SECOND-ORDER ELASTIC EFFECTS IN AN INFINITE PLANE CONTAINING A RIGID CIRCULAR INCLUSION

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SUMMARY

The problems treated in this paper deal with the second-order effects in an incompressible isotropic elastic infinite plane containing a loaded rigid circular inclusion. The analysis is based on the 'displacement function' method used for the solution of the second-order problems for incompressible elastic materials. The displacement function method is particularly suitable for the problems considered, since, in each case, the displacement boundary conditions are prescribed at the rigid inclusion–elastic medium interface. The solutions are presented in closed form, in terms of elementary functions, to a set of problems relevant to the study of stress concentrations, which occur at the fibres of reinforced rubber-like materials and in bonded rubber mountings.

1. INTRODUCTION

The theory of successive approximations has been widely applied to obtain approximate solutions to problems in finite elasticity.[1] In the method of successive approximations we assume that the stress, strain and displacement components may be expressed as power series in terms of a small real parameter, ε, the choice of this parameter depending on the problem under consideration. The non-linear differential equations of finite elasticity for the incompressibility condition, equations of equilibrium and boundary conditions can be reduced to sets of linear equations by considering the corresponding powers of ε. In second-order elasticity we take the theory as far as the second order in ε.

The solution of the second-order problem for an incompressible isotropic elastic material by making use of a 'displacement function' approach has been discussed by Selvadurai and Spencer[2] for axially symmetric problems, and Selvadurai[3] for plane strain problems. By introducing a displacement function we make

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direct use of the incompressibility of the material and reduce the solution of the second-order problem to the solution of an equation of the form $\nabla^4\Psi = g(R, \Theta)$ where ∇^2 is Laplace's operator and $g(R, \Theta)$ depends only on the first-order solution. This method of solution of the second-order problem is particularly convenient when displacement boundary conditions are prescribed.

This paper deals with the application of the displacement function technique to certain problems of an incompressible elastic infinite plane loaded by a rigid circular inclusion. A brief summary of the second-order theory is given in Section 2. In Section 3 the problem of an infinite elastic plane containing a bonded rigid circular inclusion, subjected to a combined force and a couple at its centre is considered while, in Section 4, we consider the problem of a bonded rigid circular inclusion subjected to a force at its centre, and in addition the infinite elastic plane is subjected to a state of uniform biaxial stress at infinity. Section 5 deals with a similar problem; the force at the centre of the rigid circular inclusion is replaced by a couple. Finally, in Section 6, the problem of a cylindrical cavity in an infinite plane containing a smooth oversize circular rigid inclusion is considered. In this case the infinite plane is subjected to a state of uniform biaxial stress at infinity.

In this paper the rigid circular inclusion only is considered; applications of the 'displacement function' technique to problems where the bonded inclusion has elastic characteristics will be treated in a subsequent paper.

2. FUNDAMENTAL EQUATIONS

A detailed derivation of the general theory is given by Selvadurai,[3] and in this section only the relevant results will be briefly mentioned.

(*a*) *Kinematics of deformation*

We consider the motion of a generic particle $P_0(R, \Theta)$ in the undeformed body B_0 which is in a state of zero stress or of uniform hydrostatic stress, whose polar coordinates with reference to a fixed origin 0 are R, Θ. After deformation let the same particle assume a position $P(r, \theta)$ in the deformed body B, where r, θ are the polar coordinates of P with reference to 0.

In plane strain the deformation gradients in the R, Θ directions are given by the matrix:

$$\mathbf{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Theta} \\ r\dfrac{\partial \theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial \theta}{\partial \Theta} \end{bmatrix} \tag{2.1}$$

We consider incompressible elastic materials for which:

$$\det \mathbf{F} = \frac{r}{R}\left(\frac{\partial r}{\partial R}\frac{\partial \theta}{\partial \Theta} - \frac{\partial r}{\partial \Theta}\frac{\partial \theta}{\partial R}\right) = 1 \tag{2.2}$$

We make the assumption that the displacement components u and v in the R and Θ directions, of the displacement vector connecting P_0 and P can be expanded as a power series in terms of the small real dimensionless parameter ε in the form:

$$\begin{aligned} u &= \varepsilon u_1(R, \Theta) + \varepsilon^2 u_2(R, \Theta) + 0(\varepsilon^3) \\ v &= \varepsilon v_1(R, \Theta) + \varepsilon^2 v_2(R, \Theta) + 0(\varepsilon^3) \end{aligned} \tag{2.3}$$

By considering the geometry of the deformation it can be shown (Selvadurai[3]) that in the first-order the incompressibility condition (eqn. (2.2)) reduces to:

$$\frac{\partial u_1}{\partial R} + \frac{u_1}{R} + \frac{1}{R}\frac{\partial v_1}{\partial \Theta} = 0 \tag{2.4}$$

and, in the second-order, to:

$$\begin{aligned} \frac{\partial u_2}{\partial R} + \frac{u_2}{R} + \frac{1}{R}\frac{\partial v_2}{\partial \Theta} &= \left(\frac{\partial u_1}{\partial R}\right)^2 + \frac{\partial v_1}{\partial R}\left(\frac{1}{R}\frac{\partial u_1}{\partial \Theta} - \frac{v_1}{R}\right) \\ &= H(R, \Theta) \end{aligned} \tag{2.5}$$

The first- and second-order components, $\mathbf{B}^{(1)}$ and $\mathbf{B}^{(2)}$ respectively of the Cauchy-Green strain matrix:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \tag{2.6}$$

are given by:

$$\mathbf{B} = \mathbf{I} + \varepsilon\mathbf{B}^{(1)} + \varepsilon^2\mathbf{B}^{(2)} \tag{2.7}$$

where:

$$B_{11}^{(1)} = 2\frac{\partial u_1}{\partial R}; \qquad B_{22}^{(1)} = \frac{2}{R}\frac{\partial v_1}{\partial \Theta} + 2\frac{u_1}{R};$$

$$B_{12}^{(1)} = B_{21}^{(1)} = \frac{1}{R}\frac{\partial u_1}{\partial \Theta} + \frac{\partial v_1}{\partial R} - \frac{v_1}{R}$$

$$B_{11}^{(2)} = 2\frac{\partial u_2}{\partial R} + B^*_{11}; \qquad B_{22}^{(2)} = \frac{2}{R}\frac{\partial v_2}{\partial \Theta} + 2\frac{u_2}{R} + B^*_{22}$$

$$B_{12}^{(2)} = B_{21}^{(2)} = \frac{1}{R}\frac{\partial u_2}{\partial \Theta} + \frac{\partial v_2}{\partial R} - \frac{v_2}{R} + B^*_{12}$$

and

$$B^*_{11} = \left(\frac{\partial u_1}{\partial R}\right)^2 + \left(\frac{1}{R}\frac{\partial u_1}{\partial \Theta}\right)^2 + 2\frac{v_1}{R}\frac{\partial v_1}{\partial R} - \frac{v_1^2}{R^2}$$

$$B^*_{22} = \left(\frac{\partial u_1}{\partial R}\right)^2 + \left(\frac{\partial v_1}{\partial R} - \frac{v_1}{R}\right)^2 - 2\frac{v_1}{R^2}\frac{\partial u_1}{\partial \Theta} + \frac{v_1{}^2}{R^2}$$

$$B^*_{12} = \frac{u_1 v_1}{R^2} + \frac{v_1}{R^2}\frac{\partial v_1}{\partial \Theta} - \frac{v_1}{R}\frac{\partial u_1}{\partial R} + \frac{\partial u_1}{\partial R}\left(\frac{\partial v_1}{\partial R} - \frac{v_1}{R}\right) + \frac{1}{R}\frac{\partial u_1}{\partial \Theta}\left(\frac{1}{R}\frac{\partial v_1}{\partial \Theta} + \frac{u_1}{R}\right)$$

(*b*) *Constitutive equations*

The general constitutive equation for an isotropic incompressible elastic material under plane strain conditions can be expressed in the form:

$$\mathbf{S} = -p\mathbf{I} + \mu\mathbf{B} \tag{2.8}$$

where:

$$\mathbf{S} = \begin{bmatrix} S_{rr} & S_{r\theta} \\ S_{r\theta} & S_{\theta\theta} \end{bmatrix}$$

is the symmetrical Cauchy stress matrix referred to the deformed body, p is an indeterminate scalar pressure, μ is the linear elastic shear modulus and $\mathbf{I}$ is the unit matrix.

