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SHORT COMMUNICATIONS

AN ENERGY ESTIMATE OF THE FLEXURAL BEHAVIOUR OF A CIRCULAR FOUNDATION EMBEDDED IN AN ISOTROPIC ELASTIC MEDIUM

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INTRODUCTION

The elastic analysis of circular plates embedded in soil and rock media is of importance to the geotechnical study of circular structural foundations, disc shaped anchors and in the examination of *in-situ* tests such as deep plate load tests and screw plate tests.

This note is concerned with the analytical study of the axisymmetrical flexure of a circular foundation which is embedded in bonded contact with an isotropic elastic medium of infinite extent. The flexible circular foundation is subjected to a uniformly distributed axisymmetric external load of finite extent. A theoretical solution of the above problem in the context of the linear theory of elasticity can be attempted by employing a variety of analytical and numerical techniques. The rigorous analytical treatment of the problem, which involves the solution of the governing equations of three-dimensional elasticity, can be reduced to the solution of a set of complex integro-differential equations. For example, the analogous treatments of the rigid circular plate problem are outlined by Collins[1] and Hunter and Gamblen.[2] Here, we examine the application of an energy method to the solution of the stated problem. In this development it is assumed that the deflected shape of the plate can be represented in terms of elementary functions which satisfy the symmetry and kinematic requirements of the plate flexure. This deflected shape is defined to be within a set of arbitrary constants. The circular plate is assumed to be bonded to the elastic medium at the plane surfaces. Furthermore, when examining the state of stress induced in the infinite elastic medium by the deformed circular plate, it is assumed that the thickness of the plate is negligible in comparison to its radius. The energy method centres around the development of the total potential energy functional for the flexible plate–elastic medium system, consistent with the assumed deflected shape of the plate and the imposed external loads. The total potential energy functional is composed of (i) the strain energy of the infinite elastic medium, (ii) the strain energy of the circular plate and (iii) the work component of the external loads applied to the circular plate. The strain energy of the infinite elastic medium can be developed by computing the work component of the surface tractions which comprise the bond stresses at the interface. The bond stresses associated with the imposed displacements, which are identical to the assumed plate deflections, can be determined by making use of the integral equation methods developed for mixed boundary value problems in the classical theory of elasticity. The strain energy of the plate region can be determined by considering the flexural and membrane energies of the plate region corresponding to the prescribed deflected shape.

The arbitrary constants characterizing the deflected shape of the circular plate can be uniquely determined by making use of the set of linear algebraic equations generated from the

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minimization of the total potential energy functional. The general procedure outlined above is used to analyse the flexural behaviour of the circular foundation, the deflected shape of which is represented by a second-order parabolic curve. This particular deflected shape is assumed to represent, approximately, the flexural behaviour of a moderately rigid foundation (i.e., the relative rigidity of the soil–foundation system is different from an infinite value). Using the energy method, analytical expressions are derived for the deflection and the central flexural moment of the embedded circular foundation. Numerical results presented in this note illustrate the manner in which the central deflection, the differential deflection and the central flexural moment of the embedded foundation are influenced by the extent of the applied load and the relative flexibility of the plate–elastic medium system.

ANALYSIS

We consider the axisymmetric problem of an isotropic elastic medium of infinite extent which is internally loaded by a flexible circular elastic foundation. The thickness of the foundation (h) is assumed to be small in comparison with the radius (a). The circular foundation, which is in bonded contact with the elastic medium, is subjected to an axisymmetric load of uniform intensity (p_0) which acts over a finite region (Figure 1). Owing to the axisymmetric loading of the

