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Plane cracks with frictionally constrained surfaces

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Abstract

This paper examines the class of problems dealing with the mechanics of planar cracks that are subjected to incremental loading and where the surfaces of the crack exhibit a non-linear constraint over a finite region. Since the effects of the non-linearity are restricted to pre-defined surfaces, the boundary element technique can be applied quite successfully to determine the influence of such non-linearity on the development of stress intensity factors at the crack tip. In particular, attention is focussed on the evaluation of the stress intensity factors for plane crack with a non-linear surface constraint, which is subjected in turn to an isotropic compressive stress field followed by an incremental uniaxial loading in either tension or compression.

Keywords: Boundary elements; Contact mechanics; Coulomb friction; Frictional contact in cracks; Incremental analysis of cracks; Incremental plasticity; Interface degradation; Stress intensity factors

1. Introduction

The category of problems that deal with non-linear processes that are confined to interfaces between contacting bodies is important to several branches of engineering. In classical approaches to such problems in contact mechanics, the interfaces are regarded as being either completely smooth or fully bonded. The classical solutions given in [1–6] deal with contact problems with such extremes in the interface responses. The subject matter associated with these developments and complete reviews of the mathematical developments associated with these classical contact problems are given in Refs. [7–16]. Accounts of recent advances are also given in [17–21]. The main improvement to the basic idealizations of the interface response stems from the incorporation of frictional phenomena at the contacting zones. The influence of contact friction and associated non-linear phenomena are of importance to many engineering applications including tribology, micro-mechanics of composite materials, mechanics of geomaterial interfaces including fault and fracture zones in the earth and more recently in the modelling of tactile sensors used in industrial applications. The presence of non-linearity in the mechanics of the contact zone makes the analytical approach to contact problems quite restrictive. Some earlier applications of frictional contact mechanics were made in connection with the study of the influence of friction on the contact between spheres [22]. Further examples [23,24] deal with the topic of friction at the base of a flat punch indenting a halfspace region. The analytical study of the contact problem becomes intractable when complicated forms of frictional contact phenomena, influence of separation and

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slip zones, interaction of contacting regions with inclusions and cracks, etc., need to be considered. With geomaterial regions in particular, the interfaces between contacting regions can exhibit a variety of non-linear phenomena including Coulomb friction, finite friction, interface plasticity, interface damage, asperity degradation, viscoplasticity and creep [25]. These can be complemented with constitutive laws that are derived from experimental investigations. Numerical approaches are therefore of particular interest in the study of contact phenomena where interface non-linearity becomes a dominant feature. Extensive advances have been made in the application of finite element techniques to the study of contact phenomena associated with non-linear interfaces. Detailed accounts of these developments are given in [26–31].

In this paper the boundary element method is advocated for the analysis of the contact problem that deals with non-linear effects at the contact zone. The advantage of the boundary element is self-evident in view of the fact that the non-linear contact phenomena are confined to pre-defined contact zones. The application of boundary element techniques to frictional contact problems can be traced to the earlier studies detailed in [32–34] where it was used to examine frictional contact problems. The boundary element procedure has also been used to examine a variety of problems related to non-linear interface phenomena [35–38], while other researchers [39,40] have applied the boundary element procedure to investigate the mechanics of discontinuities in geologic media. The influence of interface non-linearity on the behaviour of planar cracks located in elastic media was also investigated in Refs. [37,41]. In this paper we present a brief discussion of the incremental boundary element approach for the modelling of interface non-linearity and give results for the problem of a plane crack that has an interface non-linearity over a finite region. The non-linearity becomes effective only during the establishment of closure over this finite region. The region containing the plane crack is first subjected to an isotropic stress field and the non-linear effects are enforced during the application of an incremental uniaxial stress field at an oblique orientation to the plane of the crack. It is further assumed that the crack topography is such that the crack tips remain in an open condition to permit the evaluation of the stress intensity factors at the crack tip. The numerical results presented in the paper illustrate the manner in which these stress intensity factors are influenced by the sense of application of the uniaxial stress field and its relative magnitude.

2. The incremental boundary element procedure

The boundary element approach for the modelling of elastic continua is well established (i.e. [42–45]). For completeness, however, we shall present a brief summary of the relevant equations as they pertain to an incremental formulation. The incremental formulation is necessary for the modelling of interface non-linearity encountered at the crack surfaces. The boundary integral equation for a region ρ with volume $V^{(\rho)}$ and with surface $\Gamma^{(\rho)}$ can be written in the form

$$c_{ij}\dot{u}_j^{(\rho)} + \int_{\Gamma^{(\rho)}} P_{ij}^{*(\rho)}\dot{u}_j^{(\rho)}\,\mathrm{d}\Gamma = \int_{\Gamma^{(\rho)}} u_{ij}^{*(\rho)}\dot{P}_j^{(\rho)}\,\mathrm{d}\Gamma \tag{1}$$

where $i,j = 1,2,3$ (or x,y,z); $\dot{u}_j^{(\rho)}$ and $\dot{P}_j^{(\rho)}$ are, respectively, the incremental values of the boundary displacements and boundary tractions; $u_{ij}^{*(\rho)}$ and $P_{ij}^{*(\rho)}$ are the corresponding fundamental solutions, given by

$$u_{ij}^{*(\rho)} = \frac{1}{16\pi G_\rho(1-\nu_\rho)r}[(3-4\nu_\rho)\delta_{ij} + r_{,i}r_{,j}] \tag{2}$$

and

$$P_{ij}^{*(\rho)} = -\frac{1}{8\pi(1-\nu_\rho)r^2}\{[(1-2\nu_\rho)\delta_{ij} + 3r_{,i}r_{,j}]r_{,n} - (1-2\nu_\rho)[r_{,i}n_j - r_{,j}n_i]\} \tag{3}$$

