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THE DISPLACEMENTS OF A RIGID DISC INCLUSION EMBEDDED IN AN ISOTROPIC ELASTIC MEDIUM DUE TO THE ACTION OF AN EXTERNAL FORCE

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Introduction

This paper examines the asymmetric interaction between a rigid circular disc inclusion embedded in bonded contact with an isotropic elastic medium of infinite extent and a concentrated force located at the axis of symmetry and at a finite distance from the inclusion. The resultant displacement and rotation of the embedded inclusion are evaluated in exact closed form.

Analysis

The behaviour of rigid disc inclusions embedded in bonded contact with an elastic medium has been examined by several authors including Collins [1], Keer [2,3], Kassir and Sih [4] and Mura and Lin [5]. Solutions for these disc inclusion problems can also be recovered as limiting cases of spheroidal or ellipsoidal inclusions embedded in elastic media. Accounts of such investigations are given by Kanwal and Sharma [6] and Selvadurai [7,8]. In a majority of these investigations, it is usually assumed that the loads are applied directly to the rigid inclusion in a symmetric or asymmetric fashion. The present paper is a departure from these classical treatments in that it is assumed that the external load is located at an exterior point along the axis of geometric symmetry (Fig. 1a). The direction of the force is such that a state of asymmetric deformation exists along the z-axis. In particular, the state of asymmetry is caused by a concentrated force of magnitude 2P located at a finite distance (c) from the inclusion acting along the x-direction. The solution of this problem can be achieved by examining two sub-problems (Fig. 1b and c) which reflect the state of asymmetry of the deformation. By adopting an integral transform formulation, the