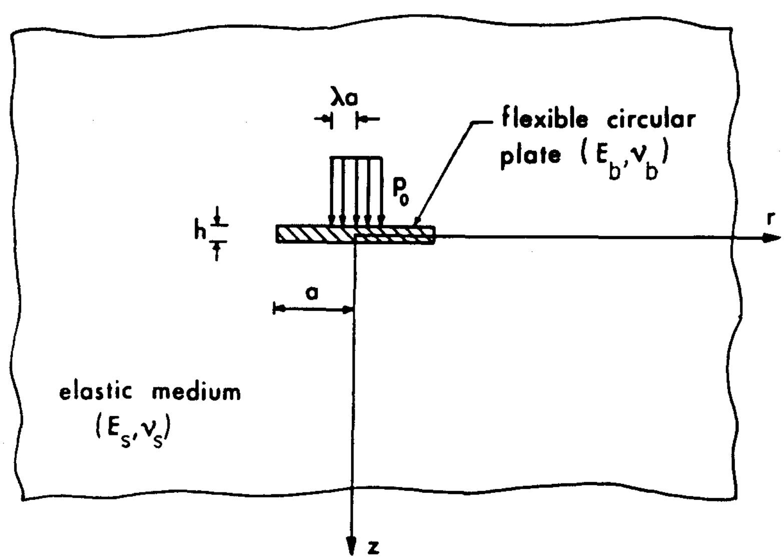

Figure 1. The geometry of the embedded circular foundation

circular foundation, displacements are induced at the bonded interface. In general, such displacements may occur in both the radial and axial directions, r and z respectively. In the ensuring development, however, it is explicitly assumed that the displacements at the interface occur only in the z-direction. The assumption of bonded contact between the foundation and the elastic medium ensures that the displacements at the interface also represent the deflected shape ($w(r)$) of the foundation. An expression for the total potential energy functional appropriate to the elastic plate–infinite elastic medium system can be developed by making use of the function $w(r)$.

The elastic strain energy of the circular plate subjected to the axisymmetric deflection $w(r)$ is composed of only the flexural energy of the plate U_F, given by

$$U_F=\frac{D}{2}\int\int_S\left[\{\nabla^2 w(r)\}^2-2(1-\nu_b)\frac{d^2w(r)}{dr^2}\left\{\frac{1}{r}\frac{dw(r)}{dr}\right\}\right]r\,dr\,d\theta \tag{1}$$

where ∇^2 is Laplace's operator given by

$$\nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr} \tag{2}$$

$D=E_b h^3/12(1-\nu_b^2)$ is the flexural rigidity of the circular foundation, E_b and ν_b are the elastic modulus and Poisson's ratio for the plate material and S corresponds to the plate region. The elastic energy of the infinite space region (U_E) can be developed by computing the work component of the surface tractions which compose the interface stresses. Since displacements at the interfaces are prescribed only in the z-direction, it is necessary to examine the work component of tractions normal to the plane surfaces of the circular plate. These normal tractions can be uniquely determined by making use of integral equation methods developed for the analysis of mixed boundary value problems in classical elasticity. We consider the problem of an isotropic elastic medium of infinite extent, which is subjected to the axisymmetric displacement field

$$u_r=0, \qquad u_z=w(r) \quad \text{for } z=0\ (0\leqslant r\leqslant a) \tag{3}$$

where u_r and u_z are the components of the displacement vector in the r and z directions respectively and $0\leqslant r\leqslant a$ corresponds to the foundation region. By employing the integral equation methods outlined by Collins[1] it can be shown that the normal stress on the bonded foundation region is given by

$$[\sigma_{zz}]_{z=0\pm}=\pm\frac{4G_s(1-\nu_s)}{r}\frac{d}{dr}\int_r^a\frac{tf(t)\,dt}{\sqrt{(t^2-r^2)}}, \quad \text{for } (0\leqslant r\leqslant a) \tag{4}$$

where

$$f(t)=\frac{2}{\pi(3-4\nu_s)}\frac{d}{dt}\int_0^t\frac{rw(r)\,dr}{\sqrt{(t^2-r^2)}}, \quad \text{for } (0\leqslant t\leqslant a) \tag{5}$$

and G_s and ν_s are respectively the linear elastic shear modulus and Poisson's ratio of the elastic medium. The upper and lower signs of (4) refer to the surfaces $z=0^+$ and $z=0^-$ of the flexible foundation. Using the above results it can be shown that the strain energy of the infinite elastic medium due to the prescribed foundation displacement $w(r)$ is given by

$$U_E=\frac{8G_s(1-\nu_s)a^3}{(3-4\nu_s)\pi}\int\int_S\frac{w(r)}{a}\left[\frac{d}{dr}\int_r^a\frac{t}{\sqrt{(t^2-r^2)}}\left\{\frac{d}{dt}\int_0^t\frac{rw(r)\,dr}{a^2\sqrt{(t^2-r^2)}}\right\}dt\right]dr\,d\theta \tag{6}$$

The potential energy of the external loads applied to the circular foundation is given by

$$U_p=-\int\int_{S_0}p(r)w(r)r\,dr\,d\theta \tag{7}$$

where S_0 is the extent of the applied loads. By combining (1), (6) and (7) we obtain the total potential energy functional for the circular foundation–elastic medium system as

$$U=U_F+U_E+U_p \tag{8}$$

From the principle of stationary total potential energy we require

$$\delta U = 0 \tag{9}$$

where δU is the variation of the total potential energy. Alternatively we may note that when a deflected shape $w(r)$ satisfies the condition $\delta U = 0$, for all $\delta w(r)$, it can be shown that $w(r)$ is the solution of the elasticity problem. To apply the principle of total potential energy to the circular foundation problem we assume that the deflected shape $w(r)$ can be represented in the form

$$w(r) = a \sum_{i=0}^{n} C_i \chi_i(r) \tag{10}$$

where C_i are arbitrary constants and $\chi_i(r)$ are arbitrary functions which render the displacement field kinematically admissible. The principle of total potential energy then requires that U be an extremum with respect to the kinematically admissible displacement field characterized by C_i (Sokolnikoff,[3] Washizu[4]). Hence

$$\frac{\partial U}{\partial C_i} = 0 \qquad (i = 0, 1, 2, \ldots, n) \tag{11}$$

This minimization procedure yields n simultaneous equations for the undetermined coefficients C_i.

EMBEDDED CIRCULAR FOUNDATION

The formal procedure developed in the previous section is now applied to the analysis of the internally loaded circular foundation problem. It is assumed that the deflected shape of the circular foundation, which is subjected to a uniform axisymmetric load over a finite area, can be represented in the form

$$w(r) = a \sum_{i=0}^{n} C_{2i}\left(\frac{r}{a}\right)^{2i} \tag{12}$$

where C_{2i} are arbitrary constants. In (12), the particular choice of functions corresponding to χ_i gives a kinematically admissible plate deflection and finite flexural moments and shearing forces in the plate region $0 \leq r \leq a$. As a first approximation of (12), we restrict our attention to a deflected shape represented by the second degree curve

$$w(r) = a\left[C_0 + C_2\left(\frac{r}{a}\right)^2\right] \tag{13}$$

The total potential energy functional U can be evaluated by making use of (8) and (13). Accordingly we obtain

$$U = \frac{16 G_s a^3 (1-\nu_s)}{(3-4\nu_s)}[C_0^2 + \beta_1 C_0 C_2 + \beta_2 C_2^2] + 4\pi D(1+\nu_b)C_2^2 - \pi p_0 \lambda^2 a^3\left[C_0 + \frac{\lambda^2}{2} C_2\right] \tag{14}$$

where

$$\beta_1 = \frac{4}{3}; \qquad \beta_2 = \frac{4}{15}$$

The constants C_0 and C_2 can be determined from the equations which are obtained by the minimization conditions

$$\frac{\partial U}{\partial C_0}=0, \qquad \frac{\partial U}{\partial C_2}=0 \tag{15}$$

Avoiding details of computation, it can be shown that the deflected shape of the circular foundation corresponding to (13) is given by

$$w(r)=\frac{P(3-4\nu_s)}{32G_s(1-\nu_s)a}\left[\frac{\{\mu_1-2\lambda^2+3R\}}{\{\mu_2+3R\}}-\frac{\{4-3\lambda^2\}}{\{\mu_2+3R\}}\left(\frac{r}{a}\right)^2\right] \tag{16}$$

where $P\ (=\pi p_0\lambda^2 a^2)$ is the total load applied to the circular foundation;

$$\mu_1=\frac{144}{30}, \qquad \mu_2=\frac{64}{30} \tag{17}$$

and R is a relative rigidity parameter of the circular foundation–elastic medium system defined by

$$R=\frac{\pi(3-4\nu_s)(1+\nu_s)}{12(1-\nu_b)(1-\nu_s)}\frac{E_b}{E_s}\left(\frac{h}{a}\right)^3 \tag{18}$$

The relative rigidity parameter R can therefore be used to examine the limits of applicability of the approximate solution (16), developed on the basis of the energy method. We note that as $R\to\infty$, the circular foundation becomes infinitely rigid; as such, the displacement (w_0) for a rigid circular plate embedded in complete bonded contact with an isotropic elastic infinite medium is given by

$$w_0=\frac{P(3-4\nu_s)}{32G_s(1-\nu_s)a} \tag{19}$$

The above expression is in agreement with equivalent results derived by Collins,[1] Kanwal and Sharma[5] and Selvadurai[6] for the rigid disc inclusion obtained by considering respectively integral equation methods, singularity methods and direct spheriodal harmonic function methods for the solution of the associated elastic problem. In the particular case when $\lambda\to 1$, and $R\to 0$ the interaction problem is reduced to that of the internal loading of an infinite elastic medium by a uniform circular area of radius a and stress intensity p_0. For this case, the central deflection $w(0)$ as determined from the energy solution (16) can be compared with the equivalent exact result obtained by an integration of the result for a Kelvin force (see e.g., Reference 3) over a circular area. We have

$$[\{w(0)\}_{\text{energy}};\{w(0)\}_{\text{exact}}]=\frac{P(3-4\nu_s)}{32G_s(1-\nu_s)a}\left[\frac{21}{16};\frac{4}{\pi}\right] \tag{20}$$

Similarly, the expressions for the deflection at the boundary of the uniformly loaded circular area yield

$$[\{w(a)\}_{\text{energy}};\{w(0)\}_{\text{exact}}]=\frac{P(3-4\nu_s)}{32G_s(1-\nu_s)a}\left[\frac{54}{64};\frac{8}{\pi^2}\right] \tag{21}$$

The flexural moments induced in the embedded foundation can, in principle, be calculated by making use of the expression for the foundation deflection given by (16) and the relationships

$$[M_r;M_\theta]=-D\left[\left\{\frac{d^2w}{dr^2}+\frac{\nu_b}{r}\frac{dw}{dr}\right\};\left\{\frac{1}{r}\frac{dw}{dr}+\nu_b\frac{d^2w}{dr^2}\right\}\right] \tag{22}$$

It may be noted that while the energy method provides an accurate estimate of the deflections of the foundation $w(r)$, the accuracy with which $w(r)$ is able to predict the flexural moments in the foundation is, in general, considerably less (see e.g., Dym and Shames[7]). Any inaccuracies that may be present in the energy expression for $w(r)$, as defined by (16), are greatly magnified in the computation of M_r and M_θ owing to the presence of derivatives of $w(r)$ up to the second order. A more accurate estimate of the flexural moments in the foundation can be determined by considering the flexural response of the foundation under the combined action of the applied stress p_0 and the contact stresses $[\sigma_{zz}]_{0\pm}$. The maximum flexural moment at the centre of the circular foundation can be computed by using the solutions developed for the flexure of a circular plate *simply supported along its boundary.* It can be shown that the flexural moment at the centre of the plate (M_0) is given by

$$\frac{M_0}{p_0 a^2 \lambda^2} = \frac{4-(1-\nu_b)\lambda^2}{16} - \frac{(1+\nu_b)}{4}\ln\lambda - \frac{(1-\nu_b)}{120}\left\{\frac{5\mu_3 - 32 - 16\lambda^2 + 15R}{\mu_2 + 3R}\right\}$$
$$+\frac{(1+\nu_b)}{36}\left\{\frac{(9\mu_3 - 96 + 27R)\ln 2}{\mu_2 + 3R}\right\} - \frac{(1+\nu_b)}{36}\left\{\frac{9\mu_3 - 80 - 12\lambda^2 + 27R}{\mu_2 + 3R}\right\} \tag{23}$$

where $\mu_3 = 64/5$. In the particular case when $\lambda \to 1$ and $R \to \infty$, (23) yields the following expression for the central flexural moment in an embedded 'rigid' foundation which is in bonded contact with an infinite elastic medium; i.e.,

$$[M_0]_{\text{rigid}} = p_0 a^2\left[\frac{(7+5\nu_b)}{48} - \frac{(1+\nu_b)}{4}\{1 - \ln 2\}\right] \tag{24}$$

The variation of the central deflection of the embedded flexible plate $\{w(0)\}$ with the relative rigidity of the plate–elastic medium system and the extent of the applied load is illustrated in Figure 2. Similar results developed for the differential deflection $\{w(0) - w(a)\}$ and the maximum flexural moment are shown in Figures 3 and 4.

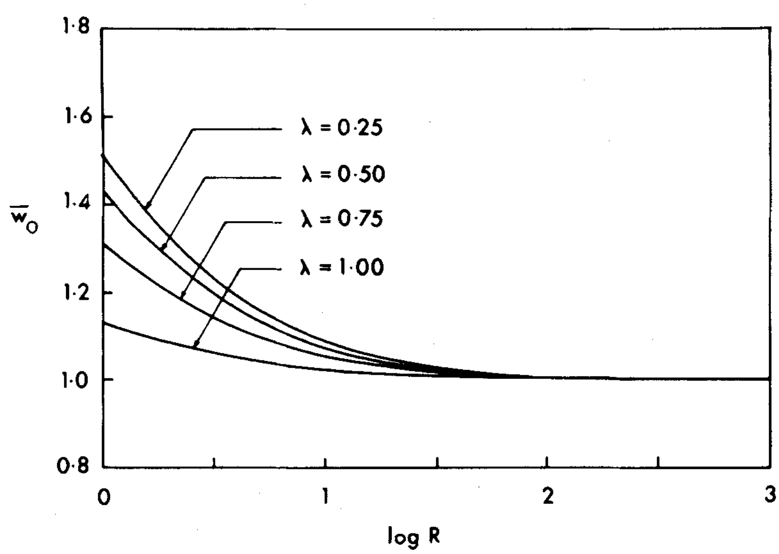

Figure 2. The variation of the central deflection of the embedded circular foundation $[w(0) = \{\bar{w}_0 P(3-4\nu_s)/32G_s(1-\nu_s)a\}]$

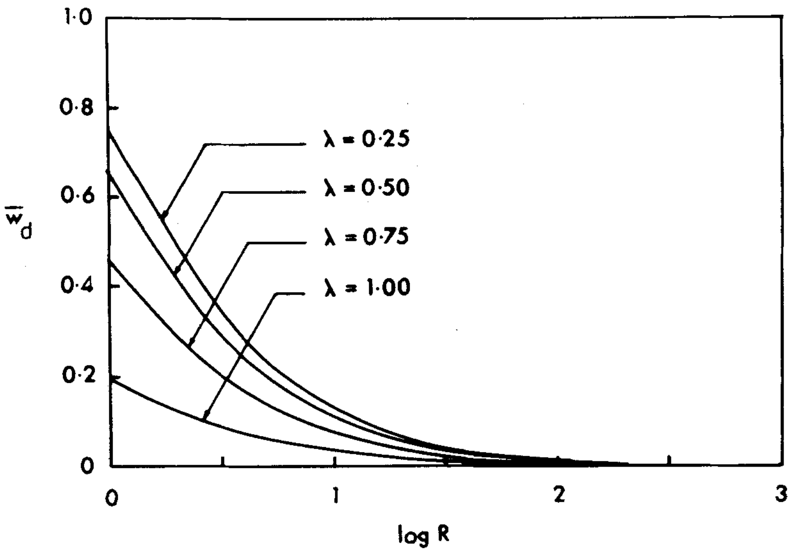

Figure 3. The variation of the differential deflection of the embedded circular foundation $[w(0) - w(a) = \{\bar{w}_d P(3-4\nu_s)/32G_s(1-\nu_s)a\}]$

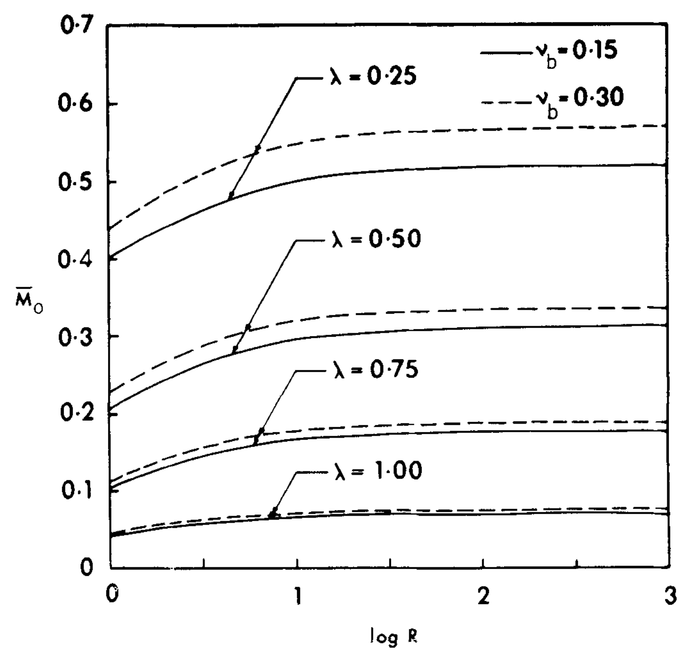

Figure 4. The variation of central flexural moment in the embedded circular foundation $[M(0) = \bar{M}_0 p_0 a^2 \lambda^2]$

CONCLUSIONS

The energy method of analysis outlined in this note provides an approximate solution to the axisymmetric flexure of a circular plate embedded in an infinite elastic solid. A second degree curve is used to represent the flexural deflection of the plate. In the limiting case of infinite relative rigidity this deflected shape yields the exact solution to the response of an infinitely rigid plate embedded in an isotropic elastic infinite space. Similarly, for a uniformly loaded plate with zero relative rigidity, the energy approximation for the deflected shape correlates well with the exact solution for the internal loading of the infinite medium with a uniform circular load. The numerical results presented in Figures 2–4 indicate that the relative rigidity (in the range $0 \to \infty$) and the extent of the external load together have a significant influence on the deflections and flexural moments developed in the embedded plate. The assumed form of the deflected shape is clearly inappropriate for situations involving localized loading ($\lambda \to 0$) of highly flexible ($R \to 0$) embedded plates. In this case, higher-order and logarithmic terms in r have to be incorporated in the assumed deflected shape to accurately predict the deflections and flexural moments in the circular plate. Alternatively, the solution of the highly flexible locally loaded plate can be investigated by analysing the appropriate infinite plate problem.

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