respectively, where r is the distance between the source and field points; n_i are the components of the outward unit normal vector to $\Gamma^{(\rho)}$; δ_{ij} is Kronecker's delta function; G_ρ and ν_ρ are the linear elastic shear modulus and Poisson's ratio respectively. Also, in (1), $c_{ij} = \delta_{ij}/2$ if the boundary is smooth. The boundary integral equation (1) is applicable to any region V consisting of sub-regions $V^{(\rho)}$, which are separated by non-linear interfaces. The bodies in contact can be subjected to the conventional displacement and traction boundary conditions as well as interface conditions. On a boundary Γ_1 where displacements are prescribed,

$$\dot{u}_i = \dot{u}_i^0 \tag{4}$$

where $\dot{u}_i^0$ is a prescribed increment of displacement. Similarly, on a boundary Γ_2 where tractions are prescribed

$$\dot{P}_i = \dot{P}_i^0 \tag{5}$$

where $\dot{P}_i^0$ is a prescribed increment of traction. On an interface region Γ_3 with non-linear constraints we have

$$\dot{P}_i = \dot{R}_i + K_{ij}^*\dot{u}_j \tag{6}$$

where $\dot{R}_i$ are increments of a residual traction and K_{ij}^* are stiffness coefficients derived through considerations of the non-linear constraints.

Upon boundary element discretization of the domain, the integral equation can be converted to its matrix equivalent, which can be written in the form

$$[\mathbf{H}]\{\dot{\mathbf{u}}\} = [\mathbf{G}]\{\dot{\mathbf{P}}\} \tag{7}$$

where $[\mathbf{H}]$ and $[\mathbf{G}]$ are the boundary element influence coefficients matrices and $\{\dot{\mathbf{u}}\}$ and $\{\dot{\mathbf{P}}\}$ are the displacement and traction vectors and sub-sets of which together will form the appropriate set of unknowns. If the configuration of the boundary and the interface conditions are defined at any level of deformation, we can obtain a final system equation in the form

$$[\mathbf{A}]\{\dot{\mathbf{U}}\} = \{\dot{\mathbf{B}}\} \tag{8}$$

from which, either the boundary or the interface unknowns can be determined.

3. Interface responses

The modelling of interface responses can be approached at a variety of levels ranging from the nano-scale atomistic models to phenomenological approaches [46–50]. The former approach has a great deal to offer in terms of improving the basic understanding of the micro-mechanical processes that contribute to interface non-linearities [49,51]. From a perspective of computational modelling, however, it is desirable to adopt a phenomenological approach to the formulation of an interface constitutive response. The classical models of completely smooth and bonded conditions represent extremes of the phenomenological approach for the characterization of an interface response; other non-linear forms are derived from considerations of conventional models of Coulomb friction, plasticity, dilatant phenomena that account for a local scale structure, and the incorporation of damage and degradation to account for deterioration of the interface with progressive deformations. The classical plasticity approaches to the modelling of continua are well documented in Refs. [52–58] while other aspects of damage and degradation of interfaces are described in [59–61]. In this paper we shall consider a treatment of the interface response that can be used to model Coulomb friction, dilatant friction and interface degradation resulting from damage to asperities at contact zones. In view of the non-linear nature of the interface response, it is necessary to adopt an incremental approach to the formulation of the constitutive responses. Also, the interface is regarded as a distinct two-dimensional surface that is void of a dimension normal to the plane. As a consequence, the interface responses must be formulated in relation to the incremental relative displacements $\dot{\varDelta}_i$ that take place at this contacting plane. We assume that these incremental relative displacements consist of an elastic or recoverable component $\dot{\varDelta}_i^{(e)}$ and an irrecoverable or plastic component $\dot{\varDelta}_i^{(p)}$: i.e.

$$\dot{\varDelta}_i = \dot{\varDelta}_i^{(e)} + \dot{\varDelta}_i^{(p)} \tag{9}$$

where, for an interface, the subscripts i (or j) can be assigned notations applicable to the local interface coordinates. For purposes of the presentation, we shall denote the values i and j applicable to an interface by x,y,z with the assumption that the direction $z$coincides with the normal to the Euclidean plane at a point on the interface. The elastic component of the incremental displacement $\dot{\varDelta}_i^{(e)}$ is related to the component of the corresponding increment of traction $\dot{t}_i$ through the linear constitutive response

$$\dot{t}_i = \tilde{k}_{ij}\dot{\varDelta}_j^{(e)} \tag{10}$$

where $\tilde{k}_{ij}$ are the linear stiffness coefficients of the interface and summation over the repeated indices is implied. This linear elastic response will persist so long as the tractions acting on the interface do not induce failure at the interface. In order to assess this limiting condition it is necessary to postulate a failure criterion for the interface. These are varied and the applicability of any phenomenological relationship to actual interfaces must be verified by recourse to experimentation.

3.1. Coulomb friction

For interfaces that exhibit Coulomb friction, the failure criterion F is given by

$$F = \sqrt{t_x^2 + t_y^2} + \mu t_z = 0 \tag{11}$$

where t_i are the value of the *total tractions* acting at the interface ($\left|\sqrt{t_x^2 + t_y^2}\right|$ is the tangential traction and t_z is the normal traction acting on the interface) and μ is the coefficient of Coulomb friction at the interface. We now assume that when the interface total tractions satisfy the failure criterion (11), the interface will experience slip in the form of irreversible displacements. It is assumed that these irreversible displacements can be determined from knowledge of a plastic potential in exactly the same way that incremental plastic strains can be determined in a continuum region [58]. The incremental plastic displacements at the interface are defined by

$$\dot{\varDelta}_i^{(p)} = \dot{\lambda}\frac{\partial \varPhi}{\partial t_i} \tag{12}$$

where $\dot{\lambda}$ is a proportionality factor referred to as the plastic/interface slip multiplier and $\varPhi(t_i)$ is the plastic/interface slip potential. In the development of an interface response for an actual interface, this plastic potential needs to be defined through experimentation. For the purposes of this paper, it is convenient to select a plastic potential that relies closely on the structure of the failure criterion. Here we assume that the plastic potential takes the form

$$\varPhi = \sqrt{t_x^2 + t_y^2} \tag{13}$$

Using (9) and (12) we can rewrite (10) in the form

$$\dot{t}_i = \tilde{k}_{ij}\left(\dot{\varDelta}_j - \dot{\lambda}\frac{\partial \varPhi}{\partial t_j}\right) \tag{14}$$

