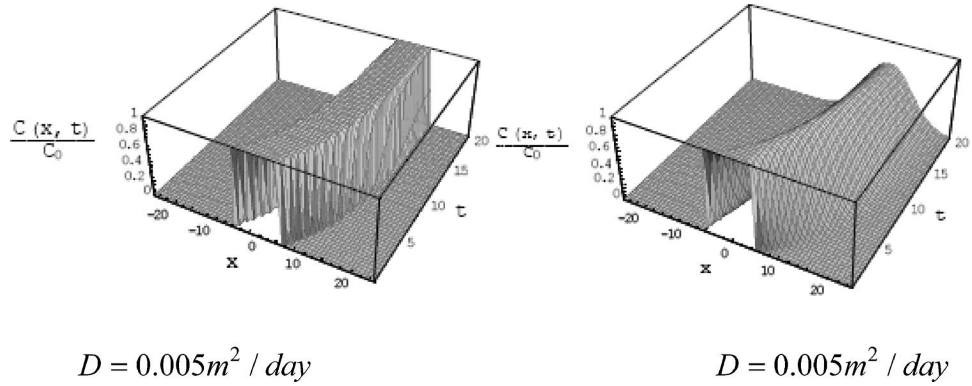

Fig. 10 The concentration profiles for the advective–diffusive transport of a plug of chemical in a porous column due to a velocity field that decays exponentially with time

diffusive transport problem in a column is to assume that, insofar as the advective–diffusive transport problem is concerned, the domain is semi-infinite. For example, for the problem involving a *constant advective flow velocity* v_0, the solution to the advective–diffusive transport in a semi-infinite porous medium with initial and boundary conditions defined by Eq. (75) is given by

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\left[\operatorname{erfc}\left(\frac{x - v_0 t}{2\sqrt{Dt}}\right) + \exp\left(\frac{x v_0}{D}\right)\operatorname{erfc}\left(\frac{x + v_0 t}{2\sqrt{Dt}}\right)\right] \quad (76)$$

where $\operatorname{erfc}(\alpha)$ is the complimentary error function defined in terms of the error function $\operatorname{erf}(\alpha)$ by

$$\operatorname{erfc}(\alpha) = 1 - \operatorname{erf}(\alpha) = 1 - \frac{2}{\sqrt{\pi}}\int_0^{\alpha} \exp(-\varsigma^2)d\varsigma \quad (77)$$

An extension of the solution to the problem involving an advective velocity with an exponential decay in time is not routine and can be achieved only through Laplace transform techniques with numerical inversion. A simpler result can be obtained by considering the *initial value problem* applicable to a domain, which is infinite. Assuming that the chemically-dosed plug region occupies the interval $x \in (-a, a)$, the initial condition can be written as

$$C(x,0) = C_0[\mathsf{H}(x + a) - \mathsf{H}(x - a)]; \quad x \in (-\infty, \infty) \quad (78)$$

Selvadurai [267] generalized the classical solution to the problem involving constant advective flow velocity to account for the time dependency in the flow velocity with an exponential form, and gave the analytical solution to the resulting advective–diffusive initial value problem defined by Eqs. (74) and (78) in the form

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\operatorname{erf}\left(\frac{-x + a + v_0\dfrac{\{1 - \exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{-x - a + v_0\dfrac{\{1 - \exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right); \quad x < v_0 t \quad (79a)$$

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\operatorname{erfc}\left(\frac{x - a - v_0\dfrac{\{1 - \exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right) - \frac{1}{2}\operatorname{erfc}\left(\frac{x + a - v_0\dfrac{\{1 - \exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right); \quad x \geqslant v_0 t \quad (79b)$$

and $\operatorname{erfc}(\alpha)$ is defined by Eq. (77).

