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International Journal of Mechanical Sciences 48 (2006) 1072–1079

International Journal of Mechanical Sciences

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Surface-stiffened elastic halfspace under the action of a horizontally directed Mindlin force

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Received 26 September 2005; received in revised form 22 April 2006; accepted 24 May 2006
Available online 24 July 2006

Abstract

This paper presents an analytical solution to the problem of the interaction between a thin plate that is adhesively bonded to the surface of an isotropic elastic halfspace and a concentrated Mindlin-type force that acts parallel to the bonded surface. The model is an idealization of a surface-stiffened region that has potential applications ranging from mechanics of thin films, thermal barrier coatings, layering created by attrition and wear of surfaces and functionally graded materials. The solution also illustrates the influence of the flexural plate model in mitigating the development of unbounded displacements during the application of localized loading directly at the bonded plate.

Keywords: Mechanics of thin films; Thermal barrier coatings; Adhesively bonded plate; Internal loading of halfspace; Cerruti's problem; Functionally graded solid

1. Introduction

The interaction between flexible plates and an elastic support constitutes an important topic in contact mechanics with investigations dating back to the works of Euler, Hertz, Winkler, Biot and others. Reviews and expositions of this subject area are given by Korenev [1], Hetenyi [2], Selvadurai [3], Sneddon et al. [4], Gladwell [5] and Johnson [6]. The recent papers by Selvadurai et al. [7] and Selvadurai [8], examine the interaction of a bonded plate and an elastic halfspace. The plate is modelled by a Poisson-Kirchhoff-Germain thin plate theory and the elastic medium is modelled as an isotropic elastic halfspace. This particular model has applications in the study of contact problems of surfaces subjected to wear where the attrition of the surface and the deposition of debris give rise to a layer with stiffer properties (Curnier [9], Hutchings [10]). The application of the plate model to represent the mechanical behaviour of the surface layer is of particular interest to material scientists working in the areas of *thin solid films* and *surface coatings technology* [11–15].

In the studies presented in Refs. [7,8], the flexure of the bonded plate of infinite extent is caused by loads that are applied either directly to the plate or acting at the interior of the elastic halfspace. These studies are, however, restricted to axisymmetric problems and it has been shown that compact closed form estimates can be obtained for results of importance to engineering applications. Other extensions include situations where the interface between the plate and the elastic halfspace can experience bilateral smooth contact, allowing flexural interaction between the stiffening plate and the halfspace, particularly in the absence of the kinematic constraint relating to zero in-plane displacements at the surface of the halfspace. Such solutions are of interest to the modelling of asperity contact between surface-stiffened elastic regions. This form of structural reinforcement of the elastic medium is considered to be only an approximation to the more complicated problem involving the representation of the surface-stiffening region by an elastic layer of finite thickness.

The present paper deals with the problem of a surface-stiffened elastic halfspace region that is subjected to a

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doi:10.1016/j.ijmecsci.2006.05.008

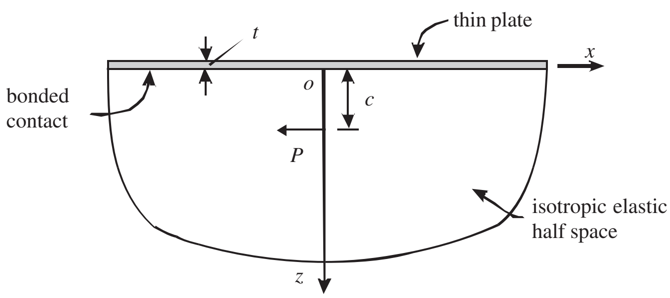

Fig. 1. Plate–halfspace interaction induced by a horizontal internal load.

concentrated force that acts parallel to the plane of the plate and is located at the interior of the halfspace region. This problem extends the classical Mindlin-type problem [16,17] to include the effects of a bonded plate located at the surface of the halfspace (Fig. 1). This example can be solved in a compact form using classical integral transform techniques applicable to asymmetric problems in the theory of elasticity [18–20]. When the interior asymmetric Mindlin-type force migrates to the surface of the halfspace region, the solution to the problem of a surface-stiffened elastic halfspace region under the action of a concentrated force acting in the plane of the surface of the stiffened halfspace region is obtained. This corresponds to the extension of the classical problem of Cerruti [21] (see also Sneddon [22,23]) to include effects of surface stiffening.