For a Coulomb frictional response, where there is no alteration in the failure characteristics of the interface with deformation, at failure

$$dF = \frac{\partial F}{\partial t_i} dt_i = 0 \tag{15}$$

This result together with (14) can be used to determine the plastic slip multiplier $\dot{\lambda}$; i.e.

$$\dot{\lambda} = \frac{1}{\psi} \frac{\partial F}{\partial t_i} \tilde{k}_{ij} \dot{\Delta}_j \tag{16}$$

where

$$\psi = \frac{\partial F}{\partial t_l} \tilde{k}_{lm} \frac{\partial \Phi}{\partial t_m} \tag{17}$$

We can now rewrite (14) in the form

$$\dot{t}_i = \left[\tilde{k}_{ij} - \frac{1}{\psi} \frac{\partial \Phi}{\partial t_l} \tilde{k}_{il} \tilde{k}_{mj} \frac{\partial F}{\partial t_m} \right] \dot{\Delta}_j = \tilde{k}_{ij}^{(ep)} \dot{\Delta}_j \tag{18}$$

If the failure criterion $F(t_i)$ and the plastic potential $\Phi(t_i)$ are known, then we can define the elastic–plastic stiffness $\tilde{k}_{ij}^{(ep)}$. In the specific case of the Coulomb frictional material with the failure criterion defined by (11) and the plastic potential defined by (13), and for the special case where

$$\begin{aligned} \tilde{k}_{xx} = \tilde{k}_{yy} &= \tilde{k}_s \\ \tilde{k}_{zz} &= \tilde{k}_n \end{aligned} \tag{19}$$

with all other $\tilde{k}_{ij} \equiv 0$, the non-symmetric elastic–plastic stiffness matrix is given by

$$[\tilde{\mathbf{k}}]^{(ep)} = \frac{1}{(t_x^2 + t_y^2)} \begin{bmatrix} \tilde{k}_s t_y^2 & -\tilde{k}_s t_x t_y & -\mu \tilde{k}_n t_x \sqrt{(t_x^2 + t_y^2)} \\ -\tilde{k}_s t_x t_y & \tilde{k}_s t_x^2 & -\mu \tilde{k}_n t_y \sqrt{(t_x^2 + t_y^2)} \\ 0 & 0 & -\tilde{k}_n (t_x^2 + t_y^2) \end{bmatrix} \tag{20}$$

and $\tilde{k}_s$ and $\tilde{k}_n$ can be interpreted as the shear and normal elastic stiffnesses at the interface. These are constitutive parameters associated with the phenomenological model of the interface response. These need to be determined either through micromechanical modelling of the local topography of the interface or through experimentation.

3.2. Dilatant interfaces

The Coulomb interface model presented previously is a classical model that is void of any representation of dilatancy of the interface during shear. The model predicts that there are no relative movements normal to the interface during shear. With geomaterials, such as rock fractures, fracture surfaces in concrete and with other rough surfaces encountered in tribological applications, the interface can experience dilatancy during shear. The Coulomb model has been modified by a number of authors [62–66]. These studies have been extended [67,68] to include fractures in fluid-saturated poroelastic media where interface fluid flow characteristics can also be influenced by shear-induced dilatancy of the interface. The model proposed in Ref. [65] can be incorporated within the current context of interface modelling. In this approach, the interface profile is represented by a periodic distribution of saw-tooth type ridges with asperity angle α (Fig. 1), where the contact between the separate material regions is governed by the friction coefficient μ. The yield condition applicable to the dilatant interface is given by

$$\begin{aligned} F = &\left| t_z \sin\alpha + (t_x^2 + t_y^2)^{1/2} \cos\alpha \right| \\ &+ \mu \left[t_z \cos\alpha - (t_x^2 + t_y^2)^{1/2} \sin\alpha \right] \end{aligned} \tag{21}$$

and

$$\Phi = \left| t_z \sin\alpha + (t_x^2 + t_y^2)^{1/2} \cos\alpha \right| \tag{22}$$

respectively. As the asperity angle $\alpha \to 0$, the results (21) and (22) reduce to the classical Coulomb interface model described by (11) and (13) respectively. The relationships presented for the dilatant interface model can be extended to account for asperity degradation. Asperity degradation is interpreted through the evolution of the asperity angle with deformation. In particular it is postulated that asperity degradation occurs only with dissipation of plastic energy. A plausible relationship for the evolution of the asperity angle α can be expressed in the form

$$\alpha = \alpha_0 \exp\left[-C W_T^{(p)} \right] \tag{23}$$

where α_0 is the initial asperity angle, C is an experimentally derived constant that describes the rate of degradation and $W_T^{(p)}$ is the total plastic work of the tangential forces at the interface. In its incremental form this rate of work is given by

$$\dot{W}_T^{(p)} = t_i \dot{\Delta}_i^{(p)} \tag{24}$$

It should be noted that such a phenomenological interpretation of plastic work assumes a certain isotropy of the interface in terms of its plastic response. The notion of two-dimensional asperities is not consistent

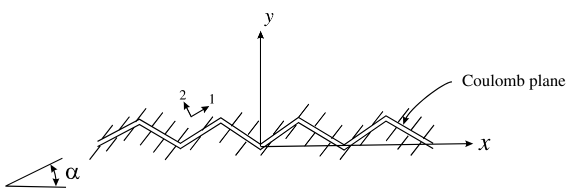


Fig. 1. Model of a dilatant interface with local Coulomb frictional effects.

with the assumed isotropy of the interface response. The model can be extended to include a three-dimensional asperity configuration; the computational implementation of such an asperity configuration is not routine and the orientation of the plastic displacement increment vector needs to be determined through consideration of a minimum plastic energy dissipation condition. For the purposes of this paper, and since all the problems that are examined in the paper refer to two-dimensional plane interfaces the models developed in the paper are sufficient.

