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AN APPROXIMATE ANALYSIS OF AN INTERNALLY LOADED ELASTIC PLATE CONTAINING AN INFINITE ROW OF CLOSELY SPACED PARALLEL CRACKS

A. P. S. SELVADURAI
Department of Civil Engineering, Carleton University, Ottawa, Canada

Abstract—The stress transfer which occurs in an internally loaded infinite elastic plate containing an array of closely spaced parallel cracks of finite width is examined. The internal loading corresponds to a doublet of concentrated forces which act at finite distances from the cracked region. The solution presented is approximate to the extent that the state of stress in the strip regions contained between adjacent cracks is considered to be one-dimensional. Such a simplification enables the derivation of certain general results for the stress distribution in a strip region contained within internally loaded half-planes of differing elastic characteristics. These solutions are obtained by Fourier transform methods. Attention is particularly focussed on the estimation of the stress magnification which occurs in the strip region.

NOTATION

b crack length
c crack spacing
x, y Cartesian coordinates
$w_q(y)$ surface displacement of the halfplane in the x-direction due to normal surface traction
$\bar{w}_q(\xi)$ Fourier cosine transform of $w_q(y)$
G linear elastic shear modulus
ν Poisson's ratio
a length parameter
$q(y)$ tensile traction; stress distribution in the strip region
$\bar{q}(\xi)$ Fourier cosine transform of $q(y)$
P_1, P_2 concentrated forces
$w_p(y)$ surface displacement of halfplane due to internal load
$\bar{w}_p(\xi)$ Fourier cosine transform of $w_p(y)$
k one-dimensional stiffness constant
E_s elastic modulus of the strip region
Γ non-dimensional stiffness parameter
h location of internal loading
$Ei(-\lambda)$ exponential integral function
$E_1(\lambda)$ modified exponential integral function
s_o stress magnification factor
λ, δ substitution parameters
ξ integration parameter

1. INTRODUCTION

THE STRESS analysis of plates weakened by an infinite row of periodic parallel cuts or cracks of finite length has been the subject of many investigations relating to fracture mechanics[1]. The problems relating to the shear and longitudinal flexibility of such plates were first investigated by Koiter[2, 3] using asymptotic methods for the solution of the associated elasticity problem. Subsequent treatments of these problems have also been presented by England and Green[4], Lowengrub[5] and Delameter and Hermann[6] who employ rigorous analyses for the solution of the governing integral equations. In a majority of these investigations, the analysis of the crack problem is greatly facilitated by the spatial symmetry of the external loading. Such symmetry enables the treatment of the periodic crack problem to be effectively restricted to the analysis of a strip region containing a single crack.

This paper is primarily concerned with the stress analysis of an internally loaded homogeneous plate which is weakened by an infinite row of very closely spaced cuts or cracks. In particular the internal loading corresponds to a pair of concentrated forces which are directed away from the layer of cracks. In order to achieve a certain degree of generalization, however, we consider the case of a composite plate (Fig. 1) containing an infinitely long region of closely spaced parallel cracks of finite width. The strip regions between the cracks are connected to two half-plane regions of differing elastic characteristics. The half-plane regions are separately subjected to a collinear system of concentrated forces each located at a finite distance from the

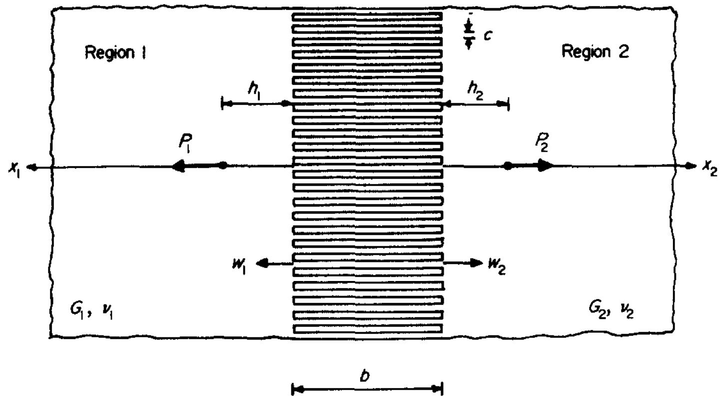

Fig. 1. Geometry and loading of the composite plate [suffixes 1 and 2 refer to the halfplane regions 1 and 2 respectively].

half-plane boundary. In the case of the system of unequal forces it is assumed that the composite plate is held "at infinity" to preserve overall equilibrium of the plate. The method of analysis presented in this paper is a departure from the classical integral equation methods of analyses mentioned earlier. Here, it is explicitly assumed that the geometric aspect ratio of the crack and its periodic spacing is such that the state of stress in the material region contained between two adjacent cracks corresponds to a state of uniaxial stress. The accuracy of this approximation will invariably depend upon the crack length to spacing ratio, b/c (Fig. 1); as b/c becomes small the effects of stress concentrations which occur at the connective regions will invalidate any assumption pertaining to the one-dimensional state of stress. In this paper we assume that the ratio c/b is such that the action of the strip region forming the cracks on the separate half-plane regions 1 and 2 can be approximated by an equivalent normal force. As a consequence, the strip region contained between the cracks can be suitably approximated by a region of spring elements (or a Winkler element) with independent mechanical action. A problem of related interest has been examined by Goodier and Kanninen[7] who employed Winkler elements to account for finite non-linear cohesive forces which occur in the cleavage crack propagation problem. The stress distribution in the strip region can be derived by considering the compatibility of displacements between the strip elements and the adjoining elastic half-planes. The analysis of the interaction problem is facilitated by the application of Fourier transform techniques[8]. Integral expressions are developed for the distribution of stress in the strip region; this stress distribution can in turn be used to determine the stress and displacements in the half-plane regions. The approximate method of analysis of the plate containing an infinite row of parallel cracks considered here, enables the derivation of certain compact results of practical interest with regard to the stress distribution in the strip region. Admittedly, the accuracy and the range of validity of such modelling can only be verified by recourse to further analytical, numerical and/or experimental treatment of the title problem. The basic problem of the weakened plate considered in this paper is of interest in connection with the behaviour of structural elements designed to absorb energy or transmit limited loads during impact or earthquake loading. Alternatively, the strip region can be visualized as a region of closely spaced stringers of greater stiffness connecting the two plate regions.

2. GENERALIZED ANALYSIS OF A COMPOSITE PLATE PROBLEM

In this section we examine the problem relating to an internally loaded composite plate which consists of two half-plane regions 1 and 2 of differing elastic characteristics and loaded in the manner described earlier (Fig. 1). In order to analyze the composite plate problem it is convenient to develop solutions related to two subproblems: namely, the surface loading of a half-plane by a symmetric distribution of normal surface traction and the Melan problem[9], corresponding to the interior loading of a half-plane by a concentrated force which acts at a finite distance from and normal to a traction free plane boundary. The solution of these two plane stress problems can be approached in a variety of ways [see 9–11]. It is, however, convenient to adopt the Fourier transform development of the two subproblems. We are particularly interested in the deformational response of the half-plane surface when subjected to these separate loads.

In considering the surface loading of the half-plane, it can be shown that the transformed value of the surface displacement of the half-plane in the x-direction, $u(o, y)$ $(= w_q(y))$, due to a tensile normal traction $q(y)$ is given by

$$\bar{w}_q(\xi) = -\frac{a\bar{q}(\xi)}{G(1+\nu)\xi} \tag{1}$$

where a is a typical length parameter of the problem and G and ν are, respectively, the linear elastic shear modulus and Poisson's ratio of the elastic material. In (1) $\bar{w}_q(\xi)$ denotes the Fourier cosine transform of $w_q(y)$ defined as [8]

$$\bar{w}_q(\xi) = F_c\{w_q(y); \xi\} = \int_0^\infty w_q(y)\cos(\xi y/a)\,d\xi. \tag{2}$$

Similarly, $\bar{q}(\xi)$ denotes the Fourier cosine transform of the normal traction $q(y)$. The corresponding Fourier inversion theorem is

$$w_q(y) = F_c^{-1}\{\bar{w}_q(\xi); y\} = \frac{2}{\pi a}\int_0^\infty \bar{w}_q(\xi)\cos(\xi y/a)\,d\xi. \tag{3}$$

The problem of a concentrated force (P) acting at the interior of an isotropic elastic half-plane with a traction free boundary was first solved by Melan [6] and later by Sneddon [12]. For this problem, the transformed value of the surface displacement $w_P(y)$, in the x-direction can be written in the form

$$\bar{w}_P(\xi) = \frac{a\bar{s}(\xi)}{G(1+\nu)\xi} \tag{4}$$

where

$$\bar{s}(\xi) = \frac{P}{2}\left\{1 + \frac{(1+\nu)}{2}\frac{\xi h}{a}\right\}e^{-\xi h/a}. \tag{5}$$

It may be noted that $q(y)$ has units of stress while P has units of force per unit length (Alternatively $P = P^*/t$ where P^* is the total load and t is the plate thickness.) The results developed above can now be applied to analyze the problem of the composite plate. The transformed value of the surface displacement of the half-plane resulting from the combined action of the surface traction and the internal concentrated force is obtained by combining (1) and (4). Identifying the stress distribution in the strip region (or the Winkler layer) as $q(y)$, the transformed expressions for the surface displacements of the elastic half-planes 1 and 2, denoted by $\bar{w}_1(\xi)$ and $\bar{w}_2(\xi)$, respectively, can be reduced to the generalized form

$$\bar{w}_n(\xi) = \frac{a}{G_n(1+\nu_n)\xi}\,[\bar{s}_n(\xi) - \bar{q}(\xi)];\ (n = 1, 2) \tag{6}$$

where G_n; ν_n $(n = 1, 2)$ are the elastic parameters of the half-planes respectively. Also,

$$\bar{s}_n(\xi) = \frac{P_n}{2}\left[1 + \frac{(1+\nu_n)}{2}\frac{\xi h_n}{a}\right]e^{-\xi h_n/a};\ (n = 1, 2) \tag{7}$$

are the equivalent representations of the effect of the Melan force given by (5), for regions 1 and 2 respectively. Considering the one-dimensional deformation of the strips in the cracked region we obtain the following relationship between the transformed variables $\bar{q}(\xi)$ and $\bar{w}_n(\xi)$;

$$\bar{q}(\xi) = k[\bar{w}_1(\xi) + \bar{w}_2(\xi)] \tag{8}$$

where $k = E_s/b$; E_s is the elastic modulus of the strip material and b is the crack length. The elimination of $\bar{w}_n(\xi)$ between (6) and (8) leads to a relationship between $\bar{q}(\xi)$ and $\bar{s}_n(\xi)$; applying the inversion theorem (3) to this result yields the following expression for the stress distribution $q(y)$ in the strip region

$$\frac{q(y)}{ka} = \frac{2}{\pi G_1(1+\nu_1)a} \int_0^\infty \frac{[\bar{s}_1(\xi) + \Gamma \bar{s}_2(\xi)]}{\left[\xi + \dfrac{ka(1+\Gamma)}{G_1(1+\nu_1)}\right]} \cos(\xi y/a)\, d\xi \tag{9}$$

where

$$\Gamma = \frac{G_1(1+\nu_1)}{G_2(1+\nu_2)}. \tag{10}$$

Similarly, the expressions for the surface displacements in the respective half-plane regions can be presented in the contracted form

$$\frac{w_m(y)}{\{2ka^2/\pi G_m(1+\nu_m)\}} = \int_0^\infty \frac{\left\{\dfrac{\{\bar{s}_m(\xi) - \bar{s}_n(\xi)\}}{\xi a G_n(1+\nu_n)} + \dfrac{\bar{s}_m(\xi)}{ka^2}\right\}}{\left[\xi + ka\left\{\dfrac{1}{G_m(1+\nu_m)} + \dfrac{1}{G_n(1+\nu_n)}\right\}\right]} \cos(\xi y/a)\, d\xi. \tag{11}$$

Expressions for the surface displacement of the separate half-plane regions 1 and 2 are recovered by assigning $m = 1$, $n = 2$ and $m = 2$, $n = 1$ in (11) respectively. In addition, the stress $q(y)$ can be directly employed to generate the stress and displacement fields in the two halfspace regions. Furthermore, the results developed here for the concentrated internal loadings can be utilized to generate solutions for other types of distributed internal loadings.

3. THE CRACKED HOMOGENEOUS PLATE

The results developed in the preceding section can now be utilized to examine the state of stress in a homogeneous isotropic elastic plate, weakened by an infinite array of closely spaced parallel cracks. The plate is loaded by a set of concentrated forces of equal magnitude (P) which are equidistant (h) from the edge of the cracked region. The stress distribution in the strip region is obtained by assigning $G_n = G$, $\nu_n = \nu$, $h_n = h$, $P_n = P$ and $k = 2G(1+\nu)/b$ in (9). Further, by setting the length parameter $a = h$, the stress distribution in the cracked region can be reduced to the form

$$q(y) = \frac{2P(1+\nu)}{\pi b} \int_0^\infty \frac{[\xi + \delta]}{[\xi + \lambda]} e^{-\xi} \cos(\xi y/h)\, d\xi \tag{12}$$

where

$$\delta = \frac{2}{(1+\nu)} \quad \lambda = 4\left(\frac{h}{b}\right). \tag{13}$$

The infinite integrals of the type (12) do not appear to reduce to any integrals known from the literature. The value of the stress at the origin $q(0)$, however, can be evaluated (Gradshteyn and Ryzhik[13]) in terms of the exponential integral function $Ei(-\lambda)$ as follows:

$$q(0) = [\sigma_{xx}(0,0)]_{\text{cracked}} = \frac{2P(1+\nu)}{\pi b}[1 + (\lambda - \delta) e^{\lambda} Ei(-\lambda)]. \tag{14}$$

The exponential integral function $Ei(-\lambda)$ is defined as

$$Ei(-\lambda) = -\int_\lambda^\infty \frac{e^{-t}}{t}\, dt = -E_1(\lambda); |\arg \lambda| < \pi \tag{15}$$

and tabulated numerical values for the function $E_1(\lambda)$ are given by Pagurova[14]. The

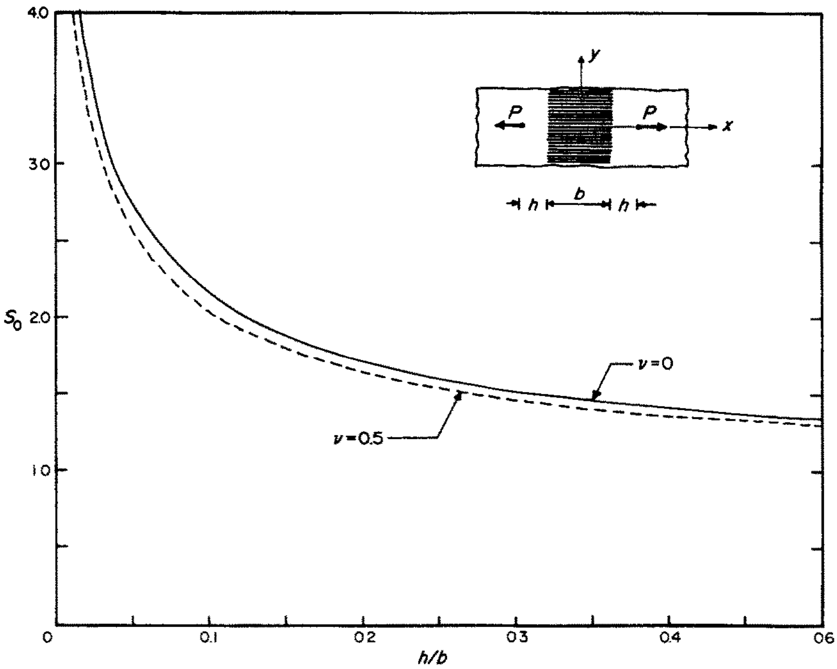

Fig. 2. The stress magnification factor, S_0, for an internally loaded homogeneous plate

$$\left[S_0 = \frac{[\sigma_{xx}(0,0)]_{\text{cracked plate}}}{[\sigma_{xx}(0,0)]_{\text{uncracked plate}}}\right].$$

numerical evaluation of the integral (12) itself can be performed using a Gauss–Legendre or another appropriate quadrature to generate the stress distribution in the strip region.

4. NUMERICAL RESULTS

The manner in which the presence of a layer of closely spaced cracks influences the stress in the infinite plate can be best illustrated by comparing the stress at the central strip of the cracked plate given by (14) with the equivalent result for the uncracked plate. The equivalent stress component in the uncracked plate $[\sigma_{xx}(0,0)]_{\text{uncracked}}$, due to a doublet of concentrated forces acting at a distance $(2h+b)$ apart is given by

$$[\sigma_{xx}(0,0)]_{\text{uncracked}} = \frac{P(3+\nu)}{\pi b\{1+2(h/b)\}}. \tag{16}$$

The corresponding stress magnification factor "S_0" at the central location due to the presence of the cracked region is given by

$$S_0 = \frac{2(1+\nu)}{(3+\nu)}\left\{1+2\frac{h}{b}\right\}\left[1+\left\{4\frac{h}{b}-\frac{2}{(1+\nu)}\right\}e^{4h/b}Ei\left(-4\frac{h}{b}\right)\right]. \tag{17}$$

From (17), it is evident that the stress magnification factor S_0 becomes singular as the concentrated force migrates to the boundary of the cracked region (i.e. $h\to 0$); the upper limit as $h\to\infty$ cannot be evaluated from (17) owing to the restriction, $|\arg\lambda|<\pi$, imposed on the evaluation of the infinite integral (12) at $y=0$. However, by examining the form of $\bar{s}(\xi)$ we note that $\bar{s}(\xi)\to 0$ as $h\to\infty$; as such the stress in the central region of the cracked plate tends to zero. Similarly, the equivalent stress components, $\sigma_{xx}(0,0)$ in the uncracked plate also tends to zero as $h\to\infty$. The variation of S_0 with h/b is shown in Fig. 2, for values of $\nu=0$ and 0.5. These results indicate that an appreciable stress magnification occurs when the concentrated force is located within a distance of approximately $h<0.4b$ from the edge of the crack region.

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