This solution provides a useful analytical result that is amenable to convenient computation (Fig. 10) and has been used to establish the accuracy of the computational schemes developed for the study of the purely advective transport problem [268–270]. Figure 11 shows the comparison of experimental, analytical, and computational results, obtained by the modified least squares (MLS) scheme, for the normalized concentration distribution, C/C_0, in the column with falling potential head over the plane of dimensionless variables $X(=x/l)$ and $T(=kt/l)$. Recently, Selvadurai [271] has shown that exact closed form results can also be

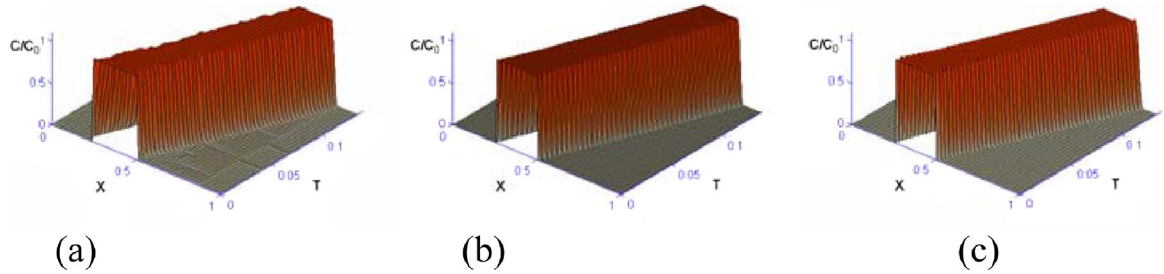

Fig. 11 A comparison of results for the time-dependent movement of a plug of chemical dye in a fluid-saturated porous column: (*a*) experimental results; (*b*) analytical results; and (*c*) numerical results derived from the MLS method

developed for the class of problems where spheroidal cavities, located in fluid-saturated porous media of infinite extent, are subjected to boundary potentials that decay exponentially with time. This category of problem is of interest to the modelling of contaminant migration during deep geological disposal of hazardous and industrial wastes from lined boreholes and hydraulically created fractures [272].

5 Plasticity and Geomechanics

The theory of plasticity has contributed significantly to the development of solutions to practical problems in geomechanics. A review of this nature would not be complete without a mention of the advances that have been made in the use of constitutive models developed within in the general framework of plasticity theories [273–280] for the development of analytical solutions. By the very nature of the nonlinearities that are inherent in the plasticity formulations, the development of analytical solutions is restricted to problems with extremely simplified geometries and loading configurations. Excluding homogeneous states of deformation, the development of analytical solutions tends to concentrate on cavity expansion problems involving both radial and spherical symmetry. The practical utility of the radially symmetric plane strain problem serves as a useful analog for the study of the pressuremeter and tunnel problems and the problem with spherical symmetry has application to the study of flow of granular materials in hoppers and as an approximate model for estimating capacity of deep foundations. The solutions dealing with incremental plasticity approaches to cavity expansion problems date back to the classical solutions by Chadwick [281], Cox et al. [282], and others and a comprehensive review of the topic is given by Hopkins [283]. The applications of these developments to problems of interest to geomechanics commences with the work of Gibson and Anderson [284], Ladanyi [285], Salençon [286], Vesic [287], Palmer [288], Baligh [289], and further references are given by Selvadurai [290,291] and Yu [292].

