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Fluid Intake Cavities in Stratified Porous Media

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ABSTRACT

Hvorslev's procedure for the estimation of the of the in-situ hydraulic conductivity characteristics of an isotropic porous medium involves the use of information concerning the rate of groundwater flow into either a cavity formed at the base of a cased borehole or a cylindrical piezometer tip. The objective of this article is to present a concise and complete development of the intake shape factor for a cavity region with either a prolate or an oblate form, located in a stratified porous medium, which has transversely isotropic hydraulic conductivity characteristics. The article presents the relevant mathematical relationships and discusses some aspects of the extension of the technique for the estimation of hydraulic conductivity of stratified porous media that are hydraulically transversely isotropic.

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1. INTRODUCTION

The hydraulic properties of sedimentary geologic materials, including varved clays, interbedded sands, and silts in fluvial and lacustrine environments, tend to display a directional dependence. Although these materials may, in general, display characteristics of hydraulic anisotropy, the transversely isotropic assumption is both a convenient and realistic first approximation in view of the fact that both hydraulic and mechanical properties of the stratified geomaterials in the plane of deposition are usually isotropic. Both microstratification due to particle shape and macrostratification due to periodic deposition can contribute to the development of a directional dependence in the fluid transport characteristics of the geomaterials (Vreedenburgh, 1938; Childs and Collis-George, 1950; Terzaghi, 1955; Mitchell, 1956; Childs, 1957a,b; Maasland, 1957; Johnson and Morris, 1962; Kenney, 1963; Davis, 1969; Bouwer, 1978). The presence of transverse isotropy in the hydraulic conductivity characteristics of sedimentary geomaterials can be more conveniently demonstrated by appeal to the following simple example. With many geomaterials that are formed as a result of a layering process, the transverse isotropy in the hydraulic conductivity properties can be identified in terms of the hydraulic conductivities in the horizontal (k_h) and vertical (k_v) directions. For example, the elementary calculations for the equivalent vertical hydraulic conductivity k_v of n layers is given by the weighted harmonic mean of the hydraulic conductivities of the individual layers as

$$k_v = \left(\sum_{i=1}^{n} t_i \right) \Big/ \left(\sum_{i=1}^{n} \frac{t_i}{k_i} \right)$$

where t_i and k_i are the thickness and hydraulic conductivity of the ith layer. The equivalent horizontal hydraulic conductivity k_h is similarly given by

$$k_h = \left(\sum_{i=1}^{n} k_i t_i \right) \Big/ \left(\sum_{i=1}^{n} t_i \right)$$

Consider a sedimentary sequence of interbedded silty sand and unweathered marine clay of equal thickness and with *isotropic* individual hydraulic conductivities of 10^{-6} m/s and 10^{-12} m/s, respectively. The equivalent vertical hydraulic conductivity of the entire sequence will be approximately 2×10^{-12} m/s and the equivalent hydraulic conductivity in the horizontal plane will be approximately 0.5×10^{-6} m/s. It is evident that significant anisotropy in the scale of a representative volume element, with dimensions significantly larger than the layer thickness, can readily materialize even with plausible choices of hydraulic conductivities of the individual layers. A similar situation relates to rock masses that are heavily fractured due to geologic stresses induced by tectonic action and stress relief. Hydraulic conductivity of such fractured media is governed both by the matrix hydraulic conductivity of the intact rock and by the hydraulic conductivity of the fractures. In instances where the parent rock is relatively impervious, the directional properties can differ by up to 10 orders of magnitude (de Marsily, 1985; Tsang, 1991; Bear et al., 1993).

The determination of the *in-situ* hydraulic conductivity characteristics of such transversely isotropic stratified materials is of importance to many geotechnical, geoenvironmental, and water resources engineering applications (De Marsily, 1986; Bear and Verruijt, 1987). The most common technique used for such purposes is the cased borehole test, in which either the transient water-level rise or water-level fall during an extraction or recharge test is used to interpret the hydraulic conductivity characteristics of the geomaterials. The rate of water entry to the casing depends on the geometric arrangement of the base of the borehole casing, or the geometric characteristics of the entry point. The most common of these is a formed cylindrical region at the base of the borehole, with a diameter roughly equal to that of the casing and a length that can be a variable quantity. To facilitate the rapid rise of the water level in the casing to equalization with the groundwater level, a piezometric standpipe of smaller diameter can be used with a bentonite sealing of the base of the casing (Kirkham, 1945; Luthin and Kirkham, 1949). The arrangement for a typical casing test is shown in Fig. 1. Hvorslev (1951) proposed that for situations where there is no consolidation of the cohesive soil region around the intake and for stationary groundwater conditions the flow rate to the entry region is given by

$$q = FkH \tag{i}$$

where H is the head inducing the flow, k is the effective hydraulic conductivity in the vicinity of the intake, and F is the intake shape factor, with dimensions of length, that is, specific to the intake geometry. Based on a result by Dachler (1936) for the potential problem related to a prolate spheroid, Hvorslev proposed a relationship for the cylindrical intake shape factor that was estimated by assuming that the prolate spheroid is inscribed within the cylindrical