2. Asymmetric problem for a surface constrained halfspace

The analysis first considers the problem of the interior loading of an isotropic elastic infinite space by a concentrated force. The solution to this problem was given by Kelvin [24] and the result of interest to this paper concerns the action of a doublet of Kelvin-type forces, each of magnitude P, acting at a separation distance $2c$ as shown in Fig. 2. The developments are referred to a system of cylindrical polar coordinates (r,θ,z), where the components of the displacement vector in the respective directions are denoted by u_r, u_θ and u_z.

Owing to the asymmetry of the deformation field induced by the doublet of Kelvin forces, the plane $z=0$ experiences the displacements

$$u_r(r,\theta,0) = u_\theta(r,\theta,0) = 0, \quad 0\leqslant r<\infty, \quad 0\leqslant\theta\leqslant 2\pi, \tag{1}$$

$$u_z(r,\theta,0) = \frac{Pcr\cos\theta}{8\pi G(1-\nu)(r^2+c^2)^{3/2}}, \quad 0\leqslant r<\infty, \quad 0\leqslant\theta\leqslant 2\pi, \tag{2}$$

where G is the linear elastic shear modulus and ν is Poisson's ratio. As is evident, due to the asymmetry of the deformation induced by the doublet of forces, the in-plane displacements of the elastic medium about the plane $z=0$, are identically zero. This is a kinematic constraint that is imposed by the inextensibility of the bonded Poisson-Kirchhoff-Germain thin plate at the surface of the halfspace. It is further noted that by virtue of the

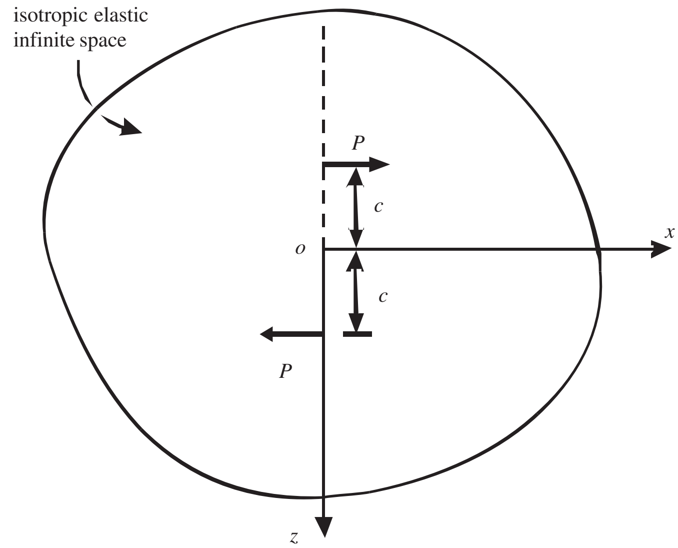

Fig. 2. Doublet of Kelvin forces in an elastic infinite space.

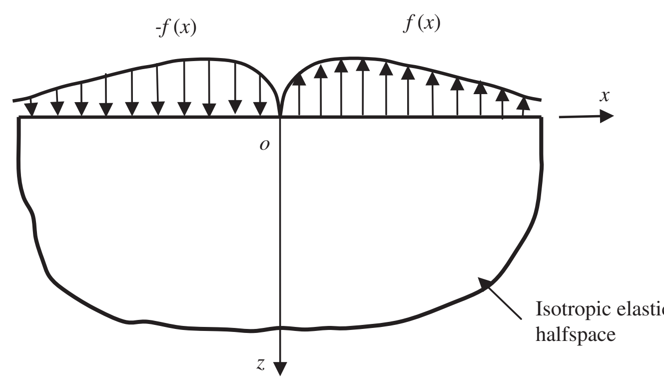

Fig. 3. Asymmetric normal loading of the surface of a halfspace region.

asymmetry of the stress state about the plane $z=0$, the normal stresses acting on the plane satisfy the condition

$$\sigma_{zz}(r,\theta,0) = 0, 0 \quad 0\leqslant r<\infty, 0\leqslant\theta\leqslant 2\pi. \tag{3}$$

