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On frictionally constrained wing-cracks

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This paper examines the mechanics of wing-cracks that are located at its extremities of an interface region with Coulomb friction. The region containing the interface and the wing-cracks is first subjected to an isotropic compression, which induces closure of the interface region. The region is then subjected to a uniaxial compression in an incremental fashion. Since the frictional effects are restricted to pre-defined surfaces, the boundary element technique can be applied quite successfully to determine the influence of Coulomb friction on the development of stress intensity factors at the tips of the wing-cracks.

1. Introduction

Classical problems dealing with the modelling of cracks in elastic media invariably assume that the faces of the cracks are smooth and that these faces do not interact during the application of the external loads. These assumptions have enabled the development of a large body of literature dealing with the mechanics of elastic bodies containing isolated and non-isolated cracks of generalized shapes that are subjected to generalized loadings (Sih [1], Murakami [2], Broberg [3]). The extension of these studies to include non-classical phenomena including interactions between the crack faces and closure of the crack tip becomes important to a number of application areas including materials science and geomechanics, where either alterations in material properties or dominant compressive loads can induce interaction between the crack surfaces. Interactions between crack surfaces are generally nonlinear and the presence of the nonlinearity in the mechanics of the contact zone makes the analytical approach to such crack problems quite restrictive. The analytical study of crack problems becomes intractable when complicated forms of frictional contact phenomena, influence of separation and slip zones, interaction of regions contacting with neighbouring cracks, etc., need to be considered. In geomechanics and materials engineering in particular, interfaces between contacting geomaterial regions can exhibit a variety of nonlinear phenomena including Coulomb friction, finite friction, interface plasticity, interface damage, asperity degradation, viscoplasticity and creep (Michałowski and Mróz [4], Selvadurai and Voyiadjis [5], Selvadurai and Boulon [6], Darve [7], Desai [8]). Numerical approaches are therefore of particular interest in the study of contact phenomena where interface nonlinearity becomes a dominant feature. Extensive advances have been made in the application of finite element techniques to the study of contact phenomena associated with nonlinear interfaces. Detailed accounts of these developments are given, among others, by Zienkiewicz and Taylor [9], Wriggers and Wagner [10], Willner [11] and Mayer and Gaul [12].

In this paper the boundary element method is used for the analysis of the contact problem that deals with nonlinear effects at the contact zone. The advantage of the boundary element method is self-evident in view of the fact that the nonlinear contact phenomena are confined to pre-defined contact zones. The application of boundary element techniques to frictional contact problems can be traced to the earlier studies by Andersson [13] and Andersson and Allan-Persson [14] who used the procedure to examine frictional contact problems. Selvadurai and Au [15–17] and Selvadurai [18] have also investigated the influence of interface nonlinearity on the behaviour of planar cracks located in elastic media.

In this paper we apply an incremental boundary element approach for the modelling of the problem of inclined wing-cracks that extend from an interface exhibiting Coulomb friction (Fig. 1). It is assumed that the frictional effects arise

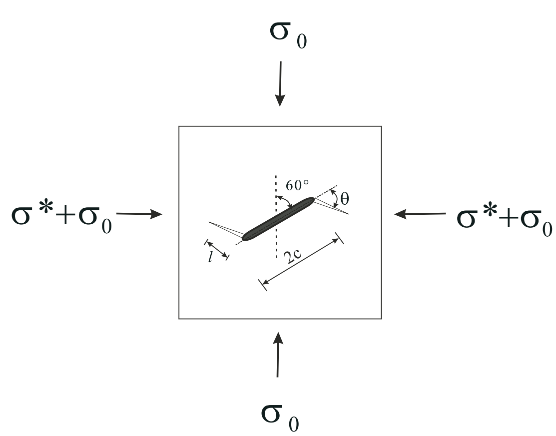

Fig. 1. Wing-cracks extending from interface with Coulomb frictional.

only upon closure of the interface region. The region containing the wing cracks and the oriented frictional interface is first subjected to an isotropic stress field σ_0 and the nonlinear effects come into effect during the application of an incremental uniaxial stress field σ^* at an orientation, as shown in Fig. 1. It is further assumed that the tips of the wing-crack remain open during the application of all external loads to permit the meaningful evaluation of the stress intensity factors at the crack tip.

2. The incremental boundary element procedure

The modelling of elastic continua with the aid of the boundary element approach is now well established (BREBBIA *et al.* [19], GAUL *et al.* [20]). The incremental form of the boundary integral equation for a region ρ with volume $V^{(\rho)}$ with surface $\Gamma^{(\rho)}$ can be written in the form

$$c_{ij}\dot{u}_j^{(\rho)} + \int\limits_{\Gamma^{(\rho)}} P_{ij}^{*(\rho)}\dot{u}_j^{(\rho)}\,d\Gamma = \int\limits_{\Gamma^{(\rho)}} u_{ij}^{*(\rho)}\dot{P}_j^{(\rho)}\,d\Gamma, \tag{2.1}$$

where $i, j = 1, 2, 3$ (or x, y, z); $\dot{u}_j^{(\rho)}$ and $\dot{P}_j^{(\rho)}$ are, respectively, the incremental values of the boundary displacements and boundary tractions; $u_{ij}^{*(\rho)}$ and $P_{ij}^{*(\rho)}$ are the corresponding fundamental solutions, given by

$$u_{ij}^{*(\rho)} = \frac{1}{16\pi G_\rho(1-\nu_\rho)r}\left[(3-4\nu_\rho)\delta_{ij} + r_{,i}\,r_{,j}\right] \tag{2.2}$$