By making use of the power series expansions for $\mathbf{S}$ and p, the first- and second-order constitutive equations reduce to:

$$\begin{aligned} \mathbf{S}^{(1)} &= -p_1\mathbf{I} + \mu\mathbf{B}^{(1)} \\ \mathbf{S}^{(2)} &= -p_2\mathbf{I} + \mu\mathbf{B}^{(2)} \end{aligned} \tag{2.9}$$

The components of $\mathbf{S}^{(1)}$ are:

$$\begin{aligned} S_{rr}{}^{(1)} &= -p_1 + \mu\left[2\frac{\partial u_1}{\partial R}\right] \\ S_{\theta\theta}{}^{(1)} &= -p_1 + \mu\left[\frac{2}{R}\frac{\partial v_1}{\partial \Theta} + 2\frac{u_1}{R}\right] \\ S_{r\theta}{}^{(1)} &= \mu\left[\frac{1}{R}\frac{\partial u_1}{\partial \Theta} + \frac{\partial v_1}{\partial R} - \frac{v_1}{R}\right] \end{aligned} \tag{2.10}$$

and the components of $\mathbf{S}^{(2)}$ can be written as:

$$\begin{aligned} S_{rr}{}^{(2)} &= -p_2 + \mu\left[2\frac{\partial u_2}{\partial R}\right] + T_{rr} \\ S_{\theta\theta}{}^{(2)} &= -p_2 + \mu\left[\frac{2}{R}\frac{\partial v_2}{\partial \Theta} + 2\frac{u_2}{R}\right] + T_{\theta\theta} \\ S_{r\theta}{}^{(2)} &= \mu\left[\frac{1}{R}\frac{\partial u_2}{\partial \Theta} + \frac{\partial v_2}{\partial R} - \frac{v_2}{R}\right] + T_{r\theta} \end{aligned} \tag{2.11}$$

where:

$$T_{rr} = \mu B^*_{11}; \qquad T_{\theta\theta} = \mu B^*_{22}; \qquad T_{r\theta} = \mu B^*_{12} \tag{2.12}$$

(*c*) *Equations of equilibrium*

By expressing the differential operators $\partial/\partial r$ and $\partial/\partial\theta$ in terms of power series expansions, the differential equations of equilibrium in the deformed configuration can also be expressed in terms of power series in ε. In the absence of body forces the first-order equations of equilibrium are

$$\frac{\partial S_{rr}^{(1)}}{\partial R} + \frac{1}{R}\frac{\partial S_{r\theta}^{(1)}}{\partial\Theta} + \frac{S_{rr}^{(1)} - S_{\theta\theta}^{(1)}}{R} = 0$$

$$\frac{\partial S_{r\theta}^{(1)}}{\partial R} + \frac{1}{R}\frac{\partial S_{\theta\theta}^{(1)}}{\partial\Theta} + 2\frac{S_{r\theta}^{(1)}}{R} = 0 \qquad (2.13)$$

and the second-order equations of equilibrium are:

$$\frac{\partial S_{rr}^{(2)}}{\partial R} + \frac{1}{R}\frac{\partial S_{r\theta}^{(2)}}{\partial\Theta} + \frac{S_{rr}^{(2)} - S_{\theta\theta}^{(2)}}{R} = N_1(R, \Theta)$$

$$\frac{\partial S_{r\theta}^{(2)}}{\partial R} + \frac{1}{R}\frac{\partial S_{\theta\theta}^{(2)}}{\partial\Theta} + 2\frac{S_{r\theta}^{(2)}}{R} = N_2(R, \Theta) \qquad (2.14)$$

where:

$$N_1(R, \Theta) = \frac{\partial u_1}{\partial R}\frac{\partial S_{rr}^{(1)}}{\partial R} + \frac{1}{R}\left(\frac{\partial v_1}{\partial R} - \frac{v_1}{R}\right)\frac{\partial S_{rr}^{(1)}}{\partial\Theta} - \frac{1}{R}\frac{\partial u_1}{\partial R}\frac{\partial S_{r\theta}^{(1)}}{\partial\Theta} + \frac{1}{R}\frac{\partial u_1}{\partial\Theta}\frac{\partial S_{r\theta}^{(1)}}{\partial R} + \frac{u_1}{R}\frac{(S_{rr}^{(1)} - S_{\theta\theta}^{(1)})}{R}$$

$$N_2(R, \Theta) = -\frac{1}{R}\frac{\partial u_1}{\partial R}\frac{\partial S_{\theta\theta}^{(1)}}{\partial\Theta} + \frac{1}{R}\frac{\partial u_1}{\partial\Theta}\frac{\partial S_{\theta\theta}^{(1)}}{\partial R} + \frac{\partial u_1}{\partial R}\frac{\partial S_{r\theta}^{(1)}}{\partial R} + \frac{1}{R}\left(\frac{\partial v_1}{\partial R} - \frac{v_1}{R}\right)\frac{\partial S_{r\theta}^{(1)}}{\partial\Theta} + 2\frac{u_1}{R^2}S_{r\theta}^{(1)} \qquad (2.15)$$

(*d*) *Displacement functions*

We observe that the first-order incompressibility condition (eqn. (2.4)) can be identically satisfied by the introduction of a displacement function, Ψ_1, such that:

$$u_1 = \frac{1}{R}\frac{\partial\Psi_1}{\partial\Theta}; \qquad v_1 = -\frac{\partial\Psi_1}{\partial R} \qquad (2.16)$$

By making use of the first-order constitutive equations (eqns. (2.10)), the first-order equations of equilibrium (eqns. (2.13)) can be written as:

$$-\frac{\partial p_1}{\partial R} + \frac{\mu}{R}\frac{\partial}{\partial\Theta}[\nabla^2\Psi_1] = 0$$

$$-\frac{1}{R}\frac{\partial p_1}{\partial\Theta} - \mu\frac{\partial}{\partial R}[\nabla^2\Psi_1] = 0 \qquad (2.17)$$

where:

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \Theta^2}$$

By eliminating p_1 and Ψ_1 in turn from eqns. (2.17), we obtain:

$$\nabla^4 \Psi_1 = 0 \tag{2.18}$$

and

$$\nabla^2 p_1 = 0 \tag{2.19}$$

It has been shown (Selvadurai[3]) that, by adopting a second-order displacement function, Ψ_2, such that:

$$u_2 = \frac{1}{R}\frac{\partial \Psi_2}{\partial \Theta} + u'_2$$

$$v_2 = -\frac{\partial \Psi_2}{\partial R} + v'_2 \tag{2.20}$$

where:

$$u'_2 = -\frac{1}{2}\frac{{u_1}^2}{R} - \frac{1}{2}\frac{{v_1}^2}{R} + \frac{v_1}{R}\frac{\partial u_1}{\partial \Theta}$$

$$v'_2 = -v_1 \frac{\partial u_1}{\partial R} \tag{2.21}$$

the second-order incompressibility condition (eqn. (2.5)) is identically satisfied, and the second-order equations of equilibrium (eqns. (2.14)) can be reduced to the form:

$$-\frac{\partial p_2}{\partial R} + \frac{\mu}{R}\frac{\partial}{\partial \Theta}[\nabla^2 \Psi_2] = F_1(R, \Theta)$$

$$-\frac{\partial p_2}{\partial \Theta} - \mu R \frac{\partial}{\partial R}[\nabla^2 \Psi_2] = F_2(R, \Theta) \tag{2.22}$$

where:

$$F_1(R, \Theta) = -\left[\frac{\partial T_{rr}}{\partial R} + \frac{1}{R}\frac{\partial T_{r\theta}}{\partial \Theta} + \frac{T_{rr} - T_{\theta\theta}}{R} + \mu\left(\frac{\partial H}{\partial R} + \nabla^2 u'_2 - \frac{u'_2}{R^2} - \frac{2}{R^2}\frac{\partial v'_2}{\partial \Theta}\right) - N_1(R, \Theta)\right]$$

$$F_2(R, \Theta) = -\left[R\frac{\partial T_{r\theta}}{\partial R} + \frac{\partial T_{\theta\theta}}{\partial \Theta} + 2T_{r\theta} + \mu\left(\frac{\partial H}{\partial \Theta} + R\nabla^2 v'_2 - \frac{v'_2}{R} + \frac{2}{R}\frac{\partial u_2}{\partial \Theta}\right) - RN_2(R, \Theta)\right]$$

By eliminating p_2 and Ψ_2 in turn from eqns. (2.22) we obtain:

$$\nabla^4\Psi_2 = \frac{1}{\mu R}\left[\frac{\partial F_1}{\partial \Theta} - \frac{\partial F_2}{\partial R}\right] \tag{2.23}$$

$$\nabla^2 p_2 = -\left[\frac{\partial F_1}{\partial R} + \frac{F_1}{R} + \frac{1}{R^2}\frac{\partial F_2}{\partial \Theta}\right] \tag{2.24}$$

A solution of eqn. (2.23) consists of a particular integral and solutions to the homogeneous equation $\nabla^4\Psi_2 = 0$. The complete second-order solution is obtained by selecting that solution which will satisfy the appropriate boundary conditions of the problem. To the second-order in ε the displacement and stress components are given by: $u = \varepsilon u_1 + \varepsilon^2 u_2$; $v = \varepsilon v_1 + \varepsilon^2 v_2$; $\mathbf{S} = \varepsilon \mathbf{S}^{(1)} + \varepsilon^2 \mathbf{S}^{(2)}$.

3. INFINITE ELASTIC PLANE CONTAINING A RIGID CIRCULAR INCLUSION SUBJECTED TO A COMBINED FORCE AND COUPLE

The second-order solution to the problem of an infinite elastic plane bounded internally by a bonded rigid circular inclusion and loaded at its centre by a concentrated force has been given by Selvadurai.[3] Here we examine the problem where the rigid circular inclusion is subjected simultaneously to a concentrated force (T) and a couple (G_0) at its centre (Fig. 1). The force T causes a rigid body translation δ of the rigid inclusion in the x-direction and the couple causes a rigid

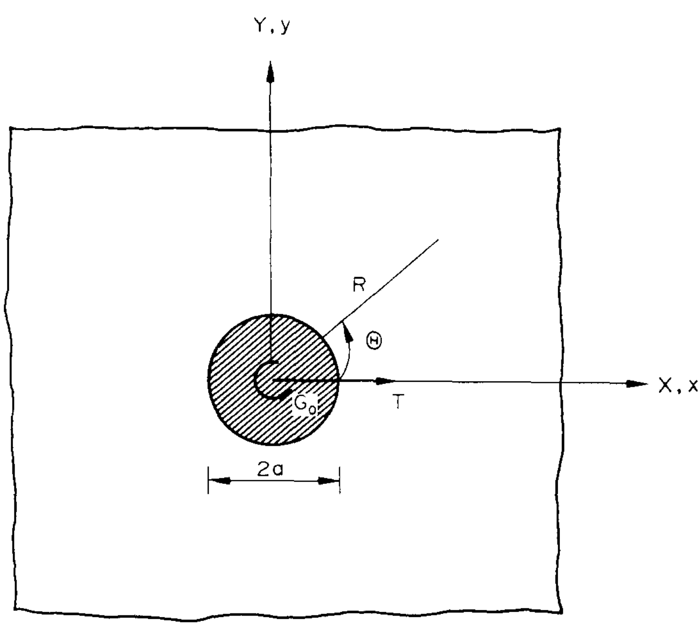

Fig. 1. Rigid circular inclusion subjected to a concentrated force and couple.

body rotation, φ. Perfect bonding between the rigid inclusion and the surrounding elastic medium is assumed, thereby prescribing displacement boundary conditions at the inclusion elastic–medium interface $R = a\,(a > 0)$. We note that, in classical elasticity, on account of the linearity of the field equations the principle of superposition of solutions is applicable. It must be emphasised that, in general, the principle of superposition is not applicable to problems of second-order elasticity theory. This is due to the presence of products, and other non-linear terms of the first-order displacement and stress components in the equations for the second-order displacement and stress components, incompressibility condition and equations of equilibrium. A first-order displacement function of the form:

$$\varepsilon\Psi_1 = \frac{\delta}{a}\left[\left(\frac{a^3}{R} + 2aR\ln\frac{R}{a}\right)\sin\Theta - \lambda_1 a^2\ln\frac{R}{a}\right] \tag{3.1}$$

which satisfies eqn. (2.18) gives the first-order displacement components:

$$\begin{aligned} \varepsilon u_1 &= \frac{\delta}{a}\left(\frac{a^3}{R^2} + 2a\ln\frac{R}{a}\right)\cos\Theta \\ \varepsilon v_1 &= \frac{\delta}{a}\left\{\left[\frac{a^3}{R^2} - 2a\left(1 + \ln\frac{R}{a}\right)\right]\sin\Theta + \lambda_1\frac{a^2}{R}\right\} \end{aligned} \tag{3.2}$$

where:

$$\lambda_1 = \frac{G_0}{4\pi\mu\delta a}$$

is a non-dimensional constant.

The first-order components are:

$$\begin{aligned} \varepsilon\frac{S_{rr}^{(1)}}{\mu} &= \frac{\delta}{a}\left(8\frac{a}{R} - 4\frac{a^3}{R^3}\right)\cos\Theta \\ \varepsilon\frac{S_{\theta\theta}^{(1)}}{\mu} &= \frac{\delta}{a}\left(4\frac{a^3}{R^3}\right)\cos\Theta \\ \varepsilon\frac{S_{r\theta}^{(1)}}{\mu} &= \frac{\delta}{a}\left(-4\frac{a^3}{R^3}\sin\Theta - 2\lambda_1\frac{a^2}{R^2}\right) \end{aligned} \tag{3.3}$$

We note that the first-order stress components (eqns. (3.3)) satisfy the boundary conditions:

$$\begin{aligned} \int_{-\pi}^{\pi}(S_{rr}^{(1)}\cos\Theta - S_{r\theta}^{(1)}\sin\Theta)\,R\,d\Theta &= T \\ \int_{-\pi}^{\pi}(S_{rr}^{(1)}\sin\Theta + S_{r\theta}^{(1)}\cos\Theta)\,R\,d\Theta &= 0 \\ \int_{-\pi}^{\pi}S_{r\theta}^{(1)}R^2\,d\Theta &= G_0 \end{aligned} \tag{3.4}$$

and $S_{rr}^{(1)}$, $S_{\theta\theta}^{(1)}$, $S_{r\theta}^{(1)}$ tend to zero as $R \to \infty$.

We choose the small real dimensionless parameter ε as δ/a. On substituting the expressions for the first-order displacement and stress components (eqns. (3.2) and (3.3)) in eqns. (2.22), the inhomogeneous differential equation for the second-order displacement function (eqn. (2.23)) reduces to:

$$\nabla^4\Psi_2 = F_{11}(R, \Theta) + \lambda_1 F_{12}(R, \Theta) \tag{3.5}$$

where:

$$F_{11}(R, \Theta) = \left(-96\frac{a^6}{R^8} + 8\frac{a^2}{R^4} + 32\frac{a^2}{R^4}\ln\frac{R}{a}\right)\sin 2\Theta$$

$$F_{12}(R, \Theta) = \left(-96\frac{a^5}{R^7} + 16\frac{a^3}{R^5}\right)\cos\Theta \tag{3.6}$$

A particular integral of eqn. (3.5) can be written in the form:

$$\Psi_{2p} = \Psi_{2p}^{(11)} + \lambda_1\Psi_{2p}^{(12)} \tag{3.7}$$

where:

$$\Psi_{2p}^{(11)} = \left[-\frac{1}{4}\frac{a^6}{R^4} + a^2\ln\frac{R}{a} + a^2\left(\ln\frac{R}{a}\right)^2\right]\sin 2\Theta$$

$$\Psi_{2p}^{(12)} = \left(-\frac{1}{2}\frac{a^5}{R^3} - \frac{a^3}{R}\ln\frac{R}{a}\right)\cos\Theta \tag{3.8}$$

The displacement components derived from the particular solution (eqns. (3.7) and (2.20)) are:

$$u_{2p} = u_{2p}^{(11)} + \lambda_1 u_{2p}^{(12)} + \lambda_1^2 u_{2p}^{(22)}; \qquad v_{2p} = v_{2p}^{(11)} + \lambda_1 v_{2p}^{(12)} \tag{3.9}$$

where:

$$u_{2p}^{(11)} = \left[-2\frac{a^4}{R^3} + \frac{a^2}{R} - \left(2\frac{a^4}{R^3} - 2\frac{a^2}{R}\right)\ln\frac{R}{a}\right]\cos 2\Theta + \left[-\frac{a^6}{R^3} + 2\frac{a^4}{R^3} - \frac{a^2}{R}\right]$$

$$u_{2p}^{(12)} = \left(-\frac{3}{2}\frac{a^5}{R^4} + 2\frac{a^3}{R^2} + \frac{a^3}{R^2}\ln\frac{R}{a}\right)\sin\Theta$$

$$u_{2p}^{(22)} = \left(-\frac{\lambda_1{}^2}{2}\frac{a^4}{R^3}\right)$$

$$v_{2p}^{(11)} = \left(-3\frac{a^4}{R^3} + \frac{a^2}{R} - 2\frac{a^4}{R^3}\ln\frac{R}{a}\right)\sin 2\Theta$$

$$v_{2p}^{(12)} = \left(-\frac{1}{2}\frac{a^5}{R^4} - \frac{a^3}{R^2} - \frac{a^3}{R^2}\ln\frac{R}{a}\right)\cos\Theta \tag{3.10}$$

By considering the kinematics of deformation of the rigid circular inclusion it can be shown that the second-order displacement components must satisfy the boundary conditions:

$$u_2(a, \Theta) = 0; \qquad v_2(a, \Theta) = 0 \tag{3.11}$$

and the second-order stress components should tend to zero as $R \to \infty$. To satisfy these boundary conditions we require five independent solutions of the homogeneous equation $\nabla^4\Psi_2 = 0$.

These solutions are:

$$\Psi_{2H} = \left(\alpha_1 + \frac{\alpha_2}{R^2}\right)\sin 2\Theta + \left(\frac{\alpha_3}{R} + \alpha_4 R\right)\cos\Theta + \alpha_5\Theta \tag{3.12}$$

where $\alpha_1, \alpha_2 \ldots$, are arbitrary constants.

On satisfying the boundary conditions (eqns. (3.11)) we obtain the final expression for the second-order displacement and stress components as follows:

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} u_2^{(11)} \\ v_2^{(11)} \end{Bmatrix} + \lambda_1 \begin{Bmatrix} u_2^{(12)} \\ v_2^{(12)} \end{Bmatrix} + \lambda_1^2 \begin{Bmatrix} u_2^{(22)} \\ v_2^{(22)} \end{Bmatrix} \tag{3.13}$$

$$\begin{Bmatrix} \dfrac{S_{rr}^{(2)}}{\mu} \\ \dfrac{S_{\theta\theta}^{(2)}}{\mu} \\ \dfrac{S_{r\theta}^{(2)}}{\mu} \end{Bmatrix} = \begin{Bmatrix} S_{rr11}^{(2)} \\ S_{\theta\theta11}^{(2)} \\ S_{r\theta11}^{(2)} \end{Bmatrix} + \lambda_1 \begin{Bmatrix} S_{rr12}^{(2)} \\ S_{\theta\theta12}^{(2)} \\ S_{r\theta12}^{(2)} \end{Bmatrix} + \lambda_1^2 \begin{Bmatrix} S_{rr22}^{(2)} \\ S_{\theta\theta22}^{(2)} \\ S_{r\theta22}^{(2)} \end{Bmatrix} \tag{3.14}$$

where:

$$u_2^{(11)} = \left(-2\frac{a^4}{R^3} + 2\frac{a^2}{R}\right)\ln\frac{R}{a}\cos 2\Theta + \left(-\frac{a}{R^5} + 2\frac{a^4}{R^3} - \frac{a^2}{R}\right)$$

$$u_2^{(12)} = \left(-\frac{3}{2}\frac{a^5}{R^4} + \frac{3}{2}\frac{a^3}{R^2} + \frac{a^3}{R^2}\ln\frac{R}{a}\right)\sin\Theta$$

$$u_2^{(22)} = \left(\frac{1}{2}\frac{a^4}{R^3} - \frac{1}{2}\frac{a^2}{R}\right)$$

$$v_2^{(11)} = \left(-\frac{a^4}{R^3} + \frac{a^2}{R} - 2\frac{a^4}{R^3}\ln\frac{R}{a}\right)\sin 2\Theta$$

$$v_2^{(12)} = \left(\frac{1}{2}\frac{a^5}{R^4} - \frac{1}{2}\frac{a^3}{R^2} - \frac{a^3}{R^2}\ln\frac{R}{a}\right)\cos\Theta$$

$$v_2^{(22)} = 0 \tag{3.15}$$

$$S_{rr11}^{(2)} = \left[4\frac{a^6}{R^6} - 8\frac{a^4}{R^4} + 2\frac{a^2}{R^2} + 4\left(\frac{a^4}{R^4} - 2\frac{a^2}{R^2}\right)\ln\frac{R}{a}\right]\cos 2\Theta$$

$$+ \left(6\frac{a^6}{R^6} - 10\frac{a^4}{R^4} + 6\frac{a^2}{R^4} + 8\frac{a^4}{R^2}\ln\frac{R}{a}\right)$$

$$S_{rr12}^{(2)} = \left(-2\frac{a^3}{R^3} + 4\frac{a^3}{R^3}\ln\frac{R}{a}\right)\sin\Theta$$

$$S_{rr22}^{(2)} = \left(-\frac{a^4}{R^4} - \frac{a^2}{R^2}\right)$$

$$S_{\theta\theta 11}^{(2)} = \left(-4\frac{a^6}{R^6} + 2\frac{a^2}{R^2} - 4\frac{a^4}{R^4}\ln\frac{R}{a}\right)\cos 2\Theta$$

$$+ \left(2\frac{a^6}{R^6} - 2\frac{a^4}{R^4} + 2\frac{a^2}{R^2} - 8\frac{a^4}{R^4}\ln\frac{R}{a}\right)$$

$$S_{\theta\theta 12}^{(2)} = \left(8\frac{a^5}{R^5} - 2\frac{a^3}{R^3} - 4\frac{a^3}{R^3}\ln\frac{R}{a}\right)\sin\Theta$$

$$S_{\theta\theta 22}^{(2)} = \left(3\frac{a^4}{R^4} + \frac{a^2}{R^2}\right)$$

$$S_{r\theta 11}^{(2)} = \left[4\frac{a^6}{R^6} - 6\frac{a^4}{R^4} + 2\frac{a^2}{R^2} + 4\left(\frac{a^4}{R^4} + \frac{a^2}{R^2}\right)\ln\frac{R}{a}\right]\sin 2\Theta$$

$$S_{r\theta 12}^{(2)} = \left(4\frac{a^5}{R^5} - 6\frac{a^3}{R^3} + 4\frac{a^3}{R^3}\ln\frac{R}{a}\right)\cos\Theta$$

$$S_{r\theta 22}^{(2)} = 0 \tag{3.16}$$

The stress components (eqn. (3.14)) tend to zero as $R \to \infty$. The resultant force exerted on the inclusion can be evaluated by considering the force resultants in the x and y directions (Fig. 1) transmitted across any contour $r = r_0$ in the deformed body. The force resultants in the x and y directions are:

$$P_x = \int_c (S_{rr}\cos\theta - S_{r\theta}\sin\theta) r_0 \, d\theta \tag{3.17}$$

and

$$P_y = \int_c (S_{rr}\sin\theta + S_{r\theta}\cos\theta) r_0 \, d\theta \tag{3.18}$$

respectively, where, on the contour, c:

$$\theta = \Theta + \varepsilon\frac{v_1}{R}(r_0, \Theta), \qquad r_0 = R + \varepsilon u_1(r_0, \Theta)$$

to order ε.

We express P_x and P_y in power series of ε (to order ε^2) as:

$$P_x = \varepsilon P_x^{(1)} + \varepsilon^2 P_x^{(2)}, \qquad P_y = \varepsilon P_y^{(1)} + \varepsilon^2 P_y^{(2)} \tag{3.19}$$

where:

$$P_y^{(1)} = \int_{-\pi}^{\pi} (S_{rr}^{(1)} \sin \Theta + S_{r\theta}^{(1)} \cos \Theta) r_0 \, d\Theta \tag{3.20}$$

$$P_x^{(1)} = \int_{-\pi}^{\pi} (S_{rr}^{(1)} \cos \Theta - S_{r\theta}^{(1)} \sin \Theta) r_0 \, d\Theta \tag{3.21}$$

$$P_y^{(2)} = \int_{-\pi}^{\pi} \left[S_{rr}^{(2)} \sin \Theta + S_{r\theta}^{(2)} \cos \Theta - u_1 \left(\frac{\partial S_{rr}^{(1)}}{\partial R} \sin \Theta + \frac{\partial S_{r\theta}^{(1)}}{\partial R} \cos \Theta \right) + \frac{v_1}{R} (S_{rr}^{(1)} \cos \Theta + S_{r\theta}^{(1)} \sin \Theta) + \frac{1}{R} \frac{\partial v_1}{\partial \Theta} (S_{rr}^{(1)} \sin \Theta + S_{r\theta}^{(1)} \cos \Theta) \right] r_0 \, d\Theta \tag{3.22}$$

$$P_x^{(2)} = \int_{-\pi}^{\pi} \left[S_{rr}^{(2)} \cos \Theta - S_{r\theta}^{(2)} \sin \Theta - u_1 \left(\frac{\partial S_{rr}^{(1)}}{\partial R} \cos \Theta - \frac{\partial S_{r\theta}^{(1)}}{\partial R} \sin \Theta \right) - \frac{v_1}{R} (S_{rr}^{(1)} \sin \Theta + S_{r\theta}^{(1)} \cos \Theta) + \frac{1}{R} \frac{\partial v_1}{\partial \Theta} (S_{rr}^{(1)} \cos \Theta - S_{r\theta}^{(1)} \sin \Theta) \right] r_0 \, d\Theta \tag{3.23}$$

Similarly, the resultant couple (M) acting on the rigid inclusion can be written as:

$$M = \varepsilon M^{(1)} + \varepsilon^2 M^{(2)} \tag{3.24}$$

to order ε^2, where:

$$M^{(1)} = \int_{-\pi}^{\pi} S_{r\theta}^{(1)} r_0^2 \, d\Theta \tag{3.25}$$

and

$$M^{(2)} = \int_{-\pi}^{\pi} \left(S_{r\theta}^{(2)} - u_1 \frac{\partial S_{r\theta}^{(1)}}{\partial R} + \frac{S_{r\theta}^{(1)}}{R} \frac{\partial v_1}{\partial \Theta} \right) r_0^2 \, d\Theta \tag{3.26}$$

On evaluating $P_x^{(1)}, P_x^{(2)}, P_y^{(1)}, \ldots, M^{(2)}$ we have:

$$\varepsilon P_x^{(1)} = 8\pi\mu a\delta, \qquad \varepsilon M^{(1)} = G_0 \frac{\delta}{a}$$

and

$$P_x^{(2)} = P_y^{(2)} = M^{(2)} = 0$$

In the problems where the rigid circular inclusion is subjected to only a concentrated force at its centre, it has been observed[3] that, by the symmetry of the problem

$P_y^{(1)}$, $P_y^{(2)}$, $M^{(1)}$ and $M^{(2)}$ are all zero and on evaluating the integral (eqn. (3.23)), $P_x^{(2)}$ was found to be equal to zero. It may also be of interest to note that, in the particular case where the rigid circular inclusion is subjected to a combined force and a couple at its centre, there is no second-order contribution to P_x, P_y and M. The applied force and couple at the centre of the inclusion, to order ε^2, are given by:

$$T = 8\pi\mu a\delta, \qquad M = G_0$$

4. INFINITE ELASTIC PLANE UNDER BIAXIAL STRESS CONTAINING A RIGID CIRCULAR INCLUSION SUBJECTED TO A CENTRAL FORCE

We consider the problem of an infinite plane, containing a bonded rigid circular inclusion, which is subjected to a state of uniform biaxial stress at infinity (Fig. 2). In addition, the rigid circular inclusion is subjected to a concentrated force at its centre which acts in the x-direction.

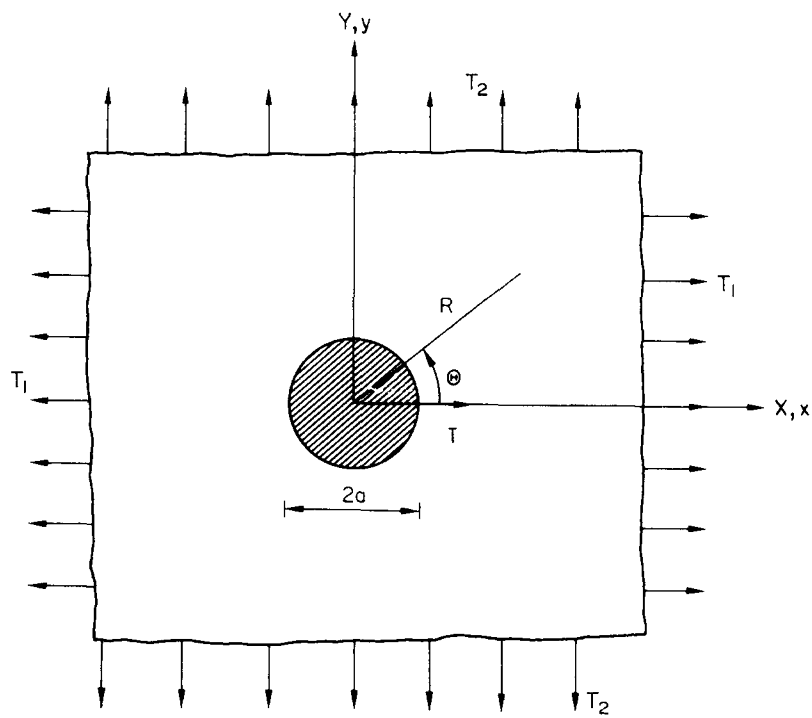

Fig. 2. Infinite plane under biaxial stress. Rigid circular inclusion subjected to concentrated force.

The first-order displacement function for this problem is given by:

$$\varepsilon\Psi_1 = \left(\frac{T_1 + T_2}{4\mu}\right)\left[a\sigma\left(\frac{1}{2}\frac{R^2}{a} - a + \frac{1}{2}\frac{a^3}{R^2}\right)\sin 2\Theta + \lambda_2\left(\frac{a^3}{R} + 2aR\ln\frac{R}{a}\right)\sin\Theta\right] \quad (4.1)$$

where T_1 and T_2 are the uniform stresses acting at infinity.

$$\sigma = \frac{T_1 - T_2}{T_1 + T_2} \quad \text{and} \quad \lambda_2 = \frac{4\mu\delta}{a(T_1 + T_2)} \quad (4.2)$$

In this, and subsequent problems treated in this paper, the small real dimensionless parameter ε is chosen to be equal to $(T_1 + T_2)/4\mu$.

The first-order displacement and stress components are:

$$u_1 = a\sigma\left(\frac{R}{a} - 2\frac{a}{R} + \frac{a^3}{R^3}\right)\cos 2\Theta + \lambda_2\left(\frac{a^3}{R^2} + 2a\ln\frac{R}{a}\right)\cos\Theta$$

$$v_1 = a\sigma\left(-\frac{R}{a} + \frac{a^3}{R^3}\right)\sin 2\Theta + \lambda_2\left[\frac{a^3}{R^2} - 2a\left(1 + \ln\frac{R}{a}\right)\right]\sin\Theta \quad (4.3)$$

and

$$S_{rr}^{(1)} = \mu\left[2 + \sigma\left(2 + 8\frac{a^2}{R^2} - 6\frac{a^4}{R^4}\right)\cos 2\Theta + \lambda_2\left(8\frac{a}{R} - 4\frac{a^3}{R^3}\right)\cos\Theta\right]$$

$$S_{\theta\theta}^{(1)} = \mu\left[2 + \sigma\left(-2 + 6\frac{a^4}{R^4}\right)\cos 2\Theta + \lambda_2\left(4\frac{a^3}{R^3}\right)\cos\Theta\right]$$

$$S_{r\theta}^{(1)} = \mu\left[\sigma\left(-2 + 4\frac{a^2}{R^2} - 6\frac{a^4}{R^4}\right)\sin 2\Theta + \lambda_2\left(-4\frac{a^3}{R^3}\right)\sin\Theta\right] \quad (4.4)$$

respectively.

It may be verified that the first-order stress components (eqns. (4.4)) reduce to a state of uniform biaxial stress at infinity and give force and couple resultants:

$$P_x^{(1)} = T, \qquad P_y^{(1)} = 0, \qquad M^{(1)} = 0$$

on the rigid circular inclusion–elastic medium interface $R = a$.

By substituting the first-order displacements and stress components (eqns. (4.3) and (4.4)) into eqns. (2.22), we obtain the inhomogeneous differential equation for the second-order displacement function:

$$\nabla^4\Psi_2 = \lambda_2^2 F_{11}(R, \Theta) + \sigma\lambda_2[F_{13}(R, \Theta) + F_{31}(R, \Theta)] + \sigma^2 F_{33}(R, \Theta) \quad (4.5)$$

where $F_{11}(R, \Theta)$ is given by eqns. (3.6) and:

$$F_{13}(R, \Theta) = \left(32\frac{a}{R^3} - 48\frac{a^3}{R^5} - 320\frac{a^7}{R^9} + 64\frac{a}{R^3}\ln\frac{R}{a}\right)\sin 3\Theta$$

$$F_{31}(R, \Theta) = \left(-16\frac{a^3}{R^5} + 32\frac{a^5}{R^7} - 192\frac{a^5}{R^7}\ln\frac{R}{a}\right)\sin\Theta$$

$$F_{33}(R, \Theta) = \left(\frac{48}{R^2} - 96\frac{a^2}{R^4} - 240\frac{a^8}{R^{10}}\right)\sin 4\Theta \tag{4.6}$$

A particular integral of eqn. (3.5) can be written in the form:

$$\Psi_{2p} = \lambda_2^2\Psi_{2p}^{(11)} + \sigma\lambda_2(\Psi_{2p}^{(13)} + \Psi_{2p}^{(31)}) + \sigma^2\Psi_{2p}^{(33)} \tag{4.7}$$

where $\Psi_{2p}^{(11)}$ is given by eqns. (3.8) and

$$\Psi_{2p}^{(13)} = \left[\frac{aR}{2} - \frac{1}{2}\frac{a^7}{R^5} + \left(aR - \frac{a^3}{R}\right)\ln\frac{R}{a}\right]\sin 3\Theta$$

$$\Psi_{2p}^{(31)} = \left[-\frac{a^5}{R^3} + \left(\frac{a^3}{R} - \frac{a^5}{R^3}\right)\ln\frac{R}{a}\right]\sin\Theta$$

$$\Psi_{2p}^{(33)} = \left(\frac{R^2}{4} - \frac{a^2}{2} - \frac{1}{4}\frac{a^8}{R^6}\right)\sin 4\Theta \tag{4.8}$$

For this problem, the second-order displacement components should satisfy the conditions:

$$u_2(R, \Theta) = v_2(R, \Theta) = 0 \tag{4.9}$$

on $R = a$, and the second-order stress components should either reduce to zero or constitute a state of uniform biaxial stress at infinity.

These boundary conditions can be identically satisfied by making use of the solutions to the homogeneous equation $\nabla^4\Psi_2 = 0$:

$$\Psi_{2H} = \sum_{m=1}^{4}\left(\frac{\alpha_m}{R^m} + \frac{\beta_m}{R^{m-2}}\right)\sin m\Theta \tag{4.10}$$

where $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$, are arbitrary constants.

The final expressions for the second-order displacement and stress components can be written in the form:

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \lambda_2^2\begin{Bmatrix} u_2^{(11)} \\ v_2^{(11)} \end{Bmatrix} + \sigma\lambda_2\begin{Bmatrix} u_2^{(13)} + u_2^{(31)} \\ v_2^{(13)} + v_2^{(31)} \end{Bmatrix} + \sigma^2\begin{Bmatrix} u_2^{(33)} \\ v_2^{(33)} \end{Bmatrix} \tag{4.11}$$

and

$$\begin{Bmatrix} \dfrac{S_{rr}^{(2)}}{\mu} \\ \dfrac{S_{\theta\theta}^{(2)}}{\mu} \\ \dfrac{S_{r\theta}^{(2)}}{\mu} \end{Bmatrix} = \lambda_2^2 \begin{Bmatrix} S_{rr11}^{(2)} \\ S_{\theta\theta11}^{(2)} \\ S_{r\theta11}^{(2)} \end{Bmatrix} + \sigma\lambda_2 \begin{Bmatrix} S_{rr13}^{(2)} + S_{rr31}^{(2)} \\ S_{\theta\theta13}^{(2)} + S_{\theta\theta31}^{(2)} \\ S_{r\theta13}^{(2)} + S_{r\theta31}^{(2)} \end{Bmatrix} + \sigma^2 \begin{Bmatrix} S_{rr33}^{(2)} \\ S_{\theta\theta33}^{(2)} \\ S_{r\theta33}^{(2)} \end{Bmatrix} \quad (4.12)$$

respectively, where:

$u_2^{(11)}$, $v_2^{(11)}$, $S_{rr11}^{(2)}$, $S_{\theta\theta11}^{(2)}$, $S_{r\theta11}^{(2)}$ are given by eqns. (3.15) and (3.16) and:

$$u_2^{(13)} = \left[\frac{a}{2} - \frac{a^3}{R^2} + \frac{1}{2}\frac{a^5}{R^4} + 3\left(\frac{a^3}{R^2} - \frac{a^5}{R^4}\right)\ln\frac{R}{a}\right]\cos 3\Theta$$

$$u_2^{(31)} = \left[-\frac{5}{2}\frac{a^3}{R^2} + 5\frac{a^5}{R^4} - \frac{5}{2}\frac{a^7}{R^6} + \left(a - \frac{a^3}{R^2}\right)\ln\frac{R}{a}\right]\cos\Theta$$

$$u_2^{(33)} = \left(\frac{a^2}{R} - 2\frac{a^4}{R^3} + \frac{a^6}{R^5}\right)\cos 4\Theta + \left(\frac{R}{2} - \frac{a^2}{R} - \frac{a^4}{R^3} + 3\frac{a^6}{R^5} - \frac{3}{2}\frac{a^8}{R^7}\right)$$

$$v_2^{(13)} = \left[\frac{a}{2} - \frac{1}{2}\frac{a^5}{R^4} + \left(\frac{a^3}{R^2} - 3\frac{a^5}{R^4}\right)\ln\frac{R}{a}\right]\sin 3\Theta$$

$$v_2^{(31)} = \left[a - \frac{3}{2}\frac{a^3}{R^2} + \frac{a^5}{R^4} - \frac{1}{2}\frac{a^7}{R^6} - \left(a + \frac{a^3}{R^2}\right)\ln\frac{R}{a}\right]\sin\Theta$$

$$v_2^{(33)} = \left(\frac{a^2}{R} - 2\frac{a^4}{R^3} + \frac{a^6}{R^5}\right)\sin 4\Theta \quad (4.13)$$

$$S_{rr13}^{(2)} = \left[-4\frac{a}{R} + 10\frac{a^3}{R^3} - 16\frac{a^5}{R^5} + 10\frac{a^7}{R^7} + \left(-4\frac{a}{R} - 12\frac{a^3}{R^3} + 12\frac{a^5}{R^5}\right)\ln\frac{R}{a}\right]\cos 3\Theta$$

$$S_{rr31}^{(2)} = \left[4\frac{a}{R} + 16\frac{a^3}{R^3} - 40\frac{a^5}{R^5} + 20\frac{a^7}{R^7} + \left(4\frac{a}{R} - 4\frac{a^3}{R^3} + 12\frac{a^5}{R^5}\right)\ln\frac{R}{a}\right]\cos\Theta$$

$$S_{rr33}{}^{(2)} = \left(-2 + 6\frac{a^4}{R^4} - 8\frac{a^6}{R^6} + 6\frac{a^8}{R^8}\right)\cos 4\Theta + \left(2 - 4\frac{a^2}{R^2} + 20\frac{a^4}{R^4} - 36\frac{a^6}{R^6} + 15\frac{a^8}{R^8}\right)$$

$$S_{\theta\theta13}{}^{(2)} = \left[4\frac{a}{R} - 6\frac{a^3}{R^3} + 4\frac{a^5}{R^5} - 10\frac{a^7}{R^7} + \left(4\frac{a}{R} - 4\frac{a^3}{R^3} - 12\frac{a^5}{R^5}\right)\ln\frac{R}{a}\right]\cos 3\Theta$$

$$S_{\theta\theta31}{}^{(2)} = \left[4\frac{a}{R} + 4\frac{a^7}{R^7} + \left(-4\frac{a}{R} + 4\frac{a^3}{R^3} - 12\frac{a^5}{R^5}\right)\ln\frac{R}{a}\right]\cos\Theta$$

$$S_{\theta\theta33}{}^{(2)} = \left(2 - 10\frac{a^4}{R^4} + 8\frac{a^6}{R^6} - 6\frac{a^8}{R^8}\right)\cos 4\Theta + \left(-2 + 4\frac{a^2}{R^2} - 4\frac{a^4}{R^4} + 4\frac{a^6}{R^6} + 3\frac{a^8}{R^8}\right)$$

$$S_{r\theta13}{}^{(2)} = \left[4\frac{a}{R} - 2\frac{a^3}{R^3} - 12\frac{a^5}{R^5} + 10\frac{a^7}{R^7} + \left(4\frac{a}{R} - 4\frac{a^3}{R^3} + 12\frac{a^5}{R^5}\right)\ln\frac{R}{a}\right]\sin 3\Theta$$

$$S_{r\theta31}{}^{(2)} = \left[-4\frac{a^3}{R^3} + 4\frac{a^7}{R^7} + \left(-4\frac{a}{R} - 4\frac{a^3}{R^3} + 12\frac{a^5}{R^5}\right)\ln\frac{R}{a}\right]\sin\Theta$$

$$S_{r\theta33}{}^{(2)} = \left(2 - 4\frac{a^2}{R^2} + 4\frac{a^4}{R^4} - 8\frac{a^6}{R^6} + 6\frac{a^8}{R^8}\right)\sin 4\Theta \tag{4.14}$$

By making use of the second-order stress components (eqn. (4.12)) and the expressions for the second-order force and couple resultants (eqns. (3.22), (3.23) and (3.26)) it can be shown that:

$$P_x^{(2)} = P_y^{(2)} = M^{(2)} = 0$$

In addition, it may be verified that, in the special case when $\delta = 0$ and $T_2 = 0$, the second-order displacement and stress components (eqns. (4.11) and (4.12)) reduce to those given by Adkins *et al.*[4] for the extension of an infinite elastic body containing a rigid circular inclusion.

5. INFINITE ELASTIC PLANE UNDER BIAXIAL STRESS CONTAINING A RIGID CIRCULAR INCLUSION SUBJECTED TO A COUPLE AT THE CENTRE

In this problem the bonded rigid circular inclusion is subjected to a concentrated couple (G_0) at its centre (Fig. 3), and the infinite plane is subjected to a state of uniform stress at infinity.

The first-order displacement function for this problem is given by:

$$\varepsilon\Psi_1 = \left(\frac{T_1 + T_2}{4\mu}\right)\left[a\sigma\left(\frac{1}{2}\frac{R^2}{a} - a + \frac{1}{2}\frac{a^3}{R^2}\right)\sin 2\Theta - \lambda_3 a^2 \ln\frac{R}{a}\right] \tag{5.1}$$

where:

$$\sigma = \frac{T_1 - T_2}{T_1 + T_2} \quad \text{and} \quad \lambda_3 = \frac{G_0}{\pi a^2(T_1 + T_2)} \tag{5.2}$$

The first-order displacement and stress components are:

$$u_1 = a\sigma\left(\frac{R}{a} - 2\frac{a}{R} + \frac{a^3}{R^3}\right)\cos 2\Theta$$

$$v_1 = a\sigma\left(-\frac{R}{a} + \frac{a^3}{R^3}\right)\sin 2\Theta + \lambda_3\frac{a^2}{R} \tag{5.3}$$

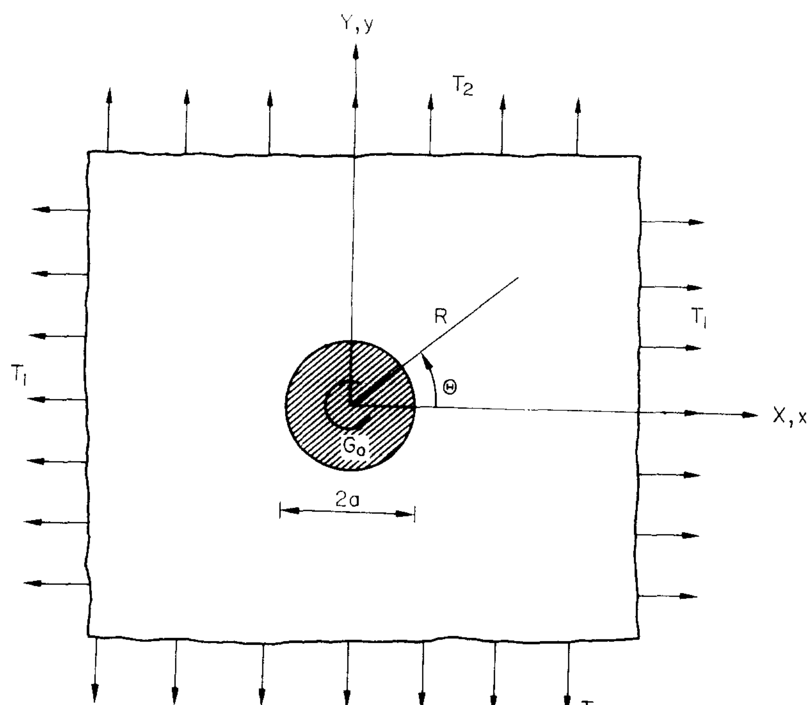

Fig. 3. Infinite plane under biaxial stress. Rigid circular inclusion subjected to couple.

and

$$S_{rr}{}^{(1)} = \mu\left[2 + \sigma\left(2 + 8\frac{a^2}{R^2} - 6\frac{a^4}{R^4}\right)\cos 2\Theta\right]$$

$$S_{\theta\theta}{}^{(1)} = \mu\left[2 + \sigma\left(-2 + 6\frac{a^4}{R^4}\right)\cos 2\Theta\right]$$

$$S_{r\theta}{}^{(1)} = \mu\left[\sigma\left(-2 + 4\frac{a^2}{R^2} - 6\frac{a^4}{R^4}\right)\sin 2\Theta - 2\lambda_3\frac{a^2}{R^2}\right] \tag{5.4}$$

respectively.

From eqns. (5.3), (5.4) and (2.22) we obtain the inhomogeneous differential equation for the second-order displacement function (eqn. (2.23)) as:

$$\nabla^4\Psi_2 = \sigma^2 F_{33}(R, \Theta) + \sigma\lambda_3 F_{23}(R, \Theta) \tag{5.5}$$

where $F_{33}(R, \Theta)$ is defined in eqns. (4.6) and

$$F_{23}(R, \Theta) = \left(-16\frac{a^2}{R^4} - 192\frac{a^6}{R^8}\right)\cos 2\Theta \tag{5.6}$$

A particular integral of eqn. (5.5) is given by:

$$\Psi_{2p} = \sigma^2\Psi_{2p}{}^{(33)} + \sigma\lambda_3\Psi_{2p}{}^{(23)} \tag{5.7}$$

where $\Psi_{2p}{}^{(33)}$ is given by eqns. (4.8) and

$$\Psi_{2p}{}^{(23)} = \left(-a^2\ln\frac{R}{a} - \frac{1}{2}\frac{a^6}{R^4}\right)\cos 2\Theta \tag{5.8}$$

The displacement boundary conditions:

$$u_2(R, \Theta) = v_2(R, \Theta) = 0 \tag{5.9}$$

at the rigid inclusion–elastic medium interface and the stress boundary conditions at infinity can be identically satisfied by making use of the homogeneous solution of eqn. (2.23)

$$\Psi_{2H} = \left(\frac{\eta_1}{R^2} + \frac{\eta_2}{R^4}\right)\sin 4\Theta + \left(\eta_3 + \frac{\eta_4}{R^2}\right)\cos 2\Theta + \eta_5\Theta \tag{5.10}$$

where $\eta_1, \eta_2, \ldots$, are arbitrary constants.

On satisfying the boundary conditions (eqn. (5.9)) we obtain the final expression for the second-order displacement and stress components, which could be written as:

$$\left.\begin{matrix} u_2 \\ v_2 \end{matrix}\right\} = \sigma^2\left\{\begin{matrix} u_2{}^{(33)} \\ v_2{}^{(33)} \end{matrix}\right\} + \sigma\lambda_3\left\{\begin{matrix} u_2{}^{(23)} \\ v_2{}^{(23)} \end{matrix}\right\} + \lambda_3{}^2\left\{\begin{matrix} u_2{}^{(22)} \\ v_2{}^{(22)} \end{matrix}\right\} \tag{5.11}$$

and

$$\begin{Bmatrix} \dfrac{S_{rr}^{(2)}}{\mu} \\ \dfrac{S_{\theta\theta}^{(2)}}{\mu} \\ \dfrac{S_{r\theta}^{(2)}}{\mu} \end{Bmatrix} = \sigma^2 \begin{Bmatrix} S_{rr33}^{(2)} \\ S_{\theta\theta33}^{(2)} \\ S_{r\theta33}^{(2)} \end{Bmatrix} + \sigma\lambda_3 \begin{Bmatrix} S_{rr23}^{(2)} \\ S_{\theta\theta23}^{(2)} \\ S_{r\theta23}^{(2)} \end{Bmatrix} + \lambda_3^2 \begin{Bmatrix} S_{rr22}^{(2)} \\ S_{\theta\theta22}^{(2)} \\ S_{r\theta22}^{(2)} \end{Bmatrix} \tag{5.12}$$

where $u_2^{(33)}$, $u_2^{(22)}$, $v_2^{(33)}$, $v_2^{(22)}$, . . ., $S_{r\theta33}^{(2)}$, $S_{r\theta22}^{(2)}$ are given by eqns. (3.15), (3.16), (4.13) and (4.14), and

$$\begin{aligned} u_2^{(23)} &= \left(-\frac{a^2}{R} + 3\frac{a^4}{R^3} - 2\frac{a^6}{R^5} + 2\frac{a^2}{R}\ln\frac{R}{a}\right)\sin 2\Theta \\ v_2^{(23)} &= \left(-\frac{a^4}{R^3} + \frac{a^6}{R^5}\right)\cos 2\Theta \end{aligned} \tag{5.13}$$

$$\begin{aligned} S_{rr23}^{(2)} &= \left(6\frac{a^2}{R^2} - 18\frac{a^4}{R^4} + 4\frac{a^6}{R^6} - 8\frac{a^2}{R^2}\ln\frac{R}{a}\right)\sin 2\Theta \\ S_{\theta\theta23}^{(2)} &= \left(-2\frac{a^2}{R^2} + 2\frac{a^4}{R^4} + 8\frac{a^6}{R^6}\right)\sin 2\Theta \\ S_{r\theta23}^{(2)} &= \left(-6\frac{a^2}{R^2} + 2\frac{a^4}{R^4} + 2\frac{a^6}{R^6} + 4\frac{a^2}{R^2}\ln\frac{R}{a}\right)\cos 2\Theta \end{aligned} \tag{5.14}$$

From the second-order stress components (eqn. (5.12)) and integrals (eqns. (3.22), (3.23) and (3.26)) we may verify that there is no second-order contribution to P_x, P_y and M.

6. OVERSIZE RIGID CIRCULAR INCLUSION IN AN INFINITE ELASTIC PLANE UNDER BIAXIAL STRESS

An infinite elastic plane contains a circular cavity of radius a. To this circular cavity is fitted a rigid circular inclusion of radius $a(1 + \xi)$ where ξ is a small quantity. The contact between the rigid circular inclusion and the infinite elastic plane is assumed to be smooth. The infinite elastic plane is also subjected to a state of uniform biaxial stress at infinity (Fig. 4). We assume that no separation takes place at the inclusion–elastic medium interface due to the action of the biaxial state of stress.

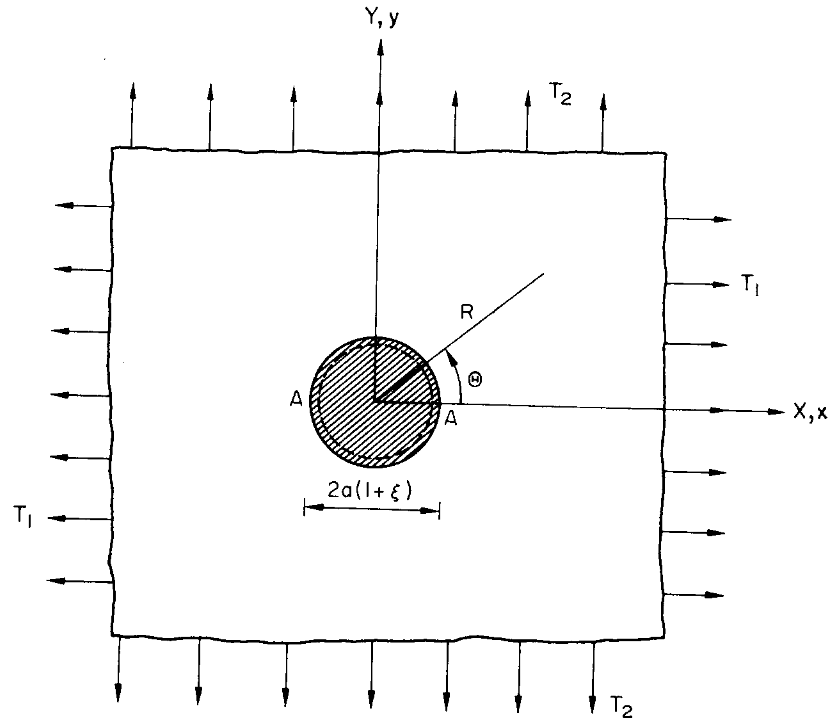

Fig. 4. Infinite plane under biaxial stress. Cylindrical cavity containing an oversize rigid inclusion.

The first-order displacement function for this problem is:

$$\varepsilon\Psi_1 = \left(\frac{T_1 + T_2}{4\mu}\right)\left[a\sigma\left(\frac{1}{2}\frac{R^2}{a} - \frac{1}{2}\frac{a}{a^2}\right)\sin 2\Theta + \lambda_4\Theta\right] \tag{6.1}$$

where:

$$\sigma = \frac{T_1 - T_2}{T_1 + T_2} \qquad \lambda_4 = \frac{4\mu\xi}{T_1 + T_2} \tag{6.2}$$

and the first-order displacement and stress components are:

$$u_1 = a\sigma\left(\frac{R}{a} - \frac{a}{R}\right)\cos 2\Theta + \lambda_4\frac{a^2}{R}$$

$$v_1 = a\sigma\left(-\frac{R}{a}\right)\sin 2\Theta \tag{6.3}$$

and

$$S_{rr}^{(1)} = \mu\left[2 + \sigma\left(2 + 4\frac{a^2}{R^2}\right)\cos 2\Theta - 2\lambda_4\frac{a^2}{R^2}\right]$$

$$S_{\theta\theta}{}^{(1)} = \mu\left(2 - 2\sigma\cos 2\Theta + 2\lambda_4 \frac{a^2}{R^2}\right)$$

$$S_{r\theta}{}^{(1)} = \mu\left[\sigma\left(-2 + 2\frac{a^2}{R^2}\right)\sin 2\Theta\right] \tag{6.4}$$

respectively.

Since the circular inclusion is rigid, the first-order displacement u_1 (eqn. (6.3)), satisfies the boundary condition:

$$u_1(a, \Theta) = a\lambda_4$$

on the surface $R = a$.

Owing to the smooth contact at the interface the shear stress component $S_{r\theta}{}^{(1)}$ is zero on $R = a$. The first-order stress components combine to form a state of uniform biaxial stress at infinity. The first-order displacement components (eqns. (6.3)) indicate that, in general, the circular cavity will assume a certain ovality due to the biaxial state of stress. For a circular inclusion of small oversize it is possible that some clearance might appear at, say, points A for $T_1 > T_2$.

From the first-order stress components (eqns. (6.4)) it is therefore possible to obtain a relationship between T_1, T_2 and ξ when separation will just occur at the points A. For a given value of ξ the 'critical values' of T_1 and T_2 are related by the equation:

$$2T_1 - T_2 - 2\mu\xi = 0 \tag{6.5}$$

In what follows we shall assume that there is no loss of contact at the rigid inclusion–elastic medium interface due to the effects of moderately large strains. The inhomogeneous differential equation (eqn. (2.23)) reduces to:

$$\nabla^4\Psi_2 = \sigma^2 F_{44}(R, \Theta) + \sigma\lambda_4 F_{45}(R, \Theta) \tag{6.6}$$

where:

$$F_{44}(R, \Theta) = \frac{48}{R^2}\left(1 - \frac{a^2}{R^2}\right)\sin 4\Theta$$

$$F_{45}(R, \Theta) = -48\frac{a^2}{R^4}\left(1 - \frac{a^2}{R^2}\right)\sin 2\Theta \tag{6.7}$$

A particular integral of eqn. (6.5) is:

$$\Psi_{2p} = \sigma^2\Psi_{2p}{}^{(44)} + \sigma\lambda_4\Psi_{2p}{}^{(45)} \tag{6.8}$$

where:

$$\Psi_{2p}{}^{(44)} = \left(\frac{R^2}{4} - \frac{a^2}{4}\right)\sin 4\Theta$$

$$\Psi_{2p}{}^{(45)} = \left(-3a^2 - \frac{a^4}{R^2}\right)\ln\frac{R}{a}\sin 2\Theta \tag{6.9}$$

In order to satisfy the boundary conditions at the rigid inclusion–elastic medium interface:

$$u_2(a, \Theta) = 0 \qquad S_{r\theta}^{(2)}(a, \Theta) = 0 \tag{6.10}$$

and at infinity, we make use of the solutions of the homogeneous equation:

$$\Psi_{2H} = \left(\frac{\rho_1}{R^2} + \frac{\rho_2}{R^4}\right) \sin 4\Theta + \left(\rho_3 + \frac{\rho_4}{R^2}\right) \sin 2\Theta + \rho_5 \Theta \tag{6.11}$$

where $\rho_1, \rho_2, \ldots$ are arbitrary constants.

Finally, the expressions for the second-order displacement and stress components can be written in the form:

$$\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \sigma^2 \begin{Bmatrix} u_2^{(44)} \\ v_2^{(44)} \end{Bmatrix} + \sigma\lambda_4 \begin{Bmatrix} u_2^{(45)} \\ v_2^{(45)} \end{Bmatrix} + \lambda_4^2 \begin{Bmatrix} u_2^{(55)} \\ v_2^{(55)} \end{Bmatrix} \tag{6.12}$$

and

$$\begin{Bmatrix} \dfrac{S_{rr}^{(2)}}{\mu} \\ \dfrac{S_{\theta\theta}^{(2)}}{\mu} \\ \dfrac{S_{r\theta}^{(2)}}{\mu} \end{Bmatrix} = \sigma^2 \begin{Bmatrix} S_{rr44}^{(2)} \\ S_{\theta\theta44}^{(2)} \\ S_{r\theta44}^{(2)} \end{Bmatrix} + \sigma\lambda_4 \begin{Bmatrix} S_{rr45}^{(2)} \\ S_{\theta\theta45}^{(2)} \\ S_{r\theta45}^{(2)} \end{Bmatrix} + \lambda_4^2 \begin{Bmatrix} S_{rr55}^{(2)} \\ S_{\theta\theta55}^{(2)} \\ S_{r\theta55}^{(2)} \end{Bmatrix} \tag{6.13}$$

respectively, where:

$$u_2^{(44)} = \left(\frac{1}{2}\frac{a^2}{R} - \frac{7}{8}\frac{a^4}{R^3} + \frac{3}{8}\frac{a^6}{R^5}\right)\cos 4\Theta + \left(\frac{R}{2} - \frac{1}{4}\frac{a^2}{R} - \frac{1}{4}\frac{a^4}{R^3}\right)$$

$$u_2^{(45)} = \left[2\frac{a^2}{R} - 2\frac{a^4}{R^3} - \left(6\frac{a^2}{R} + 2\frac{a^4}{R^3}\right)\ln\frac{R}{a}\right]\cos 2\Theta$$

$$u_2^{(55)} = \left(\frac{1}{2}\frac{a^2}{R} - \frac{1}{2}\frac{a^4}{R^3}\right)$$

$$v_2^{(44)} = \left(\frac{1}{2}\frac{a^2}{R} - \frac{5}{16}\frac{a^4}{R^3} + \frac{3}{8}\frac{a^6}{R^5}\right)\sin 4\Theta$$

$$v_2^{(45)} = \left(2\frac{a^2}{R} - 2\frac{a^4}{R^3} - 2\frac{a^4}{R^3}\ln\frac{R}{a}\right)\sin 2\Theta$$

$$v_2^{(55)} = 0 \tag{6.14}$$

$$S_{rr44}{}^{(2)} = \left(-2 + \frac{53}{8}\frac{a^4}{R^4} - \frac{15}{4}\frac{a^6}{R^6}\right)\cos 4\Theta + \left(2 - \frac{5}{2}\frac{a^2}{R^2} + 4\frac{a^4}{R^4}\right)$$

$$S_{rr45}{}^{(2)} = \left[-26\frac{a^2}{R^2} + 14\frac{a^4}{R^4} + 12\left(2\frac{a^2}{R^2} + \frac{a^4}{R^4}\right)\ln\frac{R}{a}\right]\cos 2\Theta$$

$$S_{rr55}{}^{(2)} = \left(-\frac{a^2}{R^2} + 3\frac{a^4}{R^4}\right)$$

$$S_{\theta\theta 44}{}^{(2)} = \left(2 - \frac{7}{8}\frac{a^4}{R^4} + \frac{15}{4}\frac{a^6}{R^6}\right)\cos 4\Theta + \left(-2 + \frac{5}{2}\frac{a^2}{R^2}\right)$$

$$S_{\theta\theta 45}{}^{(2)} = \left(2\frac{a^2}{R^2} - 6\frac{a^4}{R^4} - 12\frac{a^4}{R^4}\ln\frac{R}{a}\right)\cos 2\Theta$$

$$S_{\theta\theta 55}{}^{(2)} = \left(\frac{a^2}{R^2} - \frac{a^4}{R^4}\right)$$

$$S_{r\theta 44}{}^{(2)} = \left(2 - 2\frac{a^2}{R^2} + \frac{15}{4}\frac{a^4}{R^4} - \frac{15}{4}\frac{a^6}{R^6}\right)\sin 4\Theta$$

$$S_{r\theta 45}{}^{(2)} = \left[-12\frac{a^2}{R^2}\left(1 - \frac{a^2}{R^2}\right) + 12\frac{a^2}{R^2}\left(1 + \frac{a^2}{R^2}\right)\ln\frac{R}{a}\right]\sin 2\Theta$$

$$S_{r\theta 55}{}^{(2)} = 0 \tag{6.15}$$

We further observe that, owing to the symmetry of the problem and the assumed frictionless conditions at the rigid inclusion–elastic medium interface, the resultant force and couple acting on the rigid inclusion are zero. In the second-order case, the biaxial stress components T_1 and T_2 which will just cause separation at point A are related by the equation:

$$\sigma^2\left(\frac{21}{8}\varepsilon\right) + \sigma(6 - 12\lambda_4\varepsilon) + (2 - 2\lambda_4 + 2\lambda_4{}^2\varepsilon) = 0$$

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