Next, consider the problem of the normal loading of the surface of a halfspace region $(0\leqslant z<\infty)$ by a purely asymmetric normal loading (Fig. 3) while maintaining the inextensibility condition at the surface of the halfspace. Considering the form of the displacement field given by Eqs. (1) and (2), due to the doublet of forces, the boundary value problem associated with the asymmetric normal loading is defined by the displacement boundary conditions (1) and the following traction boundary conditions:

$$u_r(r,\theta,0) = 0, \quad 0\leqslant r<\infty, \quad 0\leqslant\theta\leqslant 2\pi, \tag{4}$$

$$u_\theta(r,\theta,0) = 0, \quad 0\leqslant r<\infty, \quad 0\leqslant\theta\leqslant 2\pi, \tag{5}$$

$$\sigma_{zz}(r,\theta,0) = f(r)\cos\theta, \quad 0\leqslant r<\infty, \quad 0\leqslant\theta\leqslant 2\pi, \tag{6}$$

where $f(r)$ is an arbitrary function.

In addition to these boundary conditions, the solution should also satisfy the regularity conditions that are applicable to the decay of the stress and displacement fields as $r,z\to\infty$. In the absence of body forces, the solution of the problem can be approached by making use of Muki's solution [18] for the equations of elasticity, which can be represented in terms of biharmonic and harmonic functions as follows:

$$\nabla^2\nabla^2\Phi(r,\theta,z)=0,\quad \nabla^2\Psi(r,\theta,z)=0, \tag{7}$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2} \tag{8}$$

is Laplace's operator referred to the cylindrical polar coordinate system. The displacements and the stress component relevant to the solution of the boundary value problem posed by Eqs (4)–(6) can be expressed in terms of the functions $\Phi(r,\theta,z)$ and $\Psi(r,\theta,z)$ in the following forms:

$$2Gu_r=-\frac{\partial^2\Phi}{\partial r\partial z}+\frac{2}{r}\frac{\partial\Psi}{\partial\theta},\quad 2Gu_\theta=-\frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\partial z}-2\frac{\partial\Psi}{\partial r}, \tag{9}$$

$$2Gu_z=2(1-\nu)\nabla^2\Phi-\frac{\partial^2\Phi}{\partial z^2} \tag{10}$$

and

$$\sigma_{zz}=\frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Phi-\frac{\partial^2\Phi}{\partial z^2}\right]. \tag{11}$$

Considering a single term of the Fourier–Hankel series representation of the solutions for $\Phi(r,\theta,z)$ and $\Psi(r,\theta,z)$ we have

$$\Phi(r,\theta,z)=\left\{\int_0^\infty \xi[A(\xi)+z\rightleftarrows B(\xi)]\mathrm{e}^{-\xi z}J_1(\xi r)\,\mathrm{d}\xi\right\}\cos\theta, \tag{12}$$

$$\Psi(r,\theta,z)=\left\{\int_0^\infty \xi C(\xi)\mathrm{e}^{-\xi z}J_1(\xi r)\,\mathrm{d}\xi\right\}\sin\theta, \tag{13}$$

where $A(\xi)$, $B(\xi)$ and $C(\xi)$ are arbitrary functions that can be determined by satisfying the boundary conditions (4)–(6) and ξ is an integration parameter. Substituting Eqs. (12) and (13) in the expressions (9) for $u_r(r,\theta,z)$ and $u_\theta(r,\theta,z)$, it can be shown that the boundary conditions (4) and (5) can be identically satisfied by setting

$$B(\xi)=\xi A(\xi),\quad C(\xi)=0. \tag{14}$$

The integral forms for the axial displacement u_z and normal stress σ_{zz} at the plane $z=0$ can be expressed in the following integral forms, in terms of an unknown function $B(\xi)$, i.e.

$$u_z(r,\theta,0)=\left\{-\frac{(3-4\nu)}{2G}\int_0^\infty \xi^2B(\xi)J_1(\xi r)\,\mathrm{d}\xi\right\}\cos\theta, \tag{15}$$

$$\sigma_{zz}(r,\theta,0)=\left\{2(1-\nu)\int_0^\infty \xi^3B(\xi)J_1(\xi r)\,\mathrm{d}\xi\right\}\cos\theta. \tag{16}$$