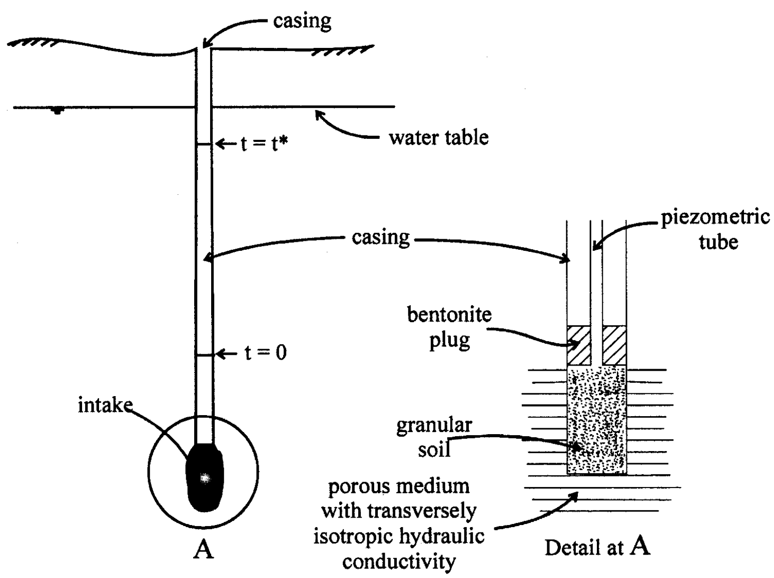

Figure 1. Casing test configuration for the in situ determination of the hydraulic properties of geomaterials with low hydraulic conductivity.

intake of diameter D and length L; the resulting expression for the intake shape factor F is given in the form

$$F = \frac{2\pi L}{\ln\left[(L/D) + \sqrt{1 + (L/D)^2}\right]} \tag{ii}$$

The limits of applicability of the above result will be discussed briefly in this introduction. The study by Hvorslev (1951) also gives details of the procedures for casing tests and the geometric relationships which govern intake regions with cylindrical, spherical, and disc shapes, which are fully embedded in porous media of infinite extent. The conventional assumption of an entry point located in a porous medium of infinite extent is not altogether unrealistic, since the intake shape factor is largely governed by fluid flow behavior in the immediate vicinity of the intake. For the assumption of a porous medium of infinite extent to be valid, the largest dimension (L) of the intake should be substantially smaller than the depth (H^*) at which the test is being conducted ($H^*/L >> 10$). In instances where the intake is located in a nondeformable porous medium with isotropic hydraulic conductivity, the intake shape factor is purely geometric in nature, due to the fact that the flow in the vicinity of the intake is governed by Laplace's equation for the hydraulic potential, which is void of any hydraulic conductivity properties of the medium. Before proceeding to discuss the literature in this area in detail, it is important make the following point. The casing test is a transient test by virtue of the time-dependent differential head inducing flow. This time dependency is quite different from that which can occur as a result of either the deformability of the porous medium, the compressibility of the pore fluid, or the flexibility that can be present in any device that is used to measure the time-dependent fluid pressure in a closed system. Such processes result in diffusive-type fluid pressure transients (see, e.g., de Josselin de Jong, 1953; Gibson, 1963, 1967, 1970; Wilkinson, 1968; Selvadurai, 2000a).

Aspects related to shape factors for intake regions located in *isotropic* porous media have been discussed and examined quite extensively in past and recent literature in geotechnical engineering and soil science. For example, the studies by Schneebli (1956) and Kallstenius and Wallgren (1956) deal with approximate representations of the intake to aid the development of the intake shape factor for cylindrical regions. Along these lines, Wilkinson (1968) suggests the modification of Hvorslev's result for the intake shape factor by considering the equivalence of *volume* between the actual cylindrical shape and an ellipsoidal (spheroidal) region. As will be shown in a subsequent section, this procedure unfortunately has no physical basis, since the flow to the intake region takes place through the surface of the intake rather than over the volume (Kallstenius and Wallgren, 1956). Lowther (1978) and others (see, e.g., Youngs, 1980) comment on the apparent discrepancy between the limiting case of the above intake shape factor as $L \to 0$, and the exact result applicable to the disc-shaped entry points. In the limit when $L \to 0$, the cited result by Hvorslev gives the intake shape factor as $2\pi D$, which is the intake shape factor for the spherical intake of diameter D. (There is no known analytical solution, in exact closed form or otherwise, for the problem of a cylindrical intake that extends beyond the base of a casing. Furthermore, it should be remarked that the mathematical treatment of the resulting potential problem with the exact geometry of a cylindrical intake of finite length terminating at the base of a borehole is a rather complex mixed boundary-value problem in potential theory, which cannot be solved without great recourse to dual integral and dual series equations.) What is important to note is that, although not specifically stated, Dachler's result is applicable only to the case of a prolate spheroidal form, thereby restricting the applicability of the result by Hvorslev only to situations where $(L/D) > 1$. The theoretical results presented in this article will identify the limits of applicability of solutions derived from spheroidal cavity intakes, which for the case of a prolate cavity is $1 \le (L/D) < \infty$ and for an oblate form corresponds to $0 \le (L/D) < 1$. The reduction to limiting cases should therefore be approached with some caution, provided basic mathematical solutions accommodate such limits.