The procedures outlined in this and previous Sections can be extended to include both frictional and adhesive effects at the contacting plane, in which case, the Coulomb model should be replaced by an appropriate model that includes interface cohesion effects. These models include the classical Mohr–Coulomb failure criterion, which accounts for both friction μ and cohesion c.

3.3. Contact and separation processes

As the incremental analysis of an interface contact problem proceeds, within an increment of loading, processes such as separation, re-establishment of contact, slip and adhesion can occur in distinct regions of the interface.

Separation: During a loading sequence normal tractions at an interface region can become tensile. Since the interfaces are considered to be unilateral in their contact response, for a region undergoing separation, the traction boundary conditions (5) are homogeneous.

Re-contact: A region of a contact zone that has experienced separation can also re-establish contact when the relative normal displacement across the separated interface region is greater than or equal to the initial gap. Then the boundary conditions change from a type given by (4) to the type given by (5).

Slip: Slip will occur when the tractions satisfy either the failure condition (11) or (21). The interface condition (6) can be applied with the stiffness coefficients defined by (20).

Adhesion: When the conditions do not violate the failure condition (11), the boundary conditions at the interface can be interpreted through (10) with the stiffness coefficients interpreted appropriately.

In a solution scheme, with all increments, all of the above four conditions must be checked in order to obtain a stable condition at the interface.

4. Localized iterative solution procedures

Considering the boundary conditions given by (4)–(6), we can rewrite the matrix equation (7) in the form

$$\begin{aligned}&\left[-\mathbf{G}^{(1)},-\mathbf{H}^{(2)},\left\{-\mathbf{H}^{(3)}-\mathbf{G}^{(3)}\mathbf{K}^{(ep)}\right\}\right]\left\{\begin{matrix}\dot{t}^{(1)}\\ \dot{\Delta}^{(2)}\\ \dot{\Delta}^{(3)}\end{matrix}\right\}\\ &\quad=\left[-\mathbf{H}^{(1)},\mathbf{G}^{(2)},\mathbf{G}^{(3)}\right]\left\{\begin{matrix}\dot{\Delta}^{0(1)}\\ \dot{t}^{0(2)}\\ \dot{R}^{(3)}\end{matrix}\right\}\end{aligned}\tag{25}$$

where the superscripts (i) $(i = 1,2,3)$ indicate the types of boundary conditions designated by (4)–(6) respectively. For a non-linear interface problem it becomes necessary to apply an efficient solution scheme to analyse the incremental and iterative matrix equation (25). Since the complete boundary consists of linear and non-linear constraints, we can use an elimination procedure on the set of linear boundary constraints; this will reduce (25) to the form

$$\left[\bar{\mathbf{A}},\left\{-\mathbf{H}^{(3)}-\mathbf{G}^{(3)}\mathbf{K}^{(ep)}\right\}\right]\left\{\begin{matrix}\dot{t}^{(1)}\\ \dot{\Delta}^{(2)}\\ \dot{\Delta}^{(3)}\end{matrix}\right\}=\left\{\bar{\mathbf{B}}\right\}+\left[\bar{\mathbf{G}}^{(3)}\right]\left\{\dot{R}^{(3)}\right\}\tag{26}$$

where $[\bar{\mathbf{A}}]$ is the reduced version of $[-\mathbf{G}^{(1)},-\mathbf{H}^{(2)}]$ with properties consistent with an upper triangle-type of matrix; $[\bar{\mathbf{B}}]$ is the reduced form of the right-hand side vector consisting of known boundary values and $[\bar{\mathbf{H}}^{(3)}]$ and $[\bar{\mathbf{G}}^{(3)}]$ are their corresponding reduced forms. The result (26) can be represented by two relations; the first corresponds to the boundary conditions associated with surfaces $\Gamma^{(1)}$ and $\Gamma^{(2)}$, which can be written as

$$\begin{aligned}&\begin{pmatrix}\bar{A}_{11} & \bar{A}_{12}\\ 0 & \bar{A}_{22}\end{pmatrix}\begin{pmatrix}\dot{t}^{(1)}\\ \dot{\Delta}^{(2)}\end{pmatrix}=\begin{pmatrix}\bar{B}_1\\ \bar{B}_2\end{pmatrix}+\begin{pmatrix}\bar{G}_1^{(3)}\\ \bar{G}_2^{(3)}\end{pmatrix}\left\{\dot{R}^{(3)}\right\}\\ &\quad-\begin{pmatrix}\bar{H}_1^{(3)} & -\bar{G}_1^{(3)}K^{(ep)}\\ \bar{H}_2^{(3)} & -\bar{G}_2^{(3)}K^{(ep)}\end{pmatrix}\left\{\dot{\Delta}^{(3)}\right\}\end{aligned}\tag{27}$$

which is essentially back-substitution form of the solution of $\{\dot{t}^{(1)}\}$ and $\{\dot{\Delta}^{(2)}\}$ if $\{\dot{\Delta}^{(3)}\}$ is known. This equation can be used at any increment when the boundary condition on $\Gamma^{(3)}$ is determined. The second result is an uncoupled equation for $\{\dot{\Delta}^{(3)}\}$, which takes the form

$$\left(\bar{\mathbf{H}}_3^{(3)}-\bar{\mathbf{G}}_3^{(3)}\mathbf{K}^{(ep)}\right)\left\{\dot{\Delta}^{(3)}\right\}=\left\{\bar{\mathbf{B}}\right\}+\left[\bar{\mathbf{G}}_3^{(3)}\right]\left\{\dot{R}^{(3)}\right\}\tag{28}$$

This equation has unknowns only on the boundary $\Gamma^{(3)}$. Hence, at any increment level, (28) can be applied in an incremental manner to determine a configuration of the boundary $\Gamma^{(3)}$. Using this procedure the non-linear boundary element problem is solved by a localized iteration procedure and the overall boundary element system matrix is factorised only once for any number of increments.