Since the applied stress at the surface of the halfspace is prescribed by Eq. (6), we can determine a corresponding result for the axial displacement of the halfspace. It is convenient to adopt a Hankel integral transform representation of the results such that the first-order Hankel transform of $\Omega(r,\theta)$ is defined by (Sneddon [25])

$$\overline{\Omega}^1(\xi,\theta,z)=H_1\{\Omega(r,\theta,z);\xi\}=\int_0^\infty r\Omega(r,\theta,z)J_1(\xi r)\,\mathrm{d}\xi \tag{17}$$

and the overbar refers to a transformed variable. Denoting the axial displacement at the surface of the halfspace due to the applied stress (6) by $w_q(r)\cos\theta$ and the applied normal load by $\hat{q}(r,\theta)=q(r)\cos\theta$, one obtains

$$\bar{u}_z^1(r,\theta,0)=\bar{w}_q^1(\xi)\cos\theta=\int_0^\infty u_z(r,\theta,0)J_1(\xi r)\,\mathrm{d}\xi, \tag{18}$$

$$\bar{\sigma}_{zz}^1(r,\theta,0)=\bar{q}^1(\xi)\cos\theta=\int_0^\infty \sigma_{zz}(r,\theta,0)J_1(\xi r)\,\mathrm{d}\xi. \tag{19}$$

The self-reciprocity properties of Hankel transforms and the results (15)–(19) give the following:

$$\bar{w}_q^1(\xi)=\frac{(3-4\nu)}{4G(1-\nu)\xi}\bar{q}^1(\xi). \tag{20}$$

Considering the surface displacement of the halfspace under the action of the interior horizontal force P given by Eq. (2) and denoting the surface displacement by $w_q(r)\cos\theta$, it can be shown that

$$\bar{w}_P^1(\xi)=\frac{(3-4\nu)}{4G(1-\nu)\xi}\bar{s}^1(\xi), \tag{21}$$

where

$$\bar{s}^1(\xi)=\frac{Pc\xi\exp(-\xi c)}{2\pi(3-4\nu)}. \tag{22}$$

The first-order Hankel transform of the combined axial displacement $w(r)\cos\theta$ of the kinematically constrained halfspace, which is due to the action of both the external normal load $q(r)\cos\theta$ and the horizontally directed concentrated force P, can be obtained by combining Eqs. (20) and (21), i.e.

$$\bar{w}^1(\xi)=\frac{(3-4\nu)}{4G(1-\nu)\xi}\left[\bar{q}^1(\xi)+\bar{s}^1(\xi)\right]. \tag{23}$$

3. The interaction between the bonded thin plate and the elastic halfspace

Now consider the problem of the flexural interaction between the Poisson-Kirchhoff-Germain thin plate and the isotropic elastic halfspace under complete bonded contact conditions at the interface. In particular the *in-plane displacements* of the interface are constrained to be *zero*. This is an approximation that is satisfactory for the flexural response associated with the thin plate governed by the small strain-small deflection assumptions. For bonded contact conditions, the axial surface displacement of the kinematically constrained halfspace is equal to the deflection of the plate. The flexure of the plate is induced by the action of an external surface load $\hat{p}(r,\theta)$ and a contact normal stress, which can be set equal to the stress $\hat{q}(r,\theta)$ applied to the surface of the halfspace. The partial differential equation governing the flexural deflection of the thin plate $\hat{w}(r,\theta)$ is [3,26,27]

$$D\hat{\nabla}^2\hat{\nabla}^2\hat{w}(r,\theta)+\hat{q}(r,\theta)=\hat{p}(r,\theta), \tag{24}$$

where $D(=E_{\rm p}t^3/12(1-v_{\rm p}^2))$, is the flexural rigidity of the plate, $E_{\rm p}$ and $v_{\rm p}$ are the elastic constants for the plate material, t is the plate thickness and

$$\hat{\nabla}^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{25}$$

The symbols with the tilde refer to quantities applicable to the plate configuration. It is assumed that $\hat{w}(r,\theta)$ and $\hat{q}(r,\theta)$ admit representations of the form $w(r)\cos\theta$ and $q(r)\cos\theta$, respectively, and for the purposes of the analysis for the title problem, the external load $\hat{p}(r,\theta)$ is set to zero. Substituting these representations in Eq. (24) and operating on the resulting equation with the first-order Hankel transform yields the following result:

$$D\xi^4\bar{w}^1(\xi)+\bar{q}^1(\xi)=0. \tag{26}$$

The function $\bar{q}^1(\xi)$ can be eliminated between Eqs. (23) and (26) to obtain an expression for $\bar{w}^1(\xi)$. Inverting this result and using the representation $\hat{w}(r,\theta)=w(r)\cos\theta$, the following expression is obtained for the deflection of the plate:

$$\frac{\hat{w}(r,\theta)}{P/8\pi G(1-v)t}=\left\{\beta\int_0^\infty\frac{\zeta\exp(-\zeta\beta)J_1(\zeta\rho)\,{\rm d}\zeta}{(1+R^*\zeta^3)}\right\}\cos\theta=\omega(\rho,\beta)\cos\theta, \tag{27}$$

where

$$R^*=\left(\frac{E_{\rm p}}{E}\right)\left[\frac{(3-4v)(1+v)}{24(1-v)(1-v_{\rm p}^2)}\right],\quad \beta=\frac{c}{t},\quad \rho=\frac{r}{t}, \tag{28}$$

where R^* is a relative stiffness parameter for the plate–elastic halfspace system. Expressions similar to (27) can be obtained for other quantities, such as the contact pressures $\hat{q}(r,\theta)$ at the interface and the flexural moments in the plate region. The stress state in the halfspace region can also be evaluated using the contact stress state $\hat{q}(r,\theta)$ and the shear tractions derived through the application of the inextensibility criterion (1). The expressions for the flexural moments and shear resultants in the plate region

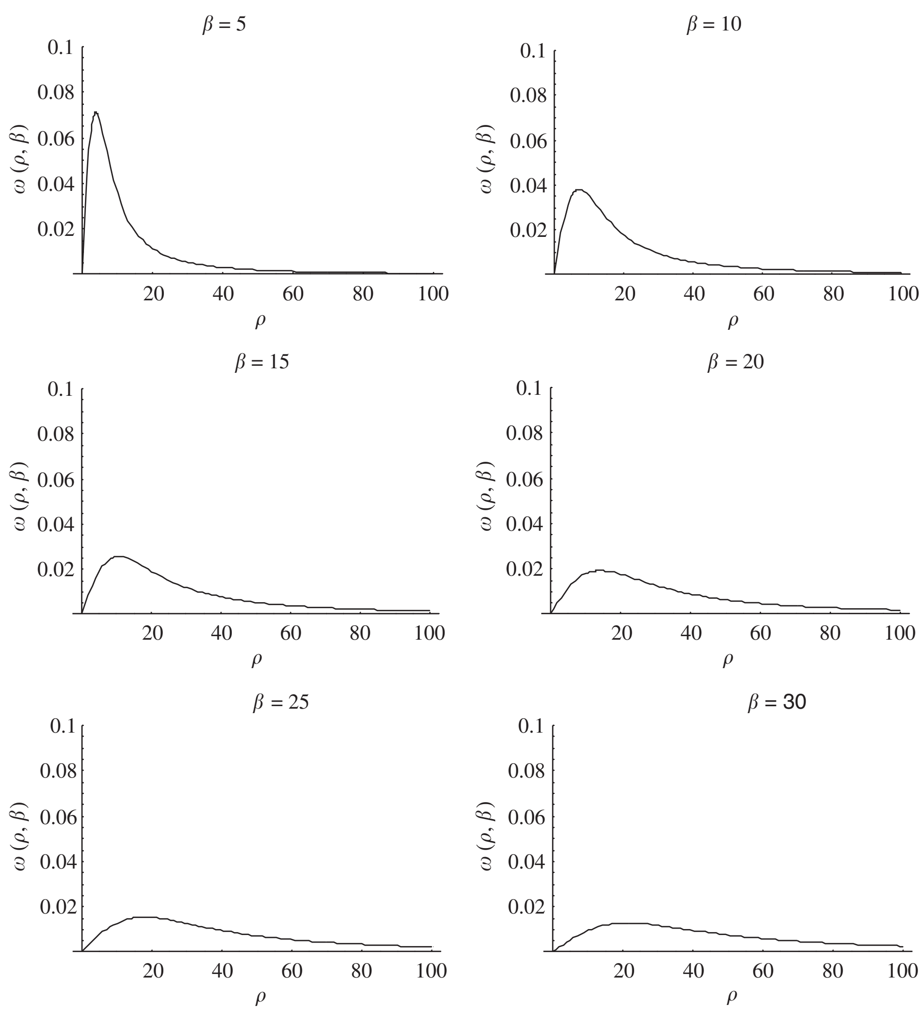

Fig. 4. Vertical deflection of the surface-stiffened halfspace ($R^*=1$).