Returning to the discussion of intake shape factor for the case where $L = 0$, the intake shape factor for a disc intake situated in a porous medium of infinite extent, i.e., $F = 4D$, provides the upper limit. Similarly, the intake shape factor for a disc-shaped entry region located at a porous medium-impervious boundary interface, i.e., $F = 2D$, (by virtue of symmetry), provides the lower limit. Hvorslev (1951) cites an intake shape factor of $F = 2.75D$, which is based on results of electrical analog studies conducted by Harza (1935) and through approximate graphical procedures given by Taylor (1948). Brand and Premchitt (1980a) suggest a value of $F = 2.63D$, which is based on both experimental and numerical procedures. The intake shape factor for the spherical intake region of $2\pi D$ is clearly outside the permissible bounds. Investigators such as Smiles and Youngs (1965), Youngs (1968), and Brand and Premchitt (1980b) have used electrical analog techniques to estimate the intake shape factors for cylindrical intakes. The latter authors also suggest the use of $1.2L$ as opposed to L in the original Hvorslev expression for the cylindrical intake problem.

The use of computational techniques and other approximate procedures for the solution of Laplace's equation in relation to the assessment of the flow pattern around the intake region also feature prominently in the estimation of the intake shape factors. The earliest use of finite-difference techniques for the evaluation of the intake shape factor for a cylindrical intake appears to be due to Wilkinson (1968). Other investigators, including Al-Dhahir and Morgenstern (1969), Raymond and Azzouz (1969), and Brand and Premchitt (1980a,b), have also conducted similar studies, and the latter authors provide quite useful comparisons between the various estimates for the intake shape factors as determined through experimental and computational considerations. The problem of determining shape factors for cylindrical piezometer tips has also been discussed by Chapuis (1989), who provides an important account of the errors associated with the various computational methods. The study by Randolph and Booker (1982) develops an approximate solution for the potential problem arising from the situation where an opening of finite length exists in an otherwise impermeable casing of infinite length located in a porous medium. Their solution is an important contribution in the sense that they appreciate the existence of a *singular velocity field* at the locations where, mathematically, the boundary conditions for the potential problem changes from one where the potential is prescribed (Dirichlet-type) to one where the gradient of the potential is prescribed (Neumann-type), along the same cylindrical boundary (see, e.g., Selvadurai, 2000a,b). Similar effects will also occur, for example, in the case of the disc-shaped intake located within a porous medium or at the interface of an impervious boundary (Selvadurai, 2003). It is a relatively easy matter to show that singular eigen-solutions are present in these circumstances. Consider, for example, a half-plane plane region occupying $r \in (0, \infty)$ and $\theta \in (0, \pi)$ in which the potential $\varphi(r, \theta)$ is to be found such that

$$\Delta^2 \varphi = 0 \quad r \in (0, \infty) \quad \theta \in (0, \pi) \tag{iii}$$

and boundary conditions

$$\varphi(r, 0) = 0 \quad \left(\frac{\partial \varphi}{\partial \theta}\right)_{\theta=\pi} = 0 \tag{iv}$$

In addition, $\varphi(r, \theta)$ should decay to zero in the entire region as $r \to \infty$. The boundary $\theta = 0$ corresponds to the entry-point region and the boundary $\theta = \pi$ corresponds to the impermeable casing. Of course, there will be a separate solution that will satisfy the potential prescribed on $\theta = 0$, consistent with the potential prescribed to initiate the flow. Considering the general solution of (iii) in plane polar coordinates (Selvadurai, 2000a), the relevant solution takes the form

$$\varphi(r, \theta) = \sum_{n=1,2,\ldots} \frac{B_n \sin n\theta}{r^n} \tag{v}$$

where B_n are arbitrary constants. To satisfy the boundary conditions so that the solution (v) is nontrivial, we require