5. Boundary element modelling of planar cracks

The application of boundary element methods to the elasto-static analysis of planar crack problems is relatively well established. Extensive accounts of both fundamental results and applications of boundary element techniques are given in Refs. [69–72]. A comprehensive survey is also in [73]. The attractiveness of the method stems from the fact that the exact stress singularity at the tip of a planar crack located in an elastic medium can be incorporated within the boundary element scheme. This enables the calculation of stress intensity factors for various modes of deformation of the crack tip, which in turn can be used to establish the conditions necessary for the further extension of the crack tip. We shall restrict attention to the in-plane deformations of the cracks where the faces can interact in a non-linear fashion through the application of generalized far-field stress states.

When modelling the discretized boundary of a domain, quadratic isoparametric elements can be employed quite successfully and efficiently. The variations of displacements and tractions within the element can be described by

$$\left.\begin{matrix} u_i \\ P_i \end{matrix}\right\} = a_0 + a_1\zeta + a_2\zeta^2 \tag{29}$$

where ζ is the local coordinate of the element and a_r ($r = 0,1,2$) are arbitrary constants of interpolation. When modelling cracks that occur either at the boundaries or at the interior of the elastic medium, it is necessary to modify (29) to account for the $1/\sqrt{\zeta}$-type stress singularity at the crack tip. In the finite element method, the quarter-point elements of the type proposed in [74] and [75] can be used to model the required variation in the displacement, which is of the $\sqrt{\zeta}$-type. If the same type of element is implemented in the boundary element scheme, we have

$$\left.\begin{matrix} u_i \\ P_i \end{matrix}\right\} = b_0 + b_1\sqrt{\zeta} + b_2\zeta \tag{30}$$

where b_i ($i = 0,1,2$) are constants, and the required stress singularity cannot be duplicated. To overcome this problem, the *singular traction quarter-point boundary element*, was introduced [69] where the expression for the traction given in (30) is multiplied by the non-dimensional parameter $\sqrt{l/\zeta}$, where l is the length of crack tip element. The variations in the tractions can thus be expressed in the form

$$P_i = \frac{c_0}{\sqrt{\zeta}} + c_1 + c_2\sqrt{\zeta} \tag{31}$$

where c_i ($i = 0,1,2$) are constants. Singular traction quarter-point boundary elements have been extensively applied in the modelling of both plane and axisymmetric crack problems in elasticity theory and the accuracy of the modelling is well documented [37,73,76–83]. The

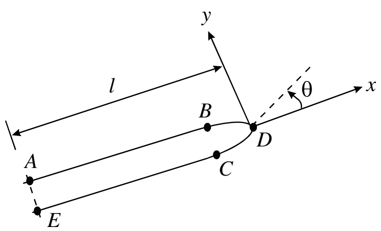


Fig. 2. Node arrangement for calculation of stress intensity factors at the crack tip.

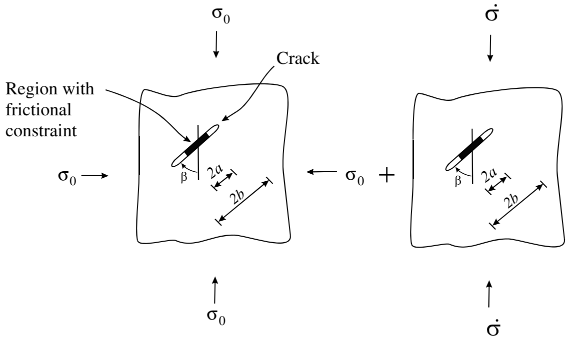


Fig. 3. The frictionally constrained inclined crack.

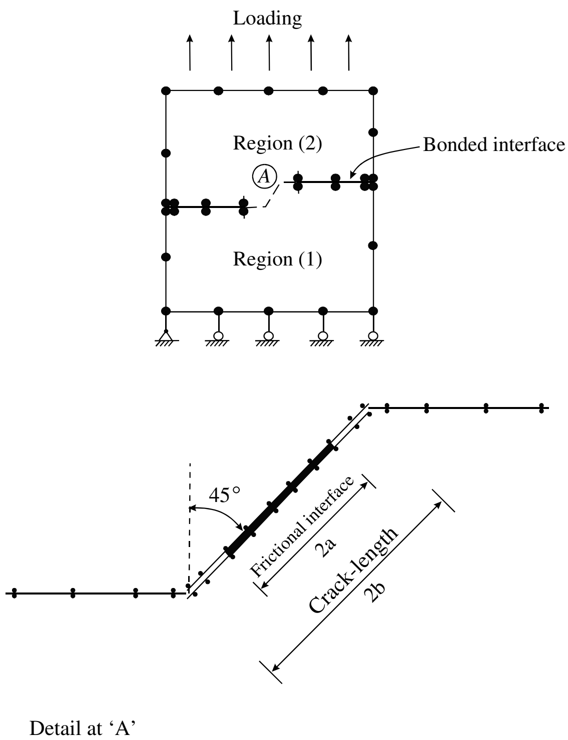


Fig. 4. The boundary element discretization of the plane domain and the crack tip.

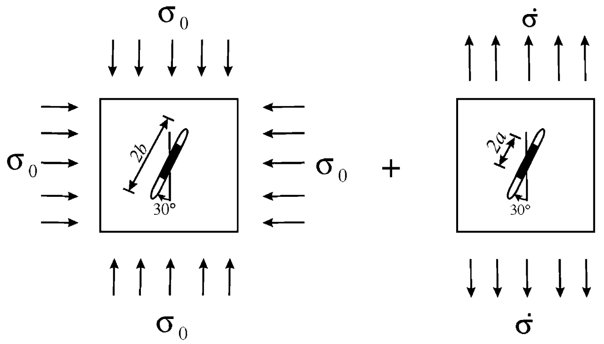
provision of the special singularity element permits the evaluation of the stress intensity factors at the crack tip. For the problems with in-plane deformations discussed here only the Mode I and Mode II stress intensity factors are relevant. These stress intensity factors can be determined by applying a displacement correlation method, which makes use of the nodal displacements at four locations A, B and E and the crack tip D (Fig. 2). The incremental estimates for the stress intensity factor are given by

$$\dot{K}_I = \frac{2G}{(k+1)}\sqrt{\frac{2\pi}{l}}\left[4\{\dot{\Delta}_y(B) - \dot{\Delta}_y(C)\} + \dot{\Delta}_y(E) - \dot{\Delta}_y(A)\right] \tag{32}$$

$$\dot{K}_{II} = \frac{2G}{(k+1)}\sqrt{\frac{2\pi}{l}}\left[4\{\dot{\Delta}_x(B) - \dot{\Delta}_x(C)\} + \dot{\Delta}_x(E) - \dot{\Delta}_x(A)\right] \tag{33}$$

where, for plane strain problems, $k = (3 - 4v)$ and for plane stress problems $k = (3 - v)/(1 + v)$.

In this paper, the boundary element technique that incorporates the special singularity element at the crack tip and interface non-linear effects at a finite region of the plane surfaces of the crack are used to examine the development of stress intensity factors at the crack tip. The specific problem deals with a Griffith-type plane crack of finite length $2b$ that is located in an elastic medium. The orientation of the crack is such that it is inclined at an angle β to the direction along which an incremental stress $\dot{\sigma}$ will be applied. The crack contains a region of finite length $2a_0$, that displays interface non-linear effects of the type discussed in the previous sections. The region that displays the non-linear effects is located symmetrically about the length of the crack (Fig. 3). First, the region containing the crack is subjected to an *isotropic compressive stress field* σ_0. During the application of this isotropic stress field, the interface



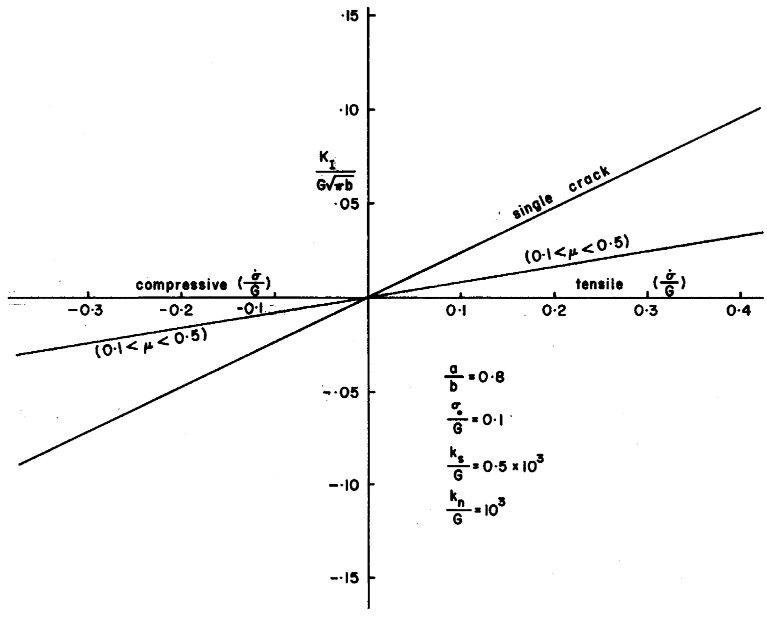


Fig. 5. Mode I stress intensity factor at the crack tip [$\beta = 30°$].

region is assumed to be inactive. This is merely an assumption invoked to keep the presentation of the results to a minimum. It is of course possible to consider a crack of finite aperture and to enforce closure over the finite region during the application of the isotropic stress field. In this case the zone of the crack with non-linear effects will possess an initial stress state other than the uniform compression associated with the application of an isotropic stress field to a crack with frictionless surfaces. With the isotropic stress field σ_0 held constant, the crack is subjected to the incremental stress $\dot{\sigma}$ that is either tensile or compressive. The boundary element discretization of both the domain and the region of the crack tip are illustrated in Fig. 4. The boundary element scheme is used to evaluate the crack opening (Mode I) and crack shearing (Mode II) stress intensity factors at the crack tip. We further assume that the crack tip always remains open during the application of both compressive and tensile incremental stresses. This is of course an idealization for the purposes of performing the computations. It is also possible to consider closure of the crack tip during application of the isotropic initial stress field and the subsequent incremental stress state in the form of either tensile or compressive incremental stresses $\dot{\sigma}$. The computational scheme is capable of examining this type of crack tip closure problem, but these studies will be discussed in some future work. An important observation for crack tip closure problems is that, in the case of two-dimensional plane crack problems, only the Mode II stress intensity factors can exist at a crack tip exhibiting frictionless closure, and a special singularity element needs to be used to accommodate such a response. The parameters required for the computational modelling include the relative dimensions of the non-linear interface region (a/b); the normalized magnitude of the isotropic stress (σ_0/G); the coefficient of friction μ and the relative magnitudes of the shear and normal stiffnesses k_s/G and k_n/G. [Note that as per (10) and (19), the stiffnesses $\tilde{k}_n$ and $\tilde{k}_n$ are expressed in units of *force*/(*length*)^3 to provide tractions with units