and

$$P_{ij}^{*(\rho)} = -\frac{1}{8\pi(1-\nu_\rho)\,r^2}\left\{[(1-2\nu_\rho)\delta_{ij} + 3r_{,i}r_{,j}]\,r_{,n} - (1-2\nu_\rho)[r_{,i}n_j - r_{,j}n_i]\right\} \tag{2.3}$$

respectively, where r is the distance between the source and field points; n_i are the components of the outward unit normal vector to $\Gamma^{(\rho)}$; δ_{ij} is Kronecker's delta function; G_ρ and ν_ρ are the linear elastic shear modulus and Poisson's ratio respectively. In (2.1), $c_{ij} = \delta_{ij}/2$ if the boundary is smooth. The boundary integral equation (2.1) is applicable to any region V consisting of sub-regions $V^{(\rho)}$, separated by nonlinear interfaces. The regions in contact can be subjected to the conventional displacement and traction boundary conditions as well as interface conditions. For example, on a boundary Γ_1 where displacements are prescribed,

$$\dot{u}_i = \dot{u}_i^0, \tag{2.4}$$

where $\dot{u}_i^0$ is a prescribed increment of displacement. On a boundary Γ_2 where tractions are prescribed, similarly,

$$\dot{P}_i = \dot{P}_i^0, \tag{2.5}$$

where $\dot{P}_i^0$ is a prescribed increment of traction. On an interface region Γ_3 with nonlinear constraints we have

$$\dot{P}_i = \dot{R}_i + K_{ij}^* \dot{u}_j, \tag{2.6}$$

where $\dot{R}_i$ are increments of a residual traction and K_{ij}^* are stiffness coefficients derived through considerations of the non-linear constraints. Considering a discretization of the domain boundary by boundary elements, the integral equation can be converted to its matrix equivalent, which can be written in the form

$$[\mathbf{H}]\{\dot{\mathbf{u}}\} = [\mathbf{G}]\left\{\dot{\mathbf{P}}\right\}, \tag{2.7}$$

where $[\mathbf{H}]$ and $[\mathbf{G}]$ are the boundary element influence coefficients matrices and $\{\dot{\mathbf{u}}\}$ and $\left\{\dot{\mathbf{P}}\right\}$ are the displacement and traction vectors and sub-sets of which together will form the appropriate set of unknowns. When the configuration of the boundary and the interface conditions are defined at any level of deformation, we obtain a final system equation of the form

$$[\mathbf{A}]\left\{\dot{\mathbf{U}}\right\} = \left\{\dot{\mathbf{B}}\right\} \tag{2.8}$$

from which, either the boundary or the interface unknowns can be determined.

3. Interface responses

The modelling of interface responses can be approached at a variety of levels ranging from the *local-scale models* to *phenomenological* approaches. The advantage of the former is that it considers a level of refinement that is not accounted for a in a phenomenological approach (BELAK [21], BUSHAN [22]). A limitation of the local-scale modelling is that it introduces a degree of refinement requiring sophisticated constitutive parameter identification through experimentation and the interpretation of such experiments through phenomenological models themselves. In computational modelling, however, it is desirable to adopt a phenomenological approach to the formulation of an interface constitutive response. The classical models of either completely *smooth* or *bonded* conditions represent extremes of the phenomenological approach; other nonlinear forms are derived from considerations of conventional models of Coulomb friction, plasticity, dilatant phenomena that account for a local scale structure, and the incorporation of damage and degradation to account for deterioration of the interface with progressive wear. In this paper we shall present a treatment of the interface response, which can be used to model Coulomb friction, dilatant friction and interface degradation resulting from damage to asperities at contact zones. In view of the nonlinear nature of the interface response, it is necessary to adopt an incremental approach to the formulation of the constitutive responses. Since the interface is regarded as a distinct two-dimensional surface that is void of a dimension normal to the plane, its response must be formulated in relation to the incremental relative displacements $\dot{\Delta}_i$ at the contacting plane. We assume that these incremental relative displacements consist of an elastic or recoverable component $\dot{\Delta}_i^{(e)}$ and an irrecoverable or plastic component $\dot{\Delta}_i^{(p)}$: i.e.

$$\dot{\Delta}_i = \dot{\Delta}_i^{(e)} + \dot{\Delta}_i^{(p)}, \tag{3.1}$$

where for an interface, the subscripts i (or j) can be assigned notations applicable to the local interface coordinates. For purposes of the presentation, we shall denote the values applicable to i and j applicable to an interface by x, y, z with the assumption that the direction z corresponds to the normal to the Euclidean plane at a point on the interface. The elastic component of the incremental displacement $\dot{\Delta}_i^{(e)}$ is related to the component of the corresponding increment of traction $\dot{t}_i$ through the linear constitutive response

$$\dot{t}_i = \tilde{k}_{ij}\dot{\Delta}_j^{(e)}, \tag{3.2}$$

where $\tilde{k}_{ij}$ are the linear stiffness coefficients of the interface and summation over the repeated indices is implied. This linear elastic response will persist until failure is induced at the interface. In order to assess this limiting condition it is necessary to postulate a failure criterion for the interface. These are varied and the applicability of any phenomenological relationship to actual interfaces must be verified by recourse to experiment.

3.1. Coulomb friction

For interfaces with Coulomb friction, the condition F for initiation of failure is given by

$$F = \sqrt{t_x^2 + t_y^2} + \mu\, t_z = 0, \tag{3.3}$$

where t_i are the value of the *total tractions* acting at the interface; $\left(\left|\sqrt{t_x^2 + t_y^2}\right|\right.$ is the tangential traction and t_z is the normal traction acting on the interface$\left.\right)$ and μ is the coefficient of Coulomb friction at the interface. We now assume that when the interface total tractions satisfy the failure criterion (3.3), it will experience slip in the form of irreversible displacements. It is assumed that these irreversible displacements can be determined from knowledge of a plastic potential in exactly the same way that incremental plastic strains can be determined in a continuum region (MICHAŁOWSKI and MRÓZ [4], DESAI and SIRIWARDANE [23], LUBLINER [24], DAVIS and SELVADURAI [25]). The incremental plastic displacements at the interface are defined by

$$\dot{\Delta}_i^{(p)} = \dot{\lambda}\frac{\partial \Phi}{\partial t_i}, \tag{3.4}$$

where $\dot{\lambda}$ is a proportionality factor referred to as the plastic/interface slip multiplier and $\Phi(t_i)$ is the plastic/interface slip potential. The plastic potential is a constitutive function as such, it needs to be defined through experimentation. For the purposes of this paper, and for consistency regarding concerning plastic strains during contact, we select a plastic potential that relies closely on the structure of the failure criterion. Here we assume that the plastic potential takes the form

$$\Phi = \sqrt{t_x^2 + t_y^2}. \tag{3.5}$$

Using (3.1) and (3.4) we can rewrite (3.2) in the form

$$\dot{t}_i = \tilde{k}_{ij}\left(\dot{\Delta}_j - \dot{\lambda}\frac{\partial \Phi}{\partial t_j}\right). \tag{3.6}$$

For Coulomb friction, (where there is no alteration in the failure characteristics in the form of hardening or softening), we require

$$dF = \frac{\partial F}{\partial t_i} dt_i = 0. \tag{3.7}$$

The plastic slip multiplier $\dot{\lambda}$ can be determined by considering the result (3.6) together with (3.7); using the result in (3.6), we obtain

$$\dot{t}_i = \left[\tilde{k}_{ij} - \frac{1}{\psi}\frac{\partial \Phi}{\partial t_l}\tilde{k}_{il}\tilde{k}_{mj}\frac{\partial F}{\partial t_m}\right]\dot{\Delta}_j = \tilde{k}_{ij}^{(ep)}\dot{\Delta}_j \tag{3.8}$$

where

$$\psi = \frac{\partial F}{\partial t_l}\tilde{k}_{lm}\frac{\partial \Phi}{\partial t_m}. \tag{3.9}$$

If the failure criterion $F(t_i)$ and the plastic potential $\Phi(t_i)$ are known, then we can define the elastic-plastic stiffness $\tilde{k}_{ij}^{(ep)}$. In the specific case of a Coulomb frictional material where the failure criterion is defined by (3.3) and the plastic potential defined by (3.5), and for the special case where

$$\tilde{k}_{xx} = \tilde{k}_{yy} = \tilde{k}_s; \qquad \tilde{k}_{zz} = \tilde{k}_n \tag{3.10}$$

with all other $\tilde{k}_{ij} \equiv 0$, the non-symmetric elastic-plastic stiffness matrix is given by

$$\left[\tilde{\mathbf{k}}\right]^{(ep)} = \frac{1}{(t_x^2+t_y^2)} \begin{bmatrix} \tilde{k}_s t_y^2 & -\tilde{k}_s t_x t_y & -\mu\,\tilde{k}_n t_x \sqrt{(t_x^2+t_y^2)} \\ -\tilde{k}_s t_x t_y & \tilde{k}_s t_x^2 & -\mu\,\tilde{k}_n t_y \sqrt{(t_x^2+t_y^2)} \\ 0 & 0 & \tilde{k}_n (t_x^2+t_y^2) \end{bmatrix} \tag{3.11}$$

and $\tilde{k}_s$ and $\tilde{k}_n$ can be interpreted as the shear and normal elastic stiffnesses at the interface. These are the constitutive parameters associated with the phenomenological model of the interface response. These need to be determined either through micromechanical modelling of the local topography of the interface or through experiment.

3.2. Contact and separation processes

As the incremental analysis of an interface contact problem proceeds, within an increment of loading, processes such as separation, re-establishment of contact, slip and adhesion can occur in distinct regions of the interface.

Separation: During a loading sequence normal tractions at an interface region can become tensile. Since the interfaces are considered to be unilateral in their contact response, for a region undergoing separation, the total tractions resulting from the summation of the incremental boundary conditions (2.5) should be homogeneous.

Re-contact: A region of a contact zone that has experienced separation can also re-establish contact when the relative normal displacement across the separated interface region is greater than or equal to the initial gap. Then the boundary conditions change from the type given by (2.4) to the type given by (2.5).

Slip: Slip will occur when the tractions satisfy the failure condition (3.3). The interface condition (2.6) can be applied with the stiffness coefficients defined by (3.11).

Adhesion: When the conditions do not violate the failure condition (3.3), the boundary conditions at the interface can be interpreted through (3.2) with the stiffness coefficients interpreted appropriately. In a solution scheme, with all increments, all of the above four conditions must be checked in order to obtain a stable condition at the interface.

4. Localized iterative solution procedures

We shall formulate the iterative analysis in relation to the incremental relative displacement components $\dot{\Delta}_i$ that consists of the difference between u_i on either side of an interface. Considering the boundary conditions given by (2.4) to (2.6), we can rewrite the matrix equation (2.7) in terms of $\dot{\Delta}_i$ and denote by $\dot{\Delta}^{(i)}$ where the superscripts (i) $(i = 1, 2, 3)$ indicate the vectors of incremental differential displacements resulting from boundary conditions of the type (2.4), (2.5) and (2.6) respectively. The resulting equation can be written in the form

$$\left[-\mathbf{G}^{(1)},\ \mathbf{H}^{(2)},\ \left\{\mathbf{H}^{(3)} - \mathbf{G}^{(3)}\mathbf{K}^{(ep)}\right\}\right] \left\{ \begin{array}{c} \dot{t}^{(1)} \\ \dot{\Delta}^{(2)} \\ \dot{\Delta}^{(3)} \end{array} \right\} = \left[-\mathbf{H}^{(1)},\ \mathbf{G}^{(2)},\ \mathbf{G}^{(3)}\right] \left\{ \begin{array}{c} \dot{\Delta}^{0(1)} \\ \dot{t}^{0(2)} \\ \dot{R}^{(3)} \end{array} \right\}. \tag{4.1}$$

For a nonlinear interface problem it becomes necessary to apply an efficient solution scheme to analyse the incremental and iterative matrix equation (4.1). Since the complete boundary consists of linear and nonlinear constraints, we can use an elimination procedure to the set of linear boundary constraints; this will reduce (4.1) to the form

$$\left[\bar{\mathbf{A}},\ \left\{-\bar{\mathbf{H}}^{(3)} - \bar{\mathbf{G}}^{(3)}\mathbf{K}^{(ep)}\right\}\right] \left\{ \begin{array}{c} \dot{t}^{(1)} \\ \dot{\Delta}^{(2)} \\ \dot{\Delta}^{(3)} \end{array} \right\} = \left\{\bar{\mathbf{B}}\right\} + \left[\bar{\mathbf{G}}^{(3)}\right] \left\{\dot{R}^{(3)}\right\}, \tag{4.2}$$

where $[\bar{\mathbf{A}}]$ is the hyper-matrix or reduced version of $[-\mathbf{G}^{(1)}, -\mathbf{H}^{(2)}]$ with properties consistent with an upper triangle-type of matrix; $[\bar{\mathbf{B}}]$ is the reduced form of the right-hand side vector consisting of known boundary values and $[\bar{\mathbf{H}}^{(3)}]$ and $[\bar{\mathbf{G}}^{(3)}]$ are their corresponding reduced forms. The result (4.2) can be represented by two relations; the first one corresponds to the boundary conditions associated with surfaces $\Gamma^{(1)}$ and $\Gamma^{(2)}$, which can be written as

$$\begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{pmatrix} \begin{pmatrix} \dot{t}^{(1)} \\ \dot{\Delta}^{(2)} \end{pmatrix} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix} + \begin{pmatrix} \bar{G}_1^{(3)} \\ \bar{G}_2^{(3)} \end{pmatrix} \left\{ \dot{R}^{(3)} \right\} - \begin{pmatrix} \bar{H}_1^{(3)} & -\bar{G}_1^{(3)}K^{(ep)} \\ \bar{H}_2^{(3)} & -\bar{G}_2^{(3)}K^{(ep)} \end{pmatrix} \left\{ \dot{\Delta}^{(3)} \right\} \tag{4.3}$$

which is essentially a back-substitution form of the solution of $\{\dot{t}^{(1)}\}$ and $\{\dot{\Delta}^{(2)}\}$ if $\{\dot{\Delta}^{(3)}\}$ is known. This equation can be used at any increment when the boundary condition on $\Gamma^{(3)}$ is determined. The second result is an uncoupled equation for $\{\dot{\Delta}^{(3)}\}$, which takes the form

$$\left[\bar{\mathbf{H}}_3^{(3)} \quad - \bar{\mathbf{G}}_3^{(3)}\mathbf{K}^{(ep)}\right] \left\{\dot{\Delta}^{(3)}\right\} = \{\bar{\mathbf{B}}_3\} + \left[\bar{\mathbf{G}}_3^{(3)}\right] \left\{\dot{R}^{(3)}\right\}. \tag{4.4}$$

This equation has unknowns only on the boundary $\Gamma^{(3)}$. Hence, at any increment level, (4.4) can be applied in an incremental manner in order to determine the configuration of the boundary $\Gamma^{(3)}$. Using this procedure, the nonlinear boundary element problem is solved by a localized iteration procedure and the overall boundary element system matrix is factorised only once for any number of increments.

5. The wing-crack problem

The application of boundary element methods to the elasto-static analysis of planar crack problems is relatively well established. A comprehensive account of developments in this area is given by ALIABADI [26]. In the boundary element procedure, the exact stress singularity at the tip of a planar crack located in an elastic medium can be incorporated within the computational scheme. This enables the calculation of stress intensity factors for various modes of deformation of the crack tip, which in turn can be used to establish the conditions necessary for studies that investigate further extension of the crack tip [27, 28]. We shall restrict attention to the in-plane deformations of the cracks, the faces of which can interact in a nonlinear fashion, through the application of generalized far-field stress states. When modelling the discretized boundary of a domain, quadratic elements with isoparametric variations of displacements and tractions of the form

$$\left.\begin{matrix} u_i \\ P_i \end{matrix}\right\} = a_0 + a_1\zeta + a_2\zeta^2 \tag{5.1}$$

within the element can be used, where ζ is the local coordinate of the element and $a_r\ (r = 0, 1, 2)$ are arbitrary constants of interpolation. For modelling cracks that occur either at the boundaries or at the interior of the elastic medium, it is necessary to modify (5.1) to account for the $1/\sqrt{\zeta}$-type stress singularity at the crack tip. In the finite element method, the quarter-point elements of the type proposed by HENSHELL and SHAW [29] and BARSOUM [30] can be used to model the required variation in the displacement, which is of the $\sqrt{\zeta}$-type. If the same type of element is implemented in the boundary element scheme, we have

$$\left.\begin{matrix} u_i \\ P_i \end{matrix}\right\} = b_0 + b_1\sqrt{\zeta} + b_2\zeta \tag{5.2}$$

where $b_i\ (i = 0, 1, 2)$ are constants, the required stress singularity cannot be duplicated. CRUSE and WILSON [31] introduced what is now known as the *singular* *traction quarter-point boundary element* where the variations in the tractions are expressed in the form

$$P_i = \frac{c_0}{\sqrt{\zeta}} + c_1 + c_2\sqrt{\zeta} \tag{5.3}$$

where $c_i\,(i = 0, 1, 2)$ are constants. Singular traction quarter-point boundary elements have been extensively applied in the modelling of both plane and axisymmetric crack problems in elasticity theory and the accuracy of the modelling is well documented ([26, 32–34]). The provision of the special singularity element permits the evaluation of the stress intensity factors at the crack tip. For the problems with in-plane deformations discussed here only the Mode I and Mode II stress intensity factors are relevant. The increments in these stress intensity factors can be determined by applying a displacement correlation method, which makes use of the increments of the nodal displacements at four locations A, B, E and D and the crack tip C (Fig. 2). The incremental estimates for the stress intensity factor are given by

$$\dot{K}_I = \frac{2G}{(k+1)}\sqrt{\frac{2\pi}{l}}\left[4\{\dot{\Delta}_y(B) - \dot{\Delta}_y(D)\} + \dot{\Delta}_y(E) - \dot{\Delta}_y(A)\right], \tag{5.4}$$

$$\dot{K}_{II} = \frac{2G}{(k+1)}\sqrt{\frac{2\pi}{l}}\left[4\{\dot{\Delta}_x(B) - \dot{\Delta}_x(D)\} + \dot{\Delta}_x(E) - \dot{\Delta}_x(A)\right], \tag{5.5}$$

where, for plane strain problems, $k = (3 - 4\nu)$ and for plane stress problems $k = (3 - \nu)/(1 + \nu)$. The calculation of the stress intensity factors through the displacement correlation technique is sensitive to the choice of the nodal points used in the computations. It is found that consistent evaluation of the stress intensity factors is ensured provided the nodal points A, B, E and D and the crack tip C are located within the wing crack section. The mesh discretizations can be structured to achieve consistent evaluation of the stress intensity factors.

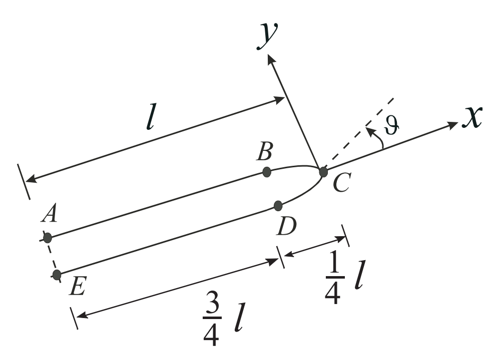

FIG. 2. Node arrangement for calculation of stress intensity factors at the crack tip.

In this paper we combine the nonlinear incremental boundary element technique described in the previous section and the modelling of the crack tip behaviour described above to examine the problem of two asymmetrically placed inclined wing-cracks that are located at the extremities of a frictional contact region in an elastic region (Fig. 1). The frictional contact region is inclined at an angle β to the axis of uniaxial incremental stress σ^*, and the entire plane region is first subjected to an isotropic stress state σ_0. We assume that during the application of this isotropic state of stress, the frictional interface region establishes contact and that the tips of the wing cracks will remain in an open condition both during the application of the isotropic stress and the incremental uniaxial stress σ^*. This is an assumption inherent in all classical models of the crack responses. To prevent closure or inter-penetration in the open crack regions, without the frictional constraint, an initial gap (approximately $0.01\, l$) is provided and during the computations, the closure between the crack surfaces is monitored to ensure that the requirement for non-inter-penetration of the crack surfaces is satisfied.

The parameters required for the computational modelling include the normalized magnitude of the isotropic stress (σ_0/G); the coefficient of friction at the interface region (μ); the length of the wing cracks relative to the half-length of the frictional interface (l/c); the orientation of the wing crack in relation to the alignment of the Coulomb friction interface region (θ), and the relative magnitudes of the shear and normal stiffnesses k_s/G and k_n/G.

[Note that as per (3.2) and (3.10), the stiffnesses $\tilde{k}_s$ and $\tilde{k}_n$ are expressed in units of $force/(length)^3$ to provide tractions with units of *stress*. In the normalization of these stiffnesses we can define non-dimensional parameters k_s/G and k_n/G, where $k_s = \tilde{k}_s \times 1$ and where $k_n = \tilde{k}_n \times 1$, where unity represents the thickness over which the plane problem is being analysed].

The following specific values of the interface stiffness parameters and the initial stress state are used in the computations:

$$\frac{k_n}{G} = 10^3; \qquad \frac{k_s}{G} = 0.5 \times 10^3; \qquad \frac{\sigma_0}{G} = 0.1. \tag{5.6}$$

The boundary element discretization of the domain is shown in Fig. 3. The computational approach considers two separate domains of the region containing the cracks and the nonlinear interface and enforces either complete continuity or frictional constraints or crack boundaries depending on the location of the boundary or interface nodal location. In order to keep the presentation of numerical results to a minimum, we will restrict attention to only the illustration of results applicable to the Mode I stress intensity factor (K_I) and for a specific orientation of the frictional interface to the direction of the incremental loading.

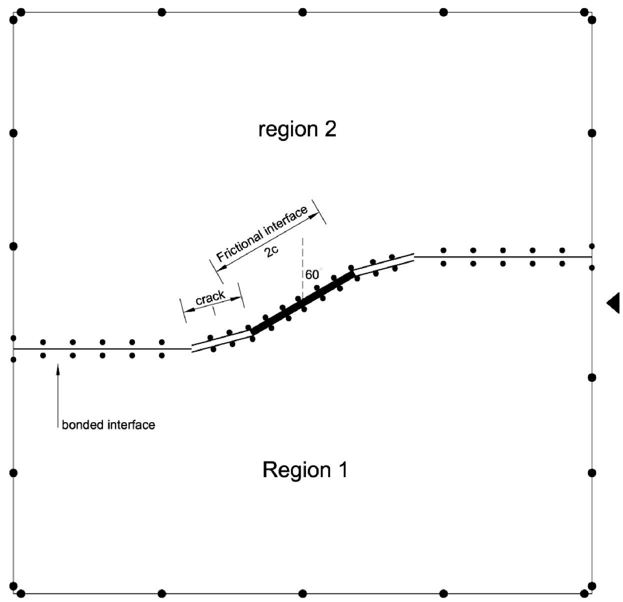

Fig. 3. Boundary element discretizations of the domain.

The emphasis on the illustration of the results applicable to (K_I) is justified in view of the fact that the further extension of the crack under the application of the axial loading is largely governed by the attainment of a critical value of K_I. Figures 4 to 6 illustrate the variation in K_I at the crack tip for various values of the normalized compressive incremental stresses σ^*/G, the coefficient of Coulomb friction μ, the length ratio l/c, when the crack is inclined at $\beta = 60°$ to the direction of application of the axial stress and for the wing-crack orientation at a deviation of $\theta = 15°$.

Computations have been carried out for range of plausible values of the Coulomb friction value; $\mu \in (0.1\,,\,1)$. Figures 7 to 9 present analogous results applicable to the case where only the wing-crack orientation is altered to $\theta = 30°$. These are effectively wing cracks that are aligned with the direction of application of σ^*. The computational results indicate that K_I is highly sensitive to the value of the coefficient of friction and the inclination θ. It must be emphasized that the range of the stresses σ^* used in the computations are purely for purposes of illustration of the computational methodology only. At high values of σ^*/σ_0, the applied stress approaches the value of the shear modulus, which may not be indicative of the stress states that can be sustained by commonly occurring brittle solids, without the development of further cracking or the nucleation of secondary cracking. The computations, however, indicate that for moderate values of σ^*/σ_0, the influence of the frictional constraint can be appreciable, depending on the orientation of the wing cracks.

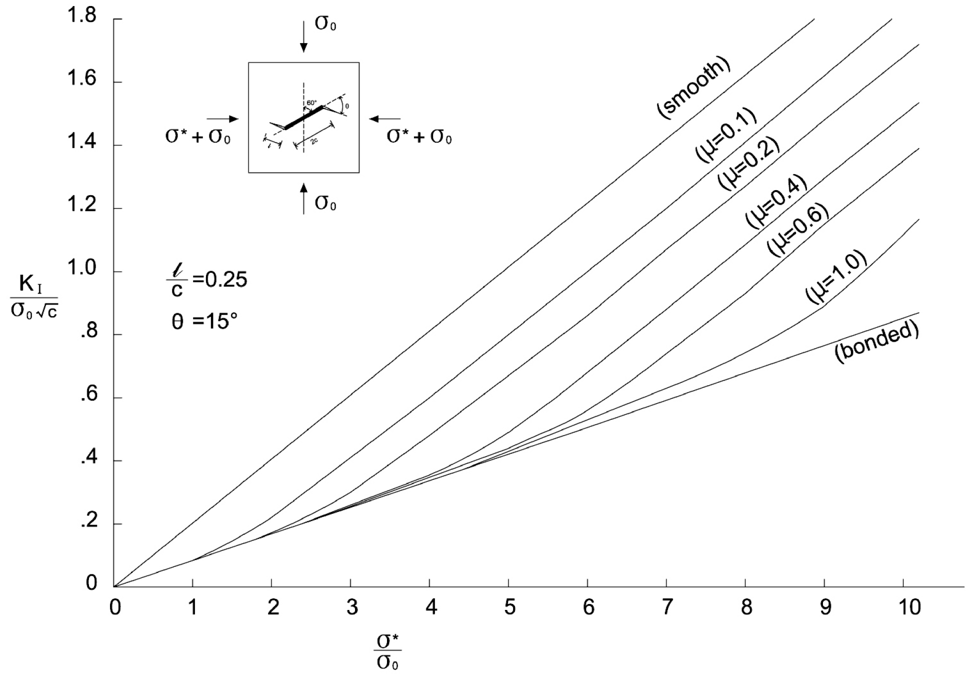

Fig. 4. Boundary element discretizations of the domain.

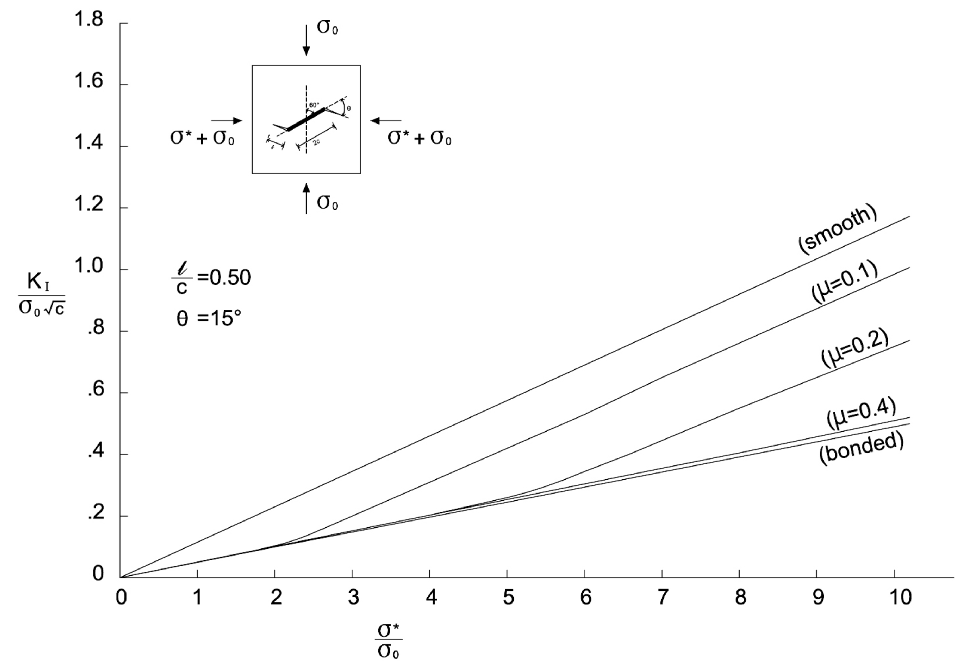

Fig. 5. Mode I stress intensity factor at the crack tip [$\beta = 60°$].

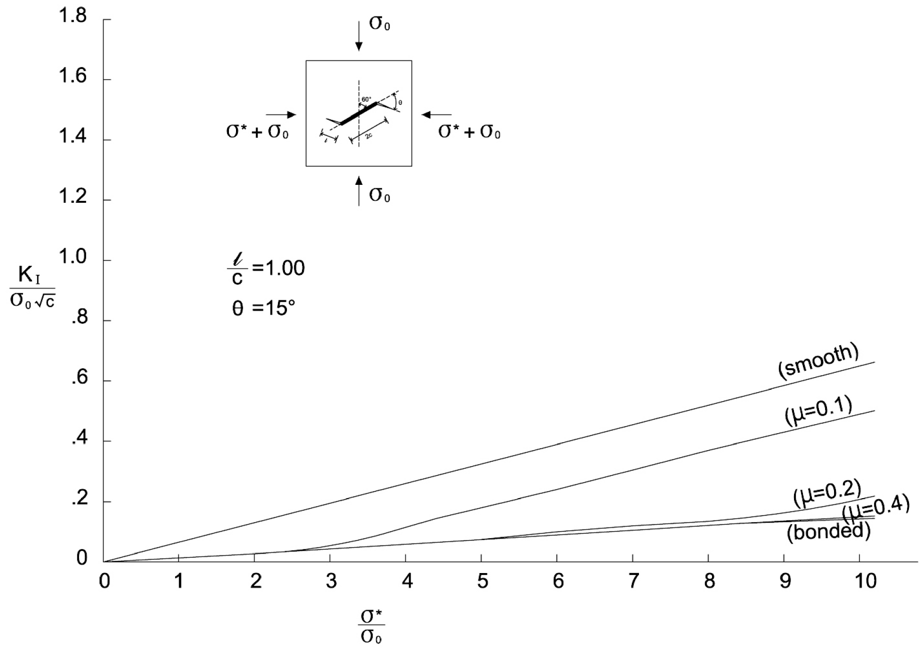

Fig. 6. Mode I stress intensity factor at the crack tip [$\beta = 60°$].

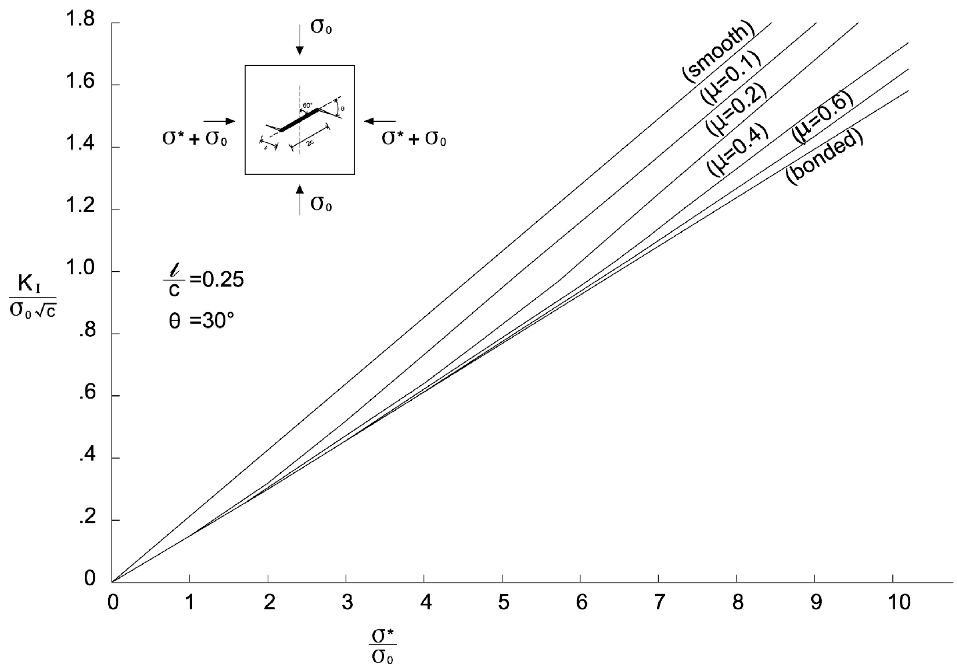

Fig. 7. Mode I stress intensity factor at the crack tip [$\beta = 60°$].

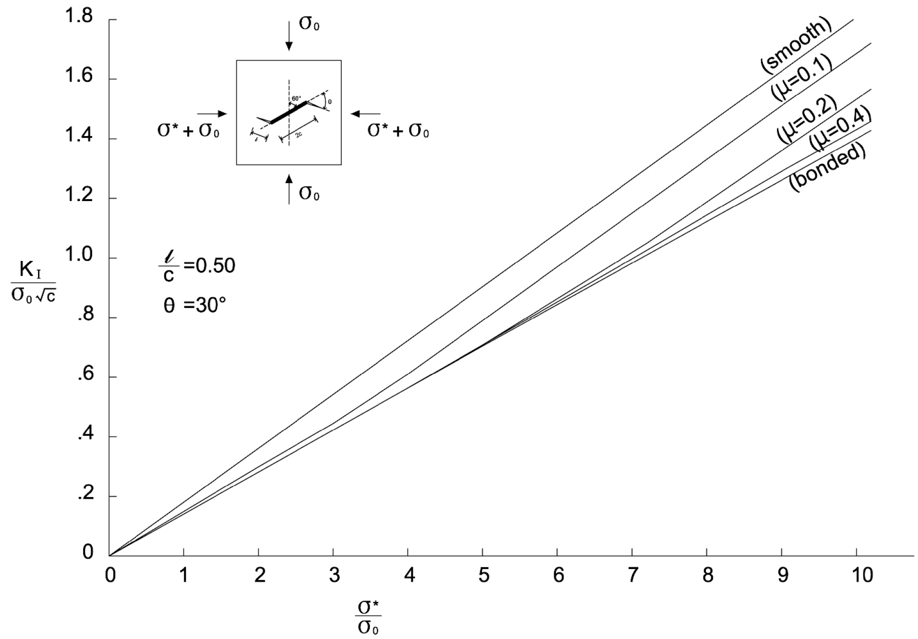

FIG. 8. Mode I stress intensity factor at the crack tip [$\beta = 60°$].

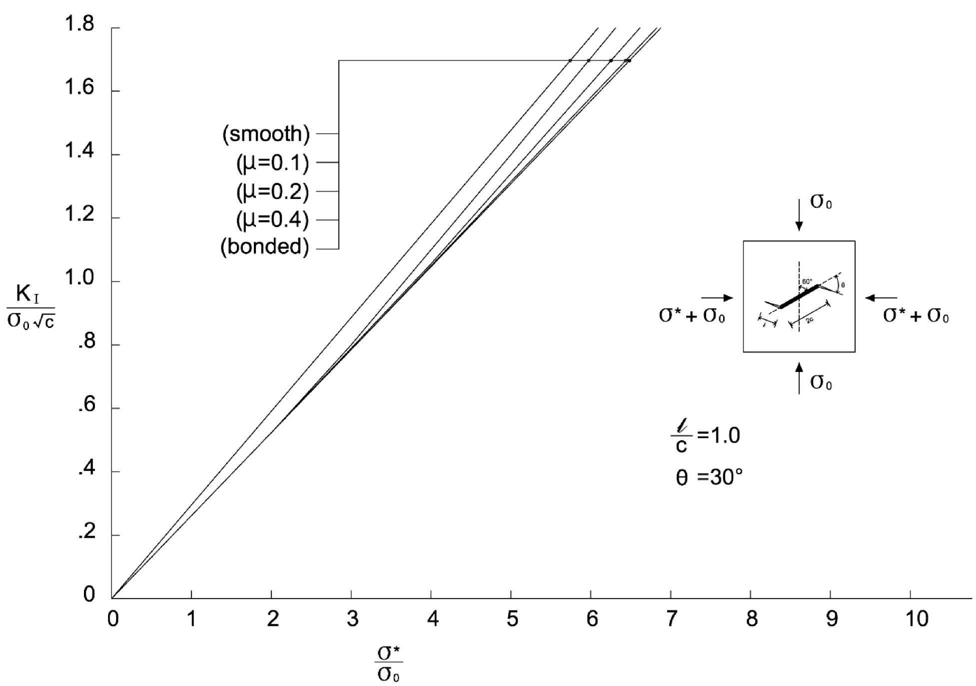

FIG. 9. Mode I stress intensity factor at the crack tip [$\beta = 60°$].

6. Concluding remarks

In classical treatments of planar crack problems it is implicitly assumed that the faces of the crack do not interact during the application of external loads. This assumption will be accurate in a majority of situations where the mode of loading is tensile and the crack is oriented normal to the direction of loading. In many other instances, particularly in the context of materials engineering and geomechanics, segments of cracks can exist in a closed condition both prior to and during the application of external stresses. In these circumstances, the interface behaviour can influence the mechanics of the crack both in terms of the crack growth and the orientation of crack growth, which are influenced by the stress intensity factors at the crack tip. The types of mechanical phenomena that can be encountered at a closed crack surface can be highly varied, with completely smooth and bonded segments constituting the extreme limits. While these extreme limits can be examined through conventional analytical and computational means, the treatment of non-classical effects including friction, slip, dilatancy and other nonlinear constraints at the crack surfaces require nonlinear computational schemes. The boundary element approach is a particularly attractive computational scheme in situations where the non-linear processes are confined only to the surfaces of the crack. It is shown that the class of plane crack problems that exhibit interactions regions that exhibit Coulomb friction can be examined through an interface plasticity formulation, which can be incorporated into an incremental boundary element formulation. The computational scheme is applied to examine the mechanics of wing-cracks that are located at the extremities of a Coulomb friction zone. The complete region is subjected to an isotropic compression followed by uniaxial loading. The computations, for the crack length ratios and the crack alignment angles considered, indicate that the general influence of frictional effects at the discontinuity is to decrease the magnitude of K_I at the tip of the wing-crack. Furthermore, the values of K_I for $\theta = 15^\circ$ are generally lower in magnitude than those for $\theta = 30^\circ$, and this is particularly so when Coulomb friction comes into effect. This observation suggests that if the wing-crack were to extend, the path of extension would deviate to align with the direction of the incremental uniaxial loading. This conclusion is in keeping with the experimental observations of Bieniawski [34] and Horii and Nemat-Nasser [35] conducted on brittle materials. The interface friction at the discontinuity generally acts as a reinforcing effect in altering the crack orientation during its quasi-static extension.

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