$$n = \frac{1}{4}, \frac{3}{4}, \ldots \tag{vi}$$

Therefore, the lowest eigenvalue contributes to singular behavior of the potential $\varphi(r, \theta)$. Since the flow potential is singular, the flow velocities are also singular at the boundary point corresponding to the demarcation between Dirichlet (potential prescribed) and Neumann (impervious) boundary conditions. This singular behavior will influence the estimation of the flow rate into the cavity through computational means, especially if, unwittingly, the boundary velocities are used to compute the flow rate. Mathematically, the singularity is integrable, in the sense that the flux, or the total flow rate, evaluated at the intake region, is finite. Finite-element techniques have also been applied to determine the intake shape factors for cylindrical and other regions. The studies by Tavenas et al. (1986a,b) deal with the application of finite-element techniques to the estimation of intake shape factors for the isotropic case. In work that followed, Tavenas et al. (1990) also provide a useful and informative commentary on the procedures that have been developed for the estimation of shape factors for cylindrical intakes, and they use the results of finite-element evaluations of the intake shape factors to evaluate results of *in-situ* permeability tests conducted on the Champlain clay. Although these authors have commented on the topic of hydraulic transverse isotropy, there was no extensive study of this aspect of the problem because of the near-hydraulic isotropy of the tested clay. Hayashi et al. (1997) and Warrick and Rojano (1999) have provided correlations between intakes with spherical, spheroidal, line source, and an open interval in an impervious cylindrical casing. In a recent study, Ratnam et al. (2001) also used the finite-element technique to reexamine the estimates for Hvorslev's intake shape factors. These authors also investigated the influence of hydraulic transverse isotropy on the intake shape factor, although no results of any generality are provided. A further recent investigation by Selvadurai and Brunelle (2001) investigated the behavior of cylindrical intake regions that straddle hydraulically inhomogeneous media. An important aspect of the application of computational methodologies, especially as they relate to the evaluation of intake shape factors for cylindrical regions incorporating the casing, is that mentioned previously concerning the change in the type of boundary condition from pervious to impervious conditions along the same surface. The flow velocities at these boundaries are singular, and any computational scheme that is used to solve the steady-state flow problem should be capable of accommodating this singular behavior in the solution scheme. Special computational schemes (Aalto, 1985) can be developed to accommodate this limitation. Such schemes, however, are not routinely available in many existing general-purpose computational codes. Mesh refinement can be used to improve the estimates, but continued mesh refinement will result in an ill-posed problem. This is not, however, an obstacle to the use of computational schemes, provided the locations where the flow velocity is singular are excluded from the calculation of the flow rates (Selvadurai and Brunelle, 2001). Since the porous medium is nondeformable and the fluid is incompressible, the flow rate to the entry point can be calculated by selecting any closed surface that encompasses the intake region. This procedure, along with suitable mesh refinements, can be used to generate computational estimates for intake shape factors for which analytical solutions are unavailable. Computational methodologies can also be used to develop solutions to a wider class of problems associated with intakes, the mathematical analysis is of which can be quite complicated. Such an example is that of a cylindrical intake located in a transversely isotropic porous medium in the general case where the planes of isotropy are inclined to the axis of the intake. This problem will be presented in a forthcoming article by Selvadurai et al. (2003). The above commentary on aspects of the shape factor for a cylindrical intake region located in a porous medium is certainly not meant to be complete, but provides a record of the approaches that have been proposed in the literature for investigating intake regions located in nondeformable porous media with predominantly isotropic hydraulic conductivity.

The situation is different in the case where the porous medium is transversely isotropic; here, *both* the geometric features of the intake and the hydraulic conductivity characteristics of the porous medium will influence the intake shape factor. This conclusion is self-evident, since the operator equation governing flow in a transversely isotropic porous medium is *pseudo-Laplacian*. The restrictions of transverse isotropy in the hydraulic conductivity characteristics of the medium are dictated not only by potential application of the results to stratified geomaterials but also by the restrictions placed by the use of a simple *in-situ* test, such as the cased borehole test. This test *cannot* be used to determine hydraulic conductivity properties associated with *general anisotropy* of the porous medium. In these circumstances, a single *in-situ* test cannot, under any circumstances, be configured to determine *all six independent constants* of the hydraulic conductivity tensor characterizing general anisotropy. In such cases recourse must also be made to complement *in-situ* tests with laboratory testing of geomaterial samples recovered with the minimum of sample disturbance. Even in circumstances where the axis of the cylindrical intake is made to coincide with the principal axes of hydraulic conductivity, the method of determining all three principal values of hydraulic conductivity is not straightforward. Such determinations invariably involve not only the *a priori* identification of the directions of principal hydraulic conductivity, but also the use of multiple boreholes with segmented water recharge/extraction locations of finite length (Louis, 1974; Hsieh and Neuman, 1985). Since the 1950s, several investigations have been made, notably in the soil science area, with the intention of developing theoretical estimates for flow rates into three-dimensional cavity regions located in soils that display transverse isotropy in their hydraulic conductivity characteristics. The assumption of transverse isotropy effectively reduces the number of hydraulic conductivity values to two, thereby offering possibilities for determining them independently by suitably adjusting the dimensions of the cylindrical intake region. For example, an early study in this area was done by Childs (1952), who examined the problem specifically in relation to the flow situation where the *in-plane* hydraulic characteristics are anisotropic. The experimental investigations arising from these studies are also described by Childs et al. (1953, 1957). Maasland and Kirkham (1955) have also reexamined a similar problem with specific reference to the study of air permeability in soil. Maasland (1957) and Maasland and Kirkham (1959) examined the problem related to a cavity located in a porous medium with orthotropic hydraulic conductivity and arrived at the rather unreliable conclusion that the in-plane hydraulic anisotropy has no significant influence on the flow into the porous cavity, which represents the piezometer cavity. Notable related studies in this area, by Philip (1985, 1986, 1987), examined the problems related to steady-state absorption from both homogeneous isotropic and transversely isotropic soils from spheroidal cavities. Again the terminology *anisotropic soils*, as used in connection with the latter investigation, is *incorrect*,*t* especially in view of the restricted form of Darcy's law used to derive the governing equations. In the case where gravity effects are neglected, the matric flux potential satisfies Laplace's equation. The general presentation adopted in this article is somewhat more detailed than the material presented by Philip (1985, 1986, 1987). One could also argue that almost any attempt along these lines, either past or present, could very well be gleaned from the classical studies in potential theory for the Dirichlet problem given, on occasions more than two centuries ago, by Legendre, Laplace, Green, Lamé, Gauss, Boussinesq, Dirichlet, and others and summarized in the literature on mathematical physics by, among others, Kellogg (1929), MacMillan (1930), Hobson (1931), and Morse and Feshbach (1953). The methodology and approach adopted here, however, is quite straightforward in scope and proceeds to develop results for the intake characteristics, with straightforward mathematical expositions, for the complete range of mismatch in the principal values of hydraulic conductivities and geometric aspect ratios which are relevant to the correct formulation of the problems. Also, the correlation between the dimensions of the cylindrical intake and the spheroidal intake is established by considering the more realistic equivalence of fluid flux between the two entry-point shapes.