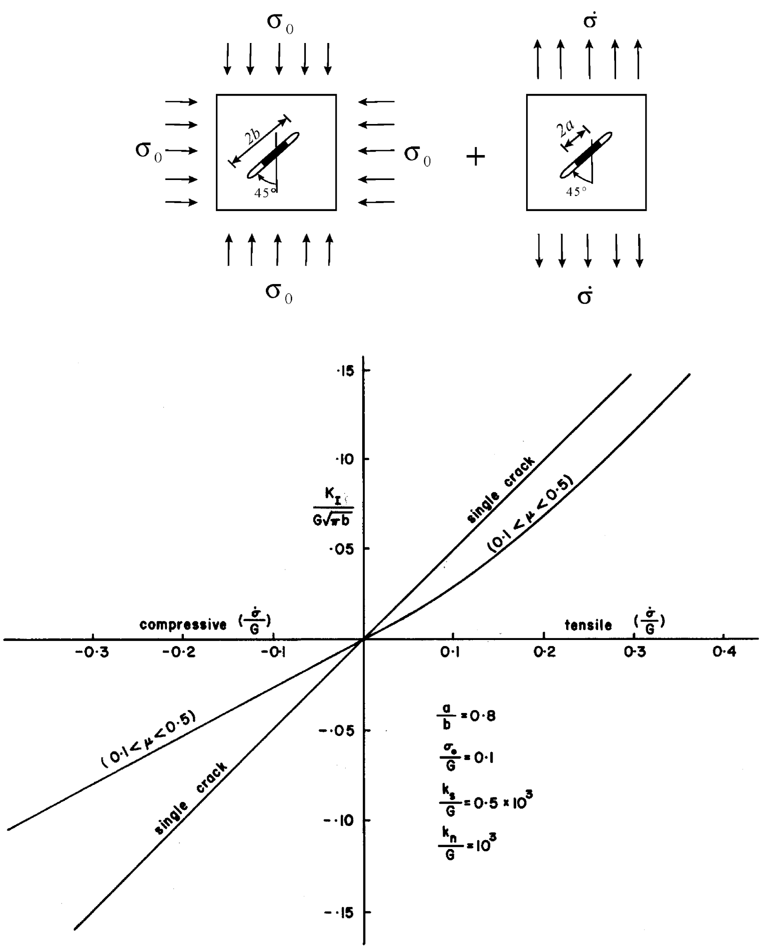


Fig. 6. Mode I stress intensity factor at the crack tip [$\beta = 45°$].

of *stress*. In the normalization of these stiffnesses we assume that we can define non-dimensional parameters k_s/G and k_n/G, where $\tilde{k}_s = \tilde{k}_s \times 1$ and $\tilde{k}_n = \tilde{k}_n \times 1$, where unity represents the thickness over which the plane problem is being analysed.] The specific values of the parameters used in the computations are indicated in the figures.

Fig. 5 illustrates the variation of the Mode I stress intensity factor at the crack tip for various values of the normalized stress incremental stresses $\dot{\sigma}/G$, both in the tensile and compressive ranges, when the crack is inclined at $\beta = 30°$ to the direction of application of the incremental axial stress. Fig. 5 also presents the analytical solution to the problem of a single crack of length $2b$ that is located in the uniaxial compressive stress field; this solution corresponds to the case where there is no interaction, frictional or otherwise, between the faces of the crack [84]. A range of plausible values is assigned for the coefficient of friction μ acting at the frictionally constrained region. The computational results indicate that the Mode I stress intensity factor is relatively insensitive to the friction coefficient in the range 0.10 to 0.50 both in the tensile and compressive ranges of the applied incremental stress and for the choices of the other input parameters. The negative value for the stress intensity factor needs to be interpreted in the light of the remarks presented previously with regard to the assumed open configuration of the crack and the sign of the compressive stress. Fig. 6 presents analogous results for the Mode I stress intensity factor at the crack tip for the case when $\beta = 45°$. Similar conclusions apply, except that in the tensile range of $\dot{\sigma}/G$, the variation in the Mode I stress intensity factor displays a non-linear trend, again for $\mu \in (0.10, 0.50)$. Figs. 7 and 8 illustrate, respectively, the variation in the Mode II stress intensity factors with $\dot{\sigma}/G$, for crack orientations of $\beta = 30°$ and $\beta = 45°$. The influence of the frictional effects and the sense of the incremental stress are more pronounced in the estimates

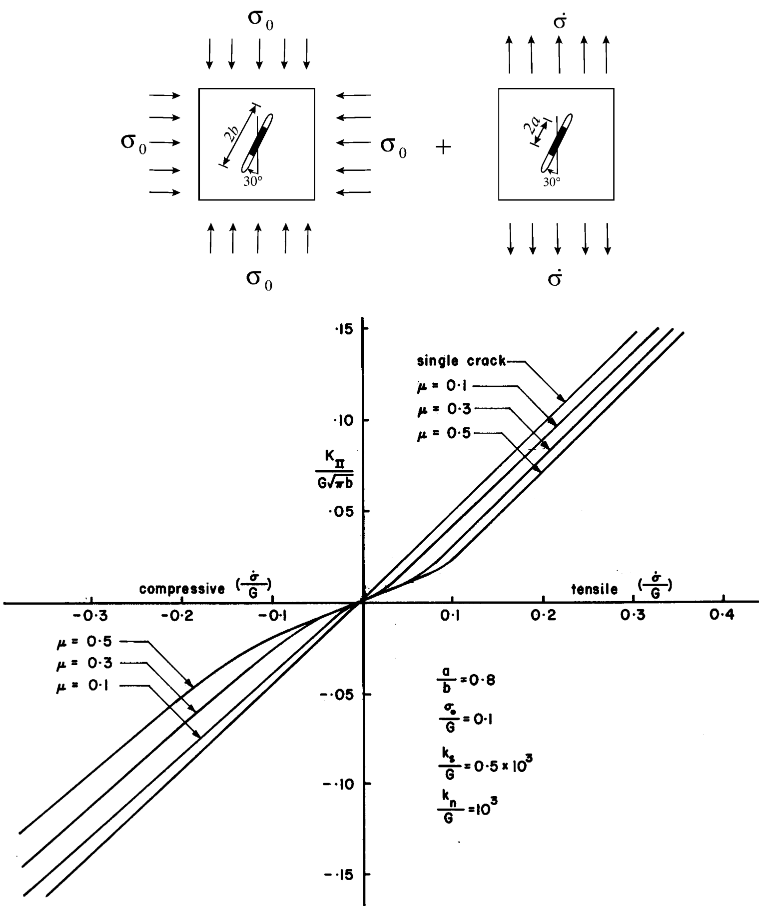


Fig. 7. Mode II stress intensity factor at the crack tip [$\beta = 30°$].

for the Mode II stress intensity factor. Again, the result for the single crack problem is provided for purposes of comparison.

6. Concluding remarks

In classical treatments of planar crack problems it is implicitly assumed that the faces of the crack do not interact during the application of external loads. This assumption will be accurate in a majority of situations where the mode of loading is tensile and the crack is oriented normal to the direction of loading. In many other instances, particularly in the context of materials engineering and geomechanics, cracks can exist in a closed condition both prior to and during the application of external stresses. In these circumstances, the interface behaviour can influence the mechanics of the crack both in terms of the crack growth and the orientation of crack growth. The types of mechanical phenomena that can be encountered at a closed crack surface can be varied with completely smooth and bonded segments constituting the extreme limits. While these extreme limits can be examined through conventional analytical and computational means, the treatment of non-classical effects including friction, slip, dilatancy and other non-linear constraints at the crack surfaces require non-linear computational schemes. The boundary element approach is a particularly attractive computational scheme in situations where the non-linear processes are confined only to the surfaces of the crack. It is shown that the class of plane crack problems that exhibit non-linear interface constraints can be examined through an interface plasticity formulation that can be incorporated into an incremental boundary element formulation. The computational scheme is applied to examine the mechanics of a plane inclined crack that is first subjected to isotropic compression with the interface condition in a relaxed condition and an axial incremental loading, which is applied in both the tension and compression modes with the non-linear interface condition invoked. The computational scheme provides estimates for the Modes I and

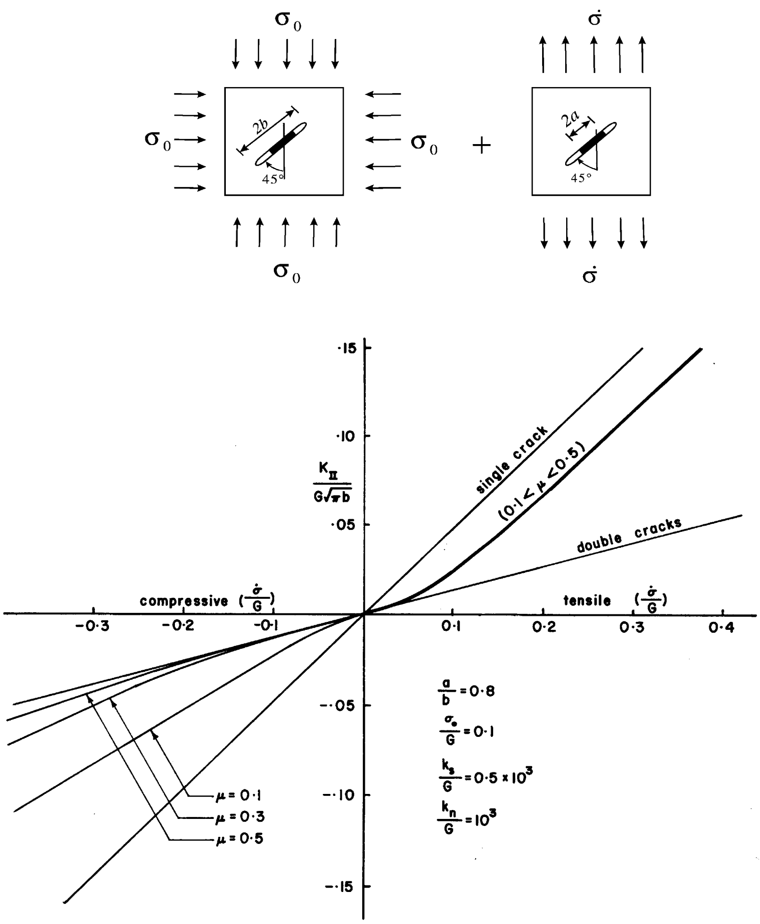


Fig. 8. Mode II stress intensity factor at the crack tip [$\beta = 45°$].

II stress intensity factors at the tip of the inclined crack under the application of the incremental axial stress. These computational results are problem specific, and as such no general conclusions can be drawn. These results do, however, indicate that the effects of interface friction at the surfaces of the crack generally have a more pronounced influence for the Mode II stress intensity factor rather than the Mode I stress intensity factor. The results of the computational modelling demonstrates that the incremental boundary element scheme is a viable technique for the treatment of a variety of interface problems with non-linear constraints encountered in applied mechanics and particularly geomechanics.

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