An early development of the application of critical state models to the study of the pressuremeter problem is due to Davis et al. [293]. These authors developed an analytical solution for the cavity expansion problem from a zero initial radius (e.g., pile driven into a soil). A rate-type hypoelastic model that captures the essential features of the critical state concepts is used to solve the plane strain problem. Although the influence of pore pressure development is considered, the transient dissipation effects are not. An interesting and complete study of the quasi-static expansion of finite cylindrical and spherical cavities in an elastic-plastic medium which satisfies the Mohr–Coulomb yield criterion and a nonassociated flow rule was presented by Bigoni and Laudiero [294]. The study also contains estimates for the limit pressure and comparisons with the analytical results derived by Chadwick [281]. The problem of the internal loading of cylindrical and spherical cavities in dilatant soil regions of infinite extent was examined by Yu and Houlsby [295]. An infinite power series expansion technique was used to develop the cavity pressure expansion relationships for an elasto-plastic material with a Mohr–Coulomb failure criterion and a nonassociated flow rule. These authors extended the analysis to develop analytical solutions for cavity contraction problems, taking into consideration Cauchy stresses and logarithmic strains of the *Hencky type* [296]. The problem of the expansion of a cavity in sand under drained conditions has been examined by Collins et al. [297]. Collins and Stimpson [298] have also developed similarity solutions for drained and undrained expansion of cylindrical and spherical cavities in soils that satisfy a rate-type constitutive relationship. The undrained cavity expansion problems for a soil that satisfies a critical state model has been examined by Collins and Yu [299]. Papanastasiou and Durban [300] examined the large-strain elasto-plastic analysis of a cylindrical cavity under radial pressure, taking into consideration a variety of phenomena including a nonassociated formulation involving both Mohr–Coulomb and Drucker–Prager failure criteria and hardening effects. Similar analyses for a spherical cavity are given by Durban and Fleck [301]. Cao et al. [302] have examined the undrained cavity expansion problem using a modified Cam clay model as the basis for the incremental analysis. Comparisons are made with existing solutions in the literature. Giraud et al. [303] studied undrained cavity contraction problems, taking into consideration a Mohr–Coulomb failure criterion and a nonassociated flow rule. Certain authors (Bolton and Whittle [304]; Cao et al. [305]) have also considered the problem of the cavity expansion problem using a nonlinear elasticity approach to describe the preyield deformation behavior. Solutions have been obtained for the loading history of the cavity boundary displacements with pressure, the undrained conditions allowing the evaluation of the large strains under conditions of radial symmetry [288,290,291]. These are by no means generalized theories of finite strain elasto-plastic behavior that accounts for product decompositions of the elastic and plastic deformation gradients to construct the correct measures for the development of elastic and plastic finite strains.

Both the cylindrical and spherical cavity problems have received extensive attention in connection with the idealized modeling of hopper flow associated with granular materials. One of the constitutive models that has been used quite extensively in this connection is the double shearing theory originally proposed by Spencer [306,307] for incompressible materials and extended by Mehrabadi and Cowin [308]. Spencer and Bradley [309] applied the theory to study the fully developed gravity flows of granular materials in contracting cylinders and tapered tubes. Hill and Cox [310–312] used the basic double shearing theory to develop exact parametric solutions for granular flows in a converging wedge and for the determination of force distributions in sand piles. Recently, Spencer [313] developed a solution to the problem of the shear loading of a granular layer under compression. Further references to the extensive application and development of analytical solutions to problems involving the double shearing theory of Spencer to both nondilatant and dilatant granular materials can be found in the article by Hill and Selvadurai [280]. More recently, the theory of hypoplasticity has been used to develop analytical solutions to problems of practical interest [314]. In particular, cavity problems for a hypoplastic granular material have been discussed by Hill [315].

6 Conclusions

The analytical approach in its classical sense has been a powerful impetus for the development of the subject of geomechanics. Concisely presented analytical solutions have several useful functions: first and foremost, they provide the geotechnical engineer with the tools to examine plausible engineering solutions to what are undoubtedly very complex problems in geomechanics and to assess more conveniently the issue of geotechnical parameter variability. Second, they provide the computational modeler with valuable benchmarking tools that will assist in the verification of the capabilities and reliability of computational approaches. By and large, such validations involve solutions to linear problems, which should be regarded as a prelude to embarking on more complicated exercises involving nonlinearities. The analytical approaches also assist in the identification of mathematical intricacies that may be glossed over in computational treatments with the result that interesting phenomena are not discovered and mathematical consistency is not rigorously enforced. The analytical method must also be recognized as a vital part of the pedagogical aspect of the education of researchers and practitioners of geomechanics.

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