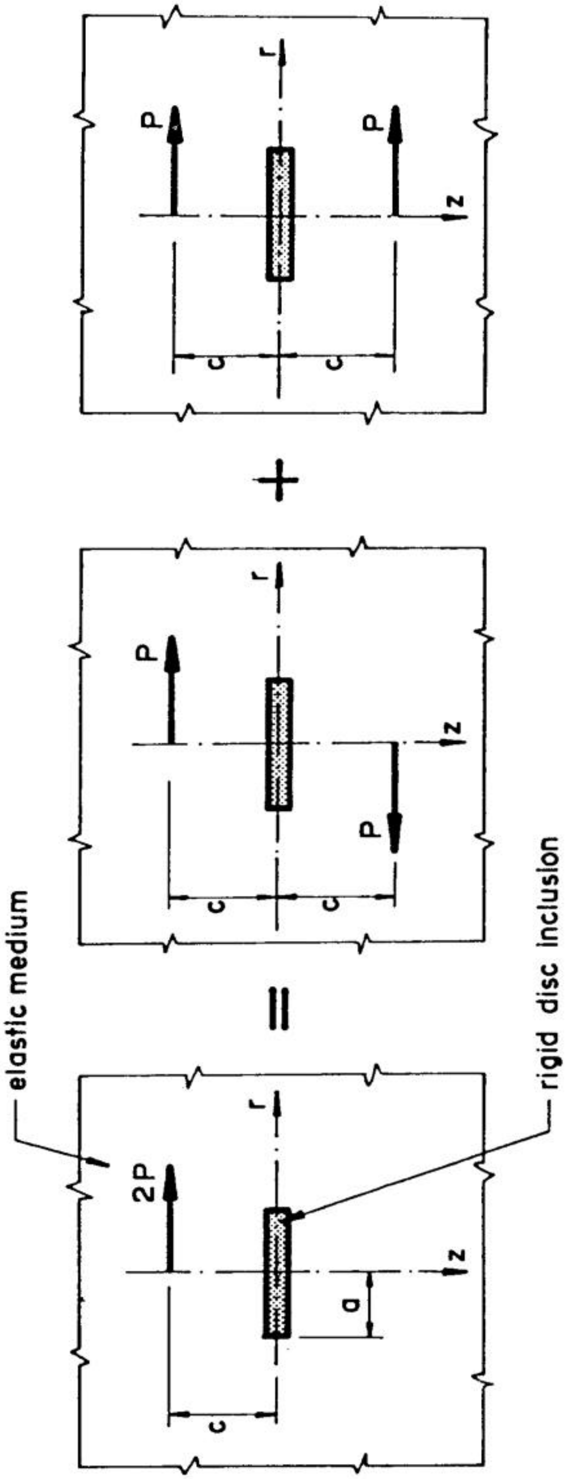

Fig.1. The geometry of the disc inclusion problem

analysis of these sub-problems can be reduced to the solution of sets of dual integral equations the solution of which is facilitated by the generalized results given by Sneddon [9]. The results for the displacement and rotation of the embedded disc inclusion can be evaluated in exact closed form.

In the ensuing we shall consider separately the problems illustrated in Figs. 1b and 1c. The rigid disc inclusion occupies a region $r \leq a$ in the plane z=0. Also, the disc inclusion is assumed to remain in bonded contact with the infinite elastic medium thereby prescribing continuity of displacements at the interfaces $z = 0^+$, $r \leq a$ and $z = 0^-$, $r \leq a$. The subscripts $z = 0^+$ and $z = 0^-$ refer to the faces of the disc inclusion in contact with the regions $z > 0$ and $z < 0$ of the elastic medium, respectively.

<u>Lateral translation of the rigid disc inclusion</u>

We consider first the problem of an inclusion-free infinite medium which is subjected to a doublet of forces P which act in the x-direction at a distance 2c ; the points of action of these forces are located along the z-axis. On account of the symmetry of the problem about the plane z = 0 (Fig. 1c), it is evident that the displacements $u_z(r,\theta,0)$ of the displacement vector $\underset{\sim}{u} = (u_r, u_\theta, u_z)$ is zero. The non zero components of u on the plane z = 0 are

$$u_r(r,\theta,0) = \frac{P\cos\theta}{8\pi G(1-\nu)}\left[\frac{r^2}{(r^2+c^2)^{3/2}} + \frac{(3-4\nu)}{(r^2+c^2)^{1/2}}\right] = u_r^0 \cos\theta \qquad (1a)$$

$$u_\theta(r,\theta,0) = \frac{P\sin\theta}{8\pi G(1-\nu)}\left[\qquad - \frac{(3-4\nu)}{(r^2+c^2)^{1/2}}\right] = u_\theta^0 \cos\theta \qquad (1b)$$

where G and ν are the shear modulus and Poisson's ratio of the elastic medium. We note that the problem related to the translation of the rigid disc inclusion also exhibits a state of symmetry about z = 0. Hence the analysis of the disc inclusion problem can be restricted to a single halfspace region in which the plane z = 0 is subjected to the following mixed boundary conditions :

$$u_r(r,\theta,0) = [\delta - u_r^0]\cos\theta \qquad 0 \leq r \leq a \qquad (2a)$$

$$u_\theta(r,\theta,0) = [-\delta - u_\theta^0]\sin\theta \qquad 0 \leq r \leq a \qquad (2b)$$

and

$$\sigma_{rz}\sin\theta + \sigma_{\theta z}\cos\theta = 0 \qquad r \geq 0 \tag{2c}$$

$$\sigma_{rz}\cos\theta - \sigma_{\theta z}\sin\theta = 0 \qquad a < r \leq \infty \tag{2d}$$

where δ is the net rigid translation of the rigid disc inclusion in the x-direction. The class of problems which displays this particular form of asymmetry can be analysed by employing the stress functions utilized by Muki [10]. These stress functions are governed by the differential equations

$$\nabla^2\nabla^2\Phi(r,\theta,z) = 0 \quad ; \quad \nabla^2\psi(r,\theta,z) = 0 \tag{3}$$

where ∇^2 is Laplace's operator. The stresses and displacements in the medium can be expressed uniquely in terms of these functions (see e.g. Muki [10]). The particular form of the stress functions appropriate for the halfspace region $z > 0$ are

$$\Phi(r,\theta,z) = \{\frac{1}{a^2}\int_0^\infty \xi[C(\xi)+zD(\xi)]e^{-\xi z/a}J_1(\xi r/a)d\xi\}\cos\theta \tag{4a}$$

$$\psi(r,\theta,z) = \{\frac{1}{a^2}\int_0^\infty \xi\, F(\xi)e^{-\xi z/a}J_1(\xi r/a)d\xi\}\sin\theta \tag{4b}$$

where $C(\xi)$, $D(\xi)$ and $F(\xi)$ are arbitrary functions. In order to satisfy the symmetry requirement $u_z(r,\theta,0) = 0$ we require $C(\xi)=-2a(1-2\nu)D(\xi)/\xi$. Similarly, in order to satisfy the traction boundary condition (2c) we require $F(\xi) = \xi C(\xi)(1-\nu)/a(1-2\nu)$. The boundary conditions (2a), (2b) and (2d) are equivalent to the pair of dual integral equations

$$\begin{aligned} &H_0\{\xi^{-1}\Psi_1(\xi)\ ;\ r\} = f_1(r) \quad ; \quad 0 \leq r \leq a \\ &H_0\{\ \ \Psi_1(\xi)\ ;\ r\} = 0 \quad ; \quad a < r < \infty \end{aligned} \tag{5}$$

where

$$\Psi_1(\xi) = \xi^2 D(\xi).$$

The nth order Hankel transform of $\phi(r,\theta,z)$ is defined by

$$\bar{\phi}^n(\xi,\theta,z) = H_n\{\phi(r,\theta,z)\ ;\ \xi\} = \int_0^\infty r\phi(r,\theta,z)J_n(\xi r/a)dr \tag{6}$$

The appropriate Hankel inversion theorem is

$$\phi(r,\theta,z) = H_n^{-1}\{\bar{\phi}^n(\xi,\theta,z)\ ;\ r\} = \frac{1}{a^2}\int_0^\infty \xi\bar{\phi}^n(\xi,\theta,z)J_n(\xi r/a)d\xi \qquad (7)$$

The function f(r) is given by

$$f_1(r) = -\frac{4Ga^3\delta}{(7-8\nu)} + \frac{Pa^3}{4\pi(1-\nu)(7-8\nu)}\left\{\frac{r^2}{(r^2+c^2)^{3/2}} + \frac{2(3-4\nu)}{(r^2+c^2)^{1/2}}\right\} \qquad (8)$$

The general solution of the dual system (5) is given by Sneddon [9] and the details of the method of analysis will not be pursued here. Briefly, by introducing a substitution of the type $D(\xi) = \xi\int_0^a \chi(t)\cos(\xi t/a)dt$, the second equation of the dual system (5) is identically satisfied. Also, the first equation of (5) is reduced to the Schlomilch integral equation [11]. The solution of this integral equation yields

$$D(\xi) = \frac{8Ga^3}{\pi\xi^2(7-8\nu)}\int_0^a \left[-\delta + \frac{Pc}{8\pi G(1-\nu)}\left\{\frac{(3-4\nu)}{(t^2+c^2)} + \frac{t^2}{(t^2+c^2)^2}\right\}\right]\cos(\xi t/a)dt \qquad (9)$$

Formal expressions for the displacement and stress components in the elastic medium can be obtained in integral form. The result of primary interest is the lateral translation of the rigid disc inclusion due to the doublet of forces acting at $z = \pm c$. To develop this result we make use of the expression for the resultant traction (T_x) acting in the z-direction on the plane faces of the inclusion. From the symmetry of the problem

$$T_x(r,\theta,0^+) = T_x(r,\theta,0^-) \qquad (10)$$

Since the resultant force (T) on the inclusion is zero, we require

$$T = \frac{4(1-\nu)}{a^4}\int_0^a\int_{-\pi}^{\pi} r\int_0^\infty \xi^3 D(\xi)J_0(\xi r/a)d\xi dr d\theta = 0 \qquad (11)$$

Evaluating (11) we obtain

$$\delta = \frac{P}{16\pi Ga(1-\nu)}\left[(7-8\nu)\tan^{-1}\left(\frac{a}{c}\right) - \frac{ac}{(a^2+c^2)}\right] \qquad (12)$$

It is evident that as $c \to 0$, the problem reduces to one in which the disc inclusion is subjected to a concentrated force of magnitude 2P acting along the x-direction. In the limit as $c \to 0$, (11) yields the following

$$\delta = \frac{P(7-8\nu)}{32Ga(1-\nu)} \tag{13}$$

This result is in agreement with the expression derived by Keer [2] for the problem of the lateral translation of a centrally loaded rigid disc inclusion. As $c \to \infty$, the inclusion experiences zero rigid displacement.

Rotation of the rigid disc inclusion

The problem related to the rigid rotation of the disc inclusion due to the asymmetric doublet of forces (Fig. 1b) can be analysed in a similar manner. Owing to the asymmetry of the deformation, the displacements in the inclusion-free infinite medium, at the plane $z = 0$, due to the doublet of forces are given by

$$u_r(r,\theta,0) = u_\theta(r,\theta,0) = 0 \tag{14}$$

$$u_z(r,\theta,0) = -\frac{P\,cr\cos\theta}{8\pi G(1-\nu)(r^2+c^2)^{3/2}} = u_z^0\cos\theta \tag{15}$$

Again, the analysis of the inclusion problem can also be referred to a single halfspace region in which the plane $z = 0$ is subjected to the following set of mixed boundary conditions

$$\left.\begin{array}{lll} u_z(r,\theta,0) = \{\Omega r - u_z^0\}\cos\theta & ; & 0 \le r \le a \\ \sigma_{zz}(r,\theta,0) = 0 & ; & a < r < \infty \end{array}\right\} \tag{16}$$

where Ω is the resultant rigid rotation of the disc inclusion. Avoiding details of analysis it can be shown that these mixed boundary conditions are equivalent to the pair of dual integral equations

$$\left.\begin{array}{lll} H_1\{\xi^{-1}\Psi_2\,;\, r\} = f_2(r) & ; & 0 \le r \le a \\ H_1\{\Psi_2\,;\, r\} = 0 & & a < r < \infty \end{array}\right\} \tag{17}$$

where

$$f_2 = -\frac{2Ga^4\Omega r}{(3-4\nu)} + \frac{Pa^4cr}{4\pi(1-\nu)(3-4\nu)(r^2+c^2)^{3/2}} \tag{18}$$

and

$$\Psi_2(\xi) = \xi^3 D(\xi)$$

The solution of the dual system (17) can be obtained from the generalized results given by Sneddon [11]. The relationship between the doublet of forces and the resulting rigid rotation Ω can be obtained by evaluating the resultant moment (M) on the inclusion. Since the inclusion is moment free, we require

$$M = \frac{4(1-\nu)}{a^4}\int_{-\pi}^{\pi}\int_0^a r^2\cos^2\theta \int_0^\infty \xi^4 D(\xi)J_1(\xi r/a)\,d\xi\,dr\,d\theta = 0 \tag{19}$$

Evaluating (19) we obtain

$$\Omega = \frac{3Pc}{16\pi Ga^3(1-\nu)}\left[\tan^{-1}\left(\frac{a}{c}\right) - \frac{ac}{(a^2+c^2)}\right] \tag{20}$$

It may be noted that the rigid rotation of the inclusion reduces to zero both as $c \to 0$ and as $c \to \infty$.

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