2. GOVERNING EQUATIONS

The theory of steadystate groundwater flow in a porous medium, which leads to the relevant potential equation, is classical and can be found in the treatises by Muskat (1937), Polubarinova-Kochina (1962), Harr (1962), Bear (1972), Scheidegger (1974), Verruijt (1982), and Philips (1991). Here we consider the problem of groundwater flow in a hydraulically transversely isotropic porous medium which is saturated with an incompressible pore fluid. Although it is possible to develop a generalized formulation of the problem, in view of the axial symmetry associated with the problems being examined, it is convenient to develop the governing equations in relation to a system of cylindrical polar coordinates (r, θ, z). The spatially averaged fluid velocity components in the porous medium referred to these coordinates are denoted by (v_r, v_θ, v_z). The potential causing fluid flow in the hydraulically transversely isotropic porous medium is taken as the Bernoulli potential consisting only of the datum head and pressure head components. Since the studies relate to considerations of fluid flow in the neighborhood of the entry point, we can, without loss of generality, assume that the pressure head $\varphi(r, \theta, z)$ is much greater than both the datum potential and the dimensions of the entry point. Also, since the problems examined will be such that the axis of symmetry will be normal to the plane of transverse isotropy, we can also assume that the pressure head is $\varphi(r, z)$. Attention is restricted to the flow of an incompressible fluid through the porous medium, which requires the velocity field to satisfy the divergence free requirement for the fluid velocities, i.e.,

$$\nabla.\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \tag{1}$$

Considering Darcy's law for fluid flow through the porous medium, we have

$$\mathbf{v} = -\mathbf{K}\nabla\varphi \tag{2}$$

where $\mathbf{K}$ is the hydraulic conductivity matrix and $\nabla\varphi$ is the gradient operator applied to the pressure potential. For a porous medium which is *hydraulically transversely isotropic*, where the principal axes of hydraulic conductivity are aligned with the coordinate axes r and z, the axisymmetric form of Darcy's law is

$$v_r = -k_{rr}\frac{\partial \varphi}{\partial r} \quad v_\theta = 0 \quad v_z = -k_{zz}\frac{\partial \varphi}{\partial z} \tag{3}$$

If we identify the z axis as the vertical direction, then the hydraulic conductivity in the z direction corresponds to the conventional hydraulic conductivity k_v in the vertical direction, and the hydraulic conductivity in the r direction corresponds to the conventional hydraulic conductivity k_h. Combining the expressions (3) with the fluid incompressibility condition (1), we obtain the partial differential equation for the flow of an ideal incompressible fluid in a hydraulically transversely isotropic porous medium as follows:

$$k_{rr}\left(\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r}\right) + k_{zz}\frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{4}$$

We now assume that $k_{rr} \neq 0$, and introduce the usual coordinate transformations

$$R = r \qquad Z = \sqrt{\frac{k_{rr}}{k_{zz}}}z \tag{5}$$

such that the governing pseudo-Laplacian partial differential equation (4) can now be rewritten as

$$\frac{\partial^2 \xi}{\partial R^2} + \frac{1}{R}\frac{\partial \xi}{\partial R} + \frac{\partial^2 \xi}{\partial Z^2} = 0 \tag{6}$$

We shall develop solutions of (6) subject to the boundary conditions that will represent the local flow patterns associated with the prolate and oblate spheroidal cavities located in hydraulically transversely isotropic porous media of infinite extent and subjected to a constant potential at the internal boundary.

3. SPHEROIDAL INTAKES

We first consider the problem of a fluid-saturated porous medium with a nondeforming porous fabric, which is of infinite extent and hydraulically transversely isotropic. The infinite medium is bounded internally by a *prolate spheroidal* cavity with semimajor axis a^* and semiminor axis b^*. The boundary of the spheroidal cavity is maintained at a constant potential φ_0. From a realistic point of view, if steady flow is to be maintained, fluid must be supplied to the boundary of the cavity, which represents a recharging of the porous medium. We shall assume that this can be done using a piezometric tube with a cross-sectional diameter significantly smaller than either the diameter or the height of the spheroidal entry point. Also, at the outset it should be remarked that the spheroidal intakes are intended to provide only an approximation to intakes such as the cylindrical piezometer tip shown in Fig. 1. The boundary-value problem requires the solution of (4) subject to the boundary condition

$$\varphi(r, z) = \varphi_0 \quad \left(\frac{r}{b^*}\right)^2 + \left(\frac{z}{a^*}\right)^2 = 1 \tag{7}$$

Also, since the porous medium is of infinite extent and the boundary value problem is *three-dimensional*, the potential should decay to zero as $\sqrt{r^2+z^2} \to \infty$. If the problem is two-dimensional, such regularity conditions cannot be applied. For the solution of the boundary-value problem we consider the transformed version of the partial differential equation which is harmonic in the region $R \in (b^*, \infty)$ and $Z \in \left(a^*\sqrt{k_{rr}/k_{zz}}, \infty\right)$. We note that by introducing the spatial transformations given by (5), we have also transformed the boundary to a spheroid with a different dimension in the z direction. In view of the spheroidal geometry of the cavity boundary, it is convenient to introduce a system of *prolate* spheroidal curvilinear coordinates (α, γ, β) to examine the flow problem. We first consider the system of prolate spheroidal coordinates defined by

$$R = c_p \sinh\alpha \sin\beta \quad Z = c_p \cosh\alpha \cos\beta \tag{8}$$

such that the parametric surfaces $\alpha = \text{const.}$, say α_0, $\beta = \beta_0$, and $\gamma = \gamma_0$, form a triple orthogonal confocal family of prolate spheroids, hyperboloids of two sheets, and

$$(ds)^2 = \left(\frac{d\alpha}{h_1^p}\right)^2 + \left(\frac{d\beta}{h_2^p}\right)^2 + \left(\frac{d\gamma}{h_3^p}\right)^2 \tag{9}$$

meridional half-planes respectively. By considering the expression for a differential arc length (ds) given by

$$h_1^p = h_2^p = \left[c_p^2(\sinh^2\alpha + \sin^2\beta)\right]^{-1/2} = h_p \tag{10}$$

where the metric or local scale coefficients are given by (Selvadurai, 2000a)

$$h_3^p = (c_p \sinh\alpha \sin\beta)^{-1} \tag{11}$$

the focal distance c_p can be expressed in terms of the dimensions of the semimajor axis and the equatorial radius of the prolate spheroid conforming to the transformed internal boundary of the porous medium, which is assumed to be defined by $\alpha = \alpha_0$, such that

$$(b_p)^2 = (b^*)^2 = c_p^2 \sinh^2\alpha_0$$

$$(a_p)^2 = \frac{k_{rr}}{k_{zz}}(a^*)^2 = c_p^2 \cosh^2\alpha_0 \tag{12}$$

and

$$c_p = a^*\sqrt{\lambda - \eta^2} \quad -\lambda = \frac{k_{rr}}{k_{zz}} \geq 1 \quad \eta = \left(\frac{b^*}{a^*}\right) \leq 1 \tag{13}$$

Since the flow problems examined exhibit a state of symmetry about the z axis of hydraulic symmetry of the porous medium, the hydraulic potential is independent of the azimuthal coordinate γ and is dependent only on the curvilinear coordinates (α, β): Laplace's equation (6) takes the form

$$\nabla^2\omega(\alpha,\beta) \tag{14}$$

$$= h_p^2\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \coth\alpha\frac{\partial}{\partial\alpha} + \cot\beta\frac{\partial}{\partial\beta}\right)\omega(\alpha,\beta) = 0$$

subject to the boundary conditions

$$\varphi(\alpha, \beta) = \varphi_0 \quad \text{on} \quad \alpha = \alpha_0 \tag{15}$$

where $\alpha = \alpha_0$ corresponds to the boundary of the prolate spheroidal cavity with semimajor axis a_p and semiminor axis b_p, and

$$\varphi(\alpha, \beta) \to 0 \quad \text{as} \quad \alpha \to \infty \tag{16}$$

For the solution of the boundary-value problem referred to the system of spheroidal coordinates, we seek Lamé products associated with spheroidal coordinate system (Hobson, 1931; Morse and Feshbach, 1953; Selvadurai, 1976; Moon and Spencer, 1988), the general expression for which can be obtained in the form

$$\varphi(\alpha,\beta) = \left[P_n^{(m)}(\cos\beta) \text{ or } Q_n^{(m)}(\cos\beta)\right]$$

$$\times\left[P_n^{(m)}(\cosh\alpha) \text{ or } Q_n^{(m)}\cosh(\alpha)\right] \tag{17}$$

with m, n = 0, 1, 2, 3 ..., and where $P_n^{(m)}$ and $Q_n^{(m)}$ are associated Legendre functions of the first and second kind, respectively. Considering the boundary conditions and the regularity conditions (15) and (16) applicable to the potential in the transformed domain, we need to select solutions of (14) for which $\varphi(\alpha, \beta) = \varphi(\alpha)$. The relevant solution, which also satisfies the regularity condition (16), can be obtained by selecting $m = n = 0$ and neglecting the remaining terms of the sequence (17); thus,

$$\varphi(\alpha) = \frac{C}{2}\ln(\xi) \tag{18}$$

where C is an arbitrary constant and

$$\xi = \left(\frac{\cosh\alpha + 1}{\cosh\alpha - 1}\right) \tag{19}$$

The arbitrary constant C can be obtained by considering the boundary condition (15); this gives

$$\varphi(\alpha) = \frac{\varphi_0}{\ln \xi_0} \ln \xi \tag{20}$$

where

$$\xi_0 = \xi(\alpha_0) \tag{21}$$

This formally completes the solution to the problem, in the sense that the coordinates R and Z can be expressed in terms of the spheroidal coordinate α in the forms

$$\cosh \alpha = \frac{(R_1 + R_2)}{2c_p} \tag{22}$$

$$R_1 = \sqrt{\left(Z + c_p\right)^2 + R^2} \quad R_2 = \sqrt{\left(Z - c_p\right)^2 + R^2} \tag{23}$$

and the original coordinates r and z can be expressed through the transformation (5). The resulting expressions can be utilized to obtain the velocity components v_r and v_z, which in turn can be used to compute the flow from the spheroidal cavity due to the constant potential φ_0 at its surface. This procedure, however, entails an inordinate amount of analytical calculations. As an alternative, it is worth noting that since the fluid is incompressible, the flow rate through *any closed surface* in the porous medium *encompassing the spheroidal cavity* will also represent the flow rate from the boundary of the spheroidal cavity. We can use this mass conservation principle to first calculate the flow at a large distance from the prolate spheroid. Therefore, at large distances from the prolate spheroid $\alpha = \alpha_0$, the coordinate α becomes equal to $\mathsf{R}/2c_p$, where $\mathsf{R} = \sqrt{R^2 + Z^2}$ is the distance from the center of the spheroid. Therefore the hydraulic potential from the prolate spheroid at large distances from it is the inverse first power potential, which takes the form

$$\varphi(R,Z) = \frac{\varphi_0}{\ln\left[\left(\cosh\alpha_0 + 1\right)/\left(\cosh\alpha_0 - 1\right)\right]} \times \left[\frac{2c_p}{\left(R^2 + Z^2\right)^{1/2}}\right] \tag{24}$$

The velocity components $v_r(r, z)$ and $v_z(r, z)$ can be obtained by using (24) along with the transformations (5) in Darcy's expressions (3) for the velocity components; we obtain

$$v_r = \frac{2\varphi_0 c_p k_{rr}}{\ln\left[\left(\sqrt{\lambda} + \sqrt{\lambda - \eta^2}\right)/\left(\sqrt{\lambda} - \sqrt{\lambda - \eta^2}\right)\right]} \times \frac{r}{\left[r^2 + \left(k_{rr}/k_{zz}\right)z^2\right]^{3/2}} \tag{25}$$

$$v_z = \frac{2\varphi_0 c_p k_{rr}}{\ln\left[\left(\sqrt{\lambda} + \sqrt{\lambda - \eta^2}\right)/\left(\sqrt{\lambda} - \sqrt{\lambda - \eta^2}\right)\right]} \times \frac{z}{\left[r^2 + \left(k_{rr}/k_{zz}\right)z^2\right]^{3/2}} \tag{26}$$

The fluid flow rate Q through any spherical surface R = const., or the flux through the surface, located at a large distance from the prolate spheroidal cavity located in the transversely isotropic porous medium, can be evaluated in terms of velocity components (25) and (26). We have

$$Q = 2\pi \int_0^{\pi} \left(v_r \sin\Theta + v_z \cos\Theta\right) \mathbb{R}^2 \sin\Theta \; d\Theta \tag{27}$$

The factor 2π in (27) accounts for the integration with respect to the azimuthal direction γ. Also, using the relations

$$R = \mathsf{R} \sin \Theta \quad Z = \mathsf{R} \cos \Theta \tag{28}$$

we can write (27) in the form

$$Q = \frac{4\pi\varphi_0 a^* \sqrt{\lambda - \eta^2}\, k_{rr} k_{zz}^{3/2}}{\ln\left[\left(\sqrt{\lambda} + \sqrt{\lambda - \eta^2}\right)/\left(\sqrt{\lambda} - \sqrt{\lambda - \eta^2}\right)\right]} \times \int_0^{\pi} \frac{\sin\Theta \, d\Theta}{\left[k_{zz} + \left(k_{rr} - k_{zz}\right)\cos^2\Theta\right]} \tag{29}$$

Evaluating (29) we obtain

$$Q = \frac{8\pi\varphi_0 a^* \sqrt{\lambda - \eta^2}\sqrt{k_{rr}k_{zz}}}{\ln\left[\left(\sqrt{\lambda} + \sqrt{\lambda - \eta^2}\right)/\left(\sqrt{\lambda} - \sqrt{\lambda - \eta^2}\right)\right]} \tag{30}$$

