Gravity-driven advective transport during deep geological disposal of contaminants

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[1] The paper presents a mathematical model for the study of the advective transport problem where gravity-induced entry of chemicals takes place from open regions of lined boreholes and fracture-type cavities that are deeply located in porous geological media. A characteristic feature of a gravity driven deep geological disposal concept is the presence of time-dependent advective flow velocities in the system. It is shown that certain exact closed-form solutions can be developed to describe the gravity-driven transport of a contaminant into a fluid-saturated host porous geologic medium. **Citation:** Selvadurai, A. P. S. (2006), Gravity-driven advective transport during deep geological disposal of contaminants, *Geophys. Res. Lett.*, *33*, L08408, doi:10.1029/2006GL025944.

1. Introduction

[2] The safe geological disposal of hazardous materials and other contaminants is a problem of importance to the general public as well as to the geosciences and geoenvironmental engineering communities. The continental deep geological disposal of such materials, including liquefied nuclear waste, biological wastes, industrial wastes involving heavy metals, hazardous chemicals, has always been an option that has been proposed for environmental management [*Testa*, 1994; *Fuenkajorn and Daemen*, 1996; *Apps and Tsang*, 1996]. Currently, disposal of any type of wastes in deep oceanic sediments and in polar Antarctic ice sheets is prohibited by International Agreements. The modes of deep geological disposal advocated usually depend on the species of hazardous material and contaminants under consideration. For example, in the context of high-level nuclear waste, disposal of liquefied waste in boreholes at depths of 1 to 2 km has been put forward as a possible option [*Chapman and McKinley*, 1987]. At more moderate depths of up to 1 km, the disposal of contaminants that exhibit innate time-dependent natural decay is considered to be an environmentally benign solution for the management of such contaminants. In deep borehole disposal schemes, the contaminant in a solution form is allowed to seep into a porous host formation through apertures that are either located in lined boreholes or from cavities that are created by hydraulic fracturing. In the conventional analysis of advective transport of liquefied hazardous wastes, it is assumed that the advective flow velocities are time-independent. Time-dependent velocity fields can be encountered in advective transport problems due to a variety of influences including, time-dependent variations in the boundary potentials governing advective flow, effects of thermally driven convection, pressure transients in the flow domain resulting from compressibility of the pore fluid and/or the porous skeleton and fully coupled poroelasticity phenomena. The objective of this paper is to present a model that can be employed to examine the advective migration of a fluidized contaminant that is introduced into a deeply located porous formation under the influence of a gravity-driven flow velocity. In particular, the contaminant is introduced into a fluid-saturated porous formation through deep boreholes, with entry points located either at the base of lined boreholes (Figure 1) or entry points created by hydraulic fracturing of the fluid-saturated porous formation. A characteristic feature of gravity driven transport is the exponentially time-dependent variation in the advective flow velocities in the porous domain. This changes the basic form of the classical advective transport problem. It is found that when the contaminant entry point is approximated by a spheroidal shape, the advective transport induced by an exponentially decaying hydraulic potential can be solved in an exact closed form. Such analytical solutions are presented for cavity regions with both prolate and oblate spheroidal shapes.

2. Governing Equations

[3] We consider the advective transport in a fluid-saturated porous medium, which is induced by Darcy flow. Attention is restricted to an isotropic medium, although the approaches adopted could be extended to include transversely isotropic media. The reduced Bernoulli potential, which consists of the pressure and datum heads, is denoted by $\varphi(\mathbf{x})$ where $\mathbf{x}$ is a position vector. The advective flow velocity vector $\mathbf{v}$ in the porous medium is given by

$$\mathbf{v} = -k\nabla\varphi \tag{1}$$

where k is the Dupuit-Forchheimer measure of the hydraulic conductivity that is related to the conventional Darcy measure of hydraulic conductivity $\tilde{k}$, through the relationship $k = \tilde{k}/n^*$, where n^* is the porosity of the porous medium. Considering (1) and mass conservation, it can be shown that the fundamental equation governing incompressible flow of the fluid through a non-deformable porous medium is

$$\nabla^2\varphi(\mathbf{x}) = 0 \tag{2}$$

where ∇^2 is Laplace's operator. The governing equation (2) can be solved by prescribing suitable Dirichlet and Neumann boundary conditions applicable to a particular flow

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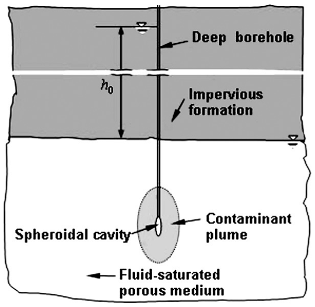

Figure 1. Gravity-driven advective transport from a deep borehole.

problem. We now consider the advective movement of a contaminant that moves through the pore space of the saturated medium. For purposes of the development of an elementary model, we assume that there is no mismatch in the fluid densities and viscosities of both the resident fluid and the contaminant that is introduced by gravity effects. By considering the conservation of mass of the contaminant in the pore space it can be shown that [see, e.g., *Bear*, 1972; *Banks*, 1994; *Charbeneau*, 1999; *Selvadurai*, 2000]

$$\frac{\partial C}{\partial t} + \nabla.(\mathbf{v}\ C) = 0 \tag{3}$$

where $C(\mathbf{x}, t)$ is the time- and space-dependent concentration of the chemical within the pore space. The general equations (1)–(3) are presented in dyadic form so that the relevant equations applicable to the spheroidal coordinate systems can be easily developed.

3. Transport From a Prolate Spheroidal Cavity

[4] We consider the problem of a prolate spheroidal cavity that is deeply located in a fluid-saturated porous medium. The gravity-driven injection to the porous medium takes place from a deep borehole, which is filled with the contaminant (Figure 1). At any instant, the potential inducing flow is denoted by φ. The resulting boundary value problem is governed by

$$\nabla^2 \varphi(\alpha, \beta) = 0 \tag{4}$$

where $0 \leq \alpha < \infty$; $0 \leq \beta \leq \pi$, are the prolate spheroidal coordinates. In (4)

$$\nabla^2 = h_p^2 \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} + \coth\alpha \frac{\partial}{\partial \alpha} \cot\beta \frac{\partial}{\partial \beta} \right) \tag{5}$$

is Laplace's operator referred to the same coordinate system with

$$h_p = \left[c_p^2 \left(\sinh^2 \alpha + \sin^2 \beta \right) \right]^{-1/2} \tag{6}$$

$$c_p = \sqrt{a_p^2 - b_p^2}; \tag{7}$$

$$a_p^2 = c_p^2 \cosh^2 \alpha_0; b_p^2 = c_p^2 \sinh^2 \alpha_0 \tag{8}$$

where $2a_p$ and $2b_p$ are respectively, the major and minor axes of the prolate spheroidal surface $\alpha = \alpha_0$, through which the contaminant enters the porous medium. The boundary condition governing the potential problem is

$$\varphi(\alpha_0, \beta) = h \tag{9}$$

where h can be identified as the differential head in the deep borehole that induces the gravity driven flow at any instant. In addition, the solution should satisfy the regularity condition applicable to three-dimensional potential problems associated with an infinite space; i.e.,

$$\varphi(\alpha, \beta) \rightarrow 0 \text{ as } \alpha \rightarrow \infty \tag{10}$$

Using results of potential theory [*Hobson*, 1931; *Moon and Spencer*, 1988; *Selvadurai*, 1976, 2003, 2004a] it can be shown that the relevant solution to the boundary value problem is

$$\varphi(\alpha) = \frac{h}{\ln \xi_0} \ln \xi \tag{11}$$

where

$$\xi = \left(\frac{\cosh\alpha + 1}{\cosh\alpha - 1} \right); \xi_0 = \xi(\alpha_0) \tag{12}$$

The corresponding velocity vector is given by

$$\mathbf{v}(\alpha, \beta) = \frac{2kh}{c_p \ln \xi_0 \sinh\alpha \sqrt{\sinh^2, \alpha + \sin^2 \beta}} \mathbf{i}_\alpha \tag{13}$$

and $\mathbf{i}_\alpha$ is the base vector.

[5] We now consider the transient effects of the potential problem, which results from the time-dependent variation in the potential corresponding to the head h. We assume that the depth of the fluid in the borehole decreases by dh in time dt. Considering mass conservation during this reduction in height

$$\left[\int\!\!\int_{S^*} \mathbf{vn} dS \right] dt = -A_0 dh \tag{14}$$

where S^* corresponds to the cavity surface α_0; $\mathbf{n}$ is the outward unit normal to the surface and A_0 is the cross sectional area of the borehole. It is implied by (14) that the cavity flow takes place over the entire surface α_0. A consequence of this assumption is that the cross-sectional area of the borehole is considered to be much smaller than the surface area of the prolate spheroidal cavity. Performing the integrations it can be shown that the head in the borehole varies with time according to the relationship

$$h = h_0 \exp(-\eta t) \tag{15}$$

where

$$\eta = \left(\frac{8\pi c_p k}{A_0 \ln \xi_0} \right) \tag{16}$$

and h_0 can be considered as the height of the contaminant in the deep borehole at the start of the disposal activity. Also, this implies that the cavity is completely filled with the contaminant at the start of the gravity-driven advective flow.

[6] The advective transport equation (3) can now be reduced to the form

$$\frac{\partial C}{\partial t}+\frac{2kh_0e^{-\eta t}}{c_p^2\sinh\alpha\left(\sinh^2\alpha+\sin^2\beta\right)\ln\xi_0}\frac{\partial C}{\partial\alpha}=0 \qquad (17)$$

The exponential variation in the advective velocity is characteristic of gravity-driven advective transport processes. A simple one-dimensional initial boundary value problem resulting in advective flow in a porous column due to a falling head was investigated by *Selvadurai* [2004b]. The problem examined here is more general, in that axisymmetric advective transport in a three-dimensional context is considered. We assume that the contaminant enters the porous medium through the surface $\alpha = \alpha_0$ at a constant concentration C_0 and that at time $t = 0$, the porous medium is contaminant free. The solution of the partial differential equation (17) is subject to the boundary condition

$$C(\alpha_0,\beta,t)=C_0\mathrm{H}(t) \qquad (18)$$

where H(t) is the Heaviside step function of time, and the initial condition

$$C(\alpha,\beta,0)=0 \qquad (19)$$

which ensures that at the start of the advective process, the porous medium is free of any contaminants. The solution of the initial boundary value problem defined by (17) to (19) can be approached in a variety of ways, including solution by the methods of characteristics and through the use of Laplace transform techniques. Here we use a trial solution approach, with the requirement that the solution to the advective transport problem with a time-dependent flow velocity converges uniformly to the solution of the advective transport problem involving time-independent flow velocities as $\eta\rightarrow 0$. The appropriate solution is sought in the general form

$$C(\alpha,\beta,t)=C_0\mathrm{H}\left[f(t)-\Omega_p(\alpha,\beta,\lambda)\right] \qquad (20)$$

where

$$\Omega_p(\alpha,\beta,\lambda)=\frac{c_p^2\ln\xi_0}{6kh_0}\left[\cosh^3\alpha-\cosh^3\alpha_0 -3\cos^2\beta\{\cosh\alpha-\cosh\alpha_0\}\right] \qquad (21)$$

and

$$\lambda=b_p/a_p \qquad (22)$$

The expression (21) can also be expressed in terms of the cylindrical coordinates r and z. Substituting (20) into (17) we obtain a first-order ordinary differential equation for $f(t)$, which can be integrated by imposing the requirement for uniform convergence to the stated limiting case. The resulting final solution to the advective transport from a prolate spheroidal cavity with a gravity-driven exponential time-dependency in the velocity field can be evaluated in the form

$$C(\alpha,\beta,t)=C_0\mathrm{H}\left[\eta^{-1}\{1-e^{-\eta t}\}-\Omega_p(\alpha,\beta,\lambda)\right] \qquad (23)$$

The validity of the solution (23) is established by appeal to a uniqueness theorem applicable to the advective-diffusive transport problem [*Selvadurai*, 2004c]. Since the result (23) is in an exact closed-form, it can be easily evaluated to generate the required numerical results. In particular, results for an elongated needle-shaped disposal cavity (Figure 1) located in a fluid-saturated porous medium can be examined by setting $\lambda \ll 1$. Similarly, the limit $\lambda\rightarrow 1$, corresponds to the case of a spherical cavity.

4. Transport From an Oblate Spheroidal Cavity

[7] The procedure outlined in the preceding section can also be extended to cover the case where the advective transport takes place, with an exponentially decaying time-dependency, from the boundary of an oblate spheroidal cavity with major axis $2b_o$ and minor axis $2a_o$. Avoiding details, it can be shown that the time- and space-dependent distribution of the chemical concentration in the porous medium is given by

$$C(\alpha,\beta,t)=C_0\mathrm{H}\left[\tilde{\eta}^{-1}\left\{1-e^{-\tilde{\eta}t}\right\}-\Omega_o(\alpha,\beta,\mu)\right] \qquad (24)$$

where

$$\Omega_o(\alpha,\beta,\mu)=\frac{c_o^2\cot^{-1}(\sinh\alpha_0)}{3kh_0}\left[\sinh^3\alpha-\sinh^3\alpha_0 +3\cos^2\beta(\sinh\alpha-\sinh\alpha_0)\right] \qquad (25)$$

$$c_o^2=b_o^2-a_o^2;\quad \mu=a_o/b_o \qquad (26)$$

$$b_o^2=c_o^2\cosh^2\alpha_0;\, a_o^2=c_o^2\sinh^2\alpha_0 \qquad (27)$$

and

$$\tilde{\eta}=\left(\frac{4\pi c_o k}{A_0\cot^{-1}(\sinh\alpha_0)}\right) \qquad (28)$$

It may also be noted that as $\lambda\rightarrow 1$ and $\mu\rightarrow 1$, in (16) and (28), respectively, both expressions converge to the result applicable to the problem for the gravity-driven advective flow from a spherical cavity. The result (24) can, however, be used to estimate the gravity-driven migration of the contaminant from a flat circular opening that is created by hydraulic fracture of the fluid-saturated porous medium. Such fractures are maintained in an open condition by using a highly porous proppant.

5. Concluding Remarks

[8] It is shown that the advective transport problems resulting from gravity-driven incompressible Darcy flow

from spheroidal cavities located in non-deformable porous media can be obtained in exact closed form. These exact solutions can be used for preliminary calculation of contaminant migration due to gravity-driven disposal at large depths in either boreholes or flat circular fracture-type features. The availability of analytical solutions enables the convenient preliminary evaluation of variability in the hydraulic properties of the porous medium and the geometry of the disposal cavity on the spatial configuration of the contaminated migration zone. The exact solutions also provide valuable benchmarking tools for the computational modelling of advection-dominated transport in porous media, which are non-routine and require time-and mesh-adaptive procedures to generate stable solutions, particularly at discontinuous fronts associated with purely advective flows [*Selvadurai and Dong*, 2006].

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References

Apps, J. A., and C.-F. Tsang (Eds.) (1996), *Deep Injection Disposal of Hazardous and Industrial Waste*, Elsevier, New York.

Banks, R. B. (1994), *Growth and Diffusion Phenomena: Mathematical Frameworks and Applications*, Springer, New York.

Bear, J. (1972), *Dynamics of Fluids in Porous Media*, Mineola, New York.

Chapman, N. A., and I. G. McKinley (1987), *The Geological Disposal of Nuclear Waste*, John Wiley, Hoboken, N. J.

Charbeneau, R. (1999), *Groundwater Hydraulics and Pollutant Transport*, Prentice-Hall, Upper Saddle River, N. J.

Fuenkajorn, K., and J. K. K. Daemen (1996), *Sealing of Boreholes and Underground Excavations*, Springer, New York.

Hobson, E. W. (1931), *The Theory of Spherical and Ellipsoidal Harmonics*, Cambridge Univ. Press, New York.

Moon, P., and D. E. Spencer (1988), *Field Theory Handbook*, Springer, New York.

Selvadurai, A. P. S. (1976), The load-deflexion characteristics of a deep rigid anchor in an elastic medium, *Geotechnique*, *26*, 603–612.

Selvadurai, A. P. S. (2000), *Partial Differential Equations in Mechanics,* vol. 1, *Fundamentals, Laplace's Equation, Diffusion Equation, Wave Equation*, Springer, New York.

Selvadurai, A. P. S. (2003), Contaminant migration from an axisymmetric source in a porous medium, *Water Resour. Res.*, *39*(8), 1203, doi:10.1029/2002WR001442.

Selvadurai, A. P. S. (2004a), Fluid intake cavities in stratified porous media, *J. Porous Media*, *7*, 1–17.

Selvadurai, A. P. S. (2004b), On the advective-diffusive transport in porous media in the presence of time-dependent velocities, *Geophys. Res. Lett.*, *31*, L13505, doi:10.1029/2004GL019646.

Selvadurai, A. P. S. (2004c), On the uniqueness theorem for advective-diffusive transport in porous media: A canonical proof, *Transp. Porous Media*, *56*, 51–60.

Selvadurai, A. P. S., and W. Dong (2006), Numerical modelling of advective transport in the presence of fluid pressure transients, *Int. J. Numer. Anal. Methods Geomech.*, doi:10.1002/nag494, in press.

Testa, S. M. (1994), *Geological Aspects of Hazardous Waste Management*, CRC Press, Boca Raton, Fla.

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