can be obtained by making use of the result (27) in the following expressions [27]:

$$M_r = -D\left[\frac{\partial^2 \hat{w}}{\partial r^2} + \nu_p\left(\frac{1}{r}\frac{\partial \hat{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \hat{w}}{\partial \theta^2}\right)\right], \tag{29}$$

$$M_\theta = -D\left[\frac{1}{r}\frac{\partial \hat{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \hat{w}}{\partial \theta^2} + \nu_p\frac{\partial^2 \hat{w}}{\partial r^2}\right], \tag{30}$$

$$M_{r\theta} = (1-\nu_p)D\left[\frac{1}{r}\frac{\partial^2 \hat{w}}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \hat{w}}{\partial \theta}\right], \tag{31}$$

$$Q_r = -D\left[\frac{\partial}{\partial r}(\tilde{\nabla}^2 \hat{w}(r))\right], \tag{32}$$

$$Q_r = -D\left[\frac{1}{r}\frac{\partial}{\partial \theta}(\tilde{\nabla}^2 \hat{w}(r))\right]. \tag{33}$$

The analysis of the interaction between the horizontally oriented Mindlin-type problem for a halfspace with a bonded flexural surface constraint is formally complete. The validity of the solution is also based on the applicability of Kirchhoff's uniqueness theorem for problems in linear elasticity [26,28–30]. It is unlikely that compact analytical solutions can be derived for all results of engineering interest; in most instances, recourse must be made to the evaluation of the infinite integrals encountered in the analytical developments using numerical techniques.

4. Limiting cases and numerical results

Attention is focused on the evaluation of the integral relationship (27) for the deflection of the thin plate under the action of the horizontally oriented concentrated force P acting at a finite distance from the surface of a halfspace stiffened with a bonded thin plate. Prior to performing any numerical evaluations, it is instructive to consider certain limiting cases associated with the result (27).

(i) In the instance when $R^* \to \infty$, the problem reduces to that of the action of the internal concentrated force in a halfspace with a rigid surface. In this case, the deflection $\hat{w}(r,\theta) \equiv 0$, over the entire surface.
(ii) When $\tilde{r} \to \infty$, the result will determine the deflection of the plate at a large distance from the point of

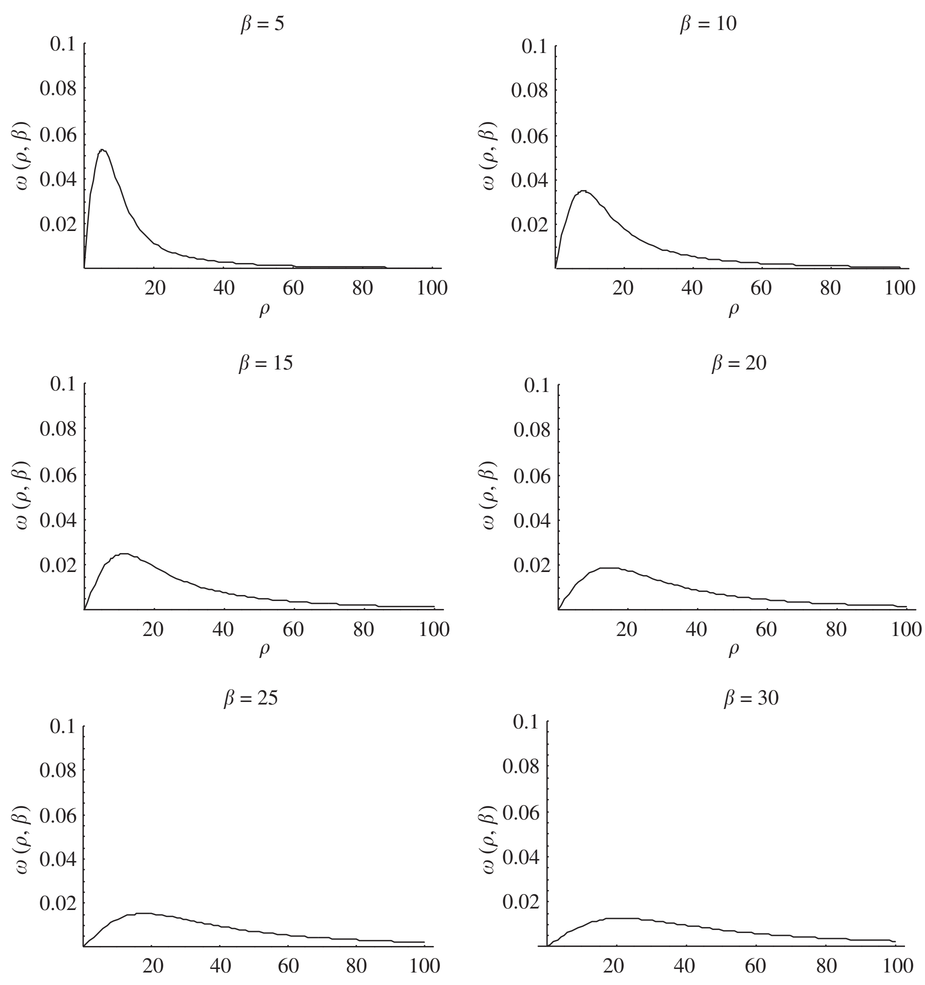

Fig. 5. Vertical deflection of the surface-stiffened halfspace ($R^* = 10$).

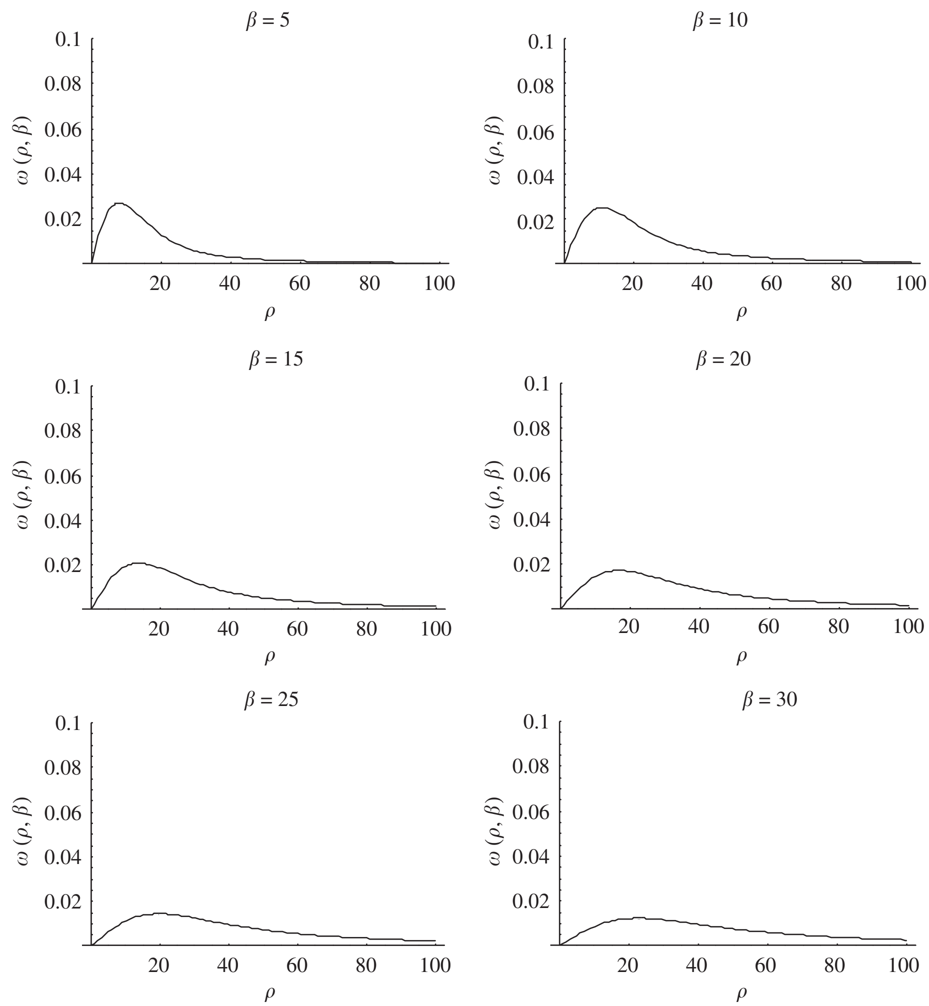

Fig. 6. Vertical deflection of the surface-stiffened halfspace ($R^* = 100$).

application of the concentrated force. In this case, since $J_1(\rho\tilde{r}) \to 0$ as $\tilde{r} \to \infty$, the deflection $\hat{w}(r, \theta) \to 0$. This result is consistent with Saint-Venant's principle in classical elasticity theory [26,29,30].

(iii) When the concentrated force is located remote from the plate–halfspace interface, $\tilde{c} \to \infty$ and, consequently, $\tilde{c}\exp(-\rho\tilde{c}) \to 0$. Here again, the deflection of the plate reduces to zero, a result that is consistent with Saint-Venant's principle.

(iv) The most interesting observation pertains to the case where the concentrated horizontal force is located at the plate–elastic halfspace interface. In this case $\tilde{c} \to 0$, and we observe that $\hat{w}(r, \theta) \to 0$. This conclusion can also be inferred from the result (2), which indicates that the axial displacement of the halfspace reduces to zero as $\tilde{c} \to 0$. This limiting result clearly indicates the importance of kinematic constraints of the type (1) in suppressing the displacement in the axial direction. For any non-zero value of $\tilde{c}$, the plate will experience a deflection. In the case of the classical Cerruti problem, the concentrated force P acts on an otherwise traction-free surface of a halfspace. In this case, the surface displacement of the halfspace is given by

$$u_z(r, \theta, 0) = \frac{P(1 - 2v)\cos\theta}{4\pi G r} \tag{34}$$

and the axial surface displacement will be non-zero, except when $v = 1/2$.

Attention will now be focused on developing certain numerical results for the deflection of the plate defined by Eq. (27). Since the dependence on θ is explicit, attention needs to be focused only on the evaluation of the integral term within the brackets in Eq. (27). From considerations of asymmetry of the deformation and from regularity considerations, it is evident that $\hat{w}(0, \theta) = \hat{w}(\infty, \theta) \equiv 0$. Therefore we shall consider the evaluation of $\hat{w}(r, 0)$ in the range $r \in (0, \infty)$. The non-dimensional parameters governing the integral term in Eq. (27) are ρ, β and R^*. Numerical results are presented in Figs. 4–6, for the variation of $\omega(\rho, \beta)$ with ρ, for specific values of the parameter β, the parameter governing the position of the concentrated force normalized with respect to the thickness of the plate and for a range of values of the relative stiffness parameter R^*. For example, the relative stiffness parameter $R^* = 1$ corresponds to a plate–substrate modular ratio of approximately 20. It appears that the infinite integral in Eq. (27) cannot be evaluated in explicit form. The integral can, however, be evaluated using a standard numerical integration routine for oscillatory integrands available in most mathematical software. The numerical results presented in Figs. 4–6 were evaluated using the numerical integration routines available in MATHEMATICA$^{\mathrm{TM}}$. The results clearly indicate the trends that can be gleaned from an inspection of the limiting cases.

5. Concluding remarks

An analytical solution can be obtained for the flexural behaviour of a thin plate bonded to an isotropic elastic halfspace, under the action of a concentrated force that acts at the interior of the halfspace and parallel to the plane of the interface. The modelling of the bonded plate by a Poisson-Kirchhoff-Germain thin plate theory places a strong kinematic constraint on the plane of bonded contact. The vanishing of the in-plane displacements at the bonded plane is a consistent requirement for the inextensibility of the neutral plane of a thin plate. The kinematic compatibility of the bonded condition is enforced between the surface of the halfspace and the neutral plane of the plate. As a result, when the internal horizontal force is located at the surface of the halfspace, the plate experiences no flexural displacements, and by virtue of the kinematic constraints, all in-plane displacements are also zero. In terms of the contact problem, it appears that when contact occurs between surface and stiffened solids, the dominant mode of interaction is derived through the normal tractions as opposed to the shear tractions. This work can be extended to the case where the inextensibility constraint is relaxed; in this case an improved theory of plates that also accommodates in-plane extensions needs to be considered.

Acknowledgements

One of the authors (APSS) is grateful to the Max Planck Gesellschaft, Berlin, for the award of the *2003 Max Planck Forschungspreis in the Engineering Sciences*, which enabled this collaborative work, and to the *Lehrstuhl für Technische Mechanik,* Friedrich-Alexander-Universität Erlangen-Nürnberg, for its kind hospitality during the visit. The authors are also grateful to the reviewers for their constructive suggestions.

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