Since this solution will be adopted to consider situations involving cylindrical intakes of diameter D and length L, with $L > D$, we will select the basic length parameter of the problem as $b^* = D/2$ and the intake shape factor will also be expressed in terms of the hydraulic conductivity in the radial direction; the expression (30) can now be written in the form

$$Q = F_{pp} k_{rr} \varphi_0 \tag{31}$$

where

$$F_{pp} = \frac{4\pi D\sqrt{\lambda-\eta^2}}{\eta\sqrt{\lambda}\,\ln\left[\left(\sqrt{\lambda}+\sqrt{\lambda-\eta^2}\right)\Big/\left(\sqrt{\lambda}-\sqrt{\lambda-\eta^2}\right)\right]}$$

$$\lambda \geq 1, \quad \eta \leq 1 \tag{32}$$

and $D = 2b^*$. As is evident, this result is developed with the specific constraint that $(b^*/a^*) \leq 1$. The subscript *pp* denotes an initial geometric configuration, corresponding to a prolate form and a hydraulic transverse isotropy that results in a transformation that maintains the prolate form. This definition of D will be maintained throughout, for consistency and to account for the fact that in the application of the results to the borehole casing entry point, the diameter D of the casing is usually constant and the length L is varied to suit the conditions of the test. The intake shape factor F_{pp} now characterizes the fluid flow rate either *from* or *to* the prolate spheroidal cavity located in the hydraulically transversely isotropic porous medium. The case of $\eta = 1$ corresponds to the problem of a spherical cavity which is located in a hydraulically transversely isotropic porous medium, i.e.,

$$[F]_{\substack{\text{spherical cavity}\\ \text{trans. isotropic}}} = \frac{4\pi D\sqrt{\lambda-1}}{\sqrt{\lambda}\,\ln\left[\left(\sqrt{\lambda}+\sqrt{\lambda-1}\right)\Big/\left(\sqrt{\lambda}-\sqrt{\lambda-1}\right)\right]} \tag{33}$$

where D is the diameter of the spherical cavity. As $\lambda \to 1$, the above result reduces to the solution for the intake shape factor for the spherical cavity which is located in an isotropic porous medium, i.e.,

$$[F]_{\substack{\text{spherical cavity}\\ \text{isotropic}}} = 2\pi D \tag{34}$$

In developing the preceding analysis we have explicitly assumed that the hydraulic conductivity ratio $(k_{rr}/k_{zz}) \geq 1$. There can be situations where $(k_{rr}/k_{zz}) \leq 1$. Examples of such materials could include a soil containing a dense arrangement of *wick drains*, which are usually installed to accelerate consolidation of clays and silts of low hydraulic conductivity and in instances where an otherwise hydraulically isotropic geomaterial is altered by the creation of dissolution channels due to the vertical migration (z direction) of a reactive chemical (Philips, 1991). In such instances, we need to formulate the problem in relation to a system of oblate spheroidal coordinates defined by

$$R = c_o \cosh\alpha \sin\beta \quad Z = c_o \sinh\alpha \cos\beta \tag{35}$$

The procedures associated with the analysis of the cavity with an oblate shape follow the same basic approach discussed previously, and we shall present here only the final result for the fluid transport rate from the oblate spheroidal cavity which is subjected to a constant potential φ_0, i.e.,

$$Q = \frac{2\pi D\varphi_0\sqrt{\eta^2-\lambda}\,\sqrt{k_{rr}k_{zz}}}{\eta\cot^{-1}\left(\sqrt{\lambda}\big/\sqrt{\eta^2-\lambda}\right)}$$

$$\lambda = \frac{k_{rr}}{k_{zz}} \leq 1 \qquad \eta = \frac{b^*}{a^*} \geq 1 \tag{36}$$

Alternatively, for consistency and comparison, the flow rate can be defined as previously by taking into consideration the hydraulic conductivity k_{rr}. This gives rise to the definition of the intake shape factor F_{oo} for an oblate spheroidal cavity, i.e.,

$$Q = F_{oo}k_{rr}\varphi_0 \tag{37}$$

where

$$F_{oo} = \frac{2\pi D\sqrt{\eta^2-\lambda}}{\eta\sqrt{\lambda}\,\cot^{-1}\left(\sqrt{\lambda}\big/\sqrt{\eta^2-\lambda}\right)}$$

$$\lambda = \frac{k_{rr}}{k_{zz}} \leq 1 \qquad \eta = \frac{b^*}{a^*} \geq 1 \tag{38}$$

where a^* is the half-length of the minor axis (i.e., in the z direction) and b^* is the equatorial radius of the oblate spheroidal cavity.

In the case when $\eta = 1$, the solution (38) corresponds to that of the intake shape factor for a spherical cavity which is located in a hydraulically transversely isotropic porous medium of infinite extent, i.e.,

$$[F]_{\substack{\text{spherical cavity}\\ \text{trans. isotropic}}} = \frac{2\pi D\sqrt{1-\lambda}}{\sqrt{\lambda}\,\cot^{-1}\left(\sqrt{\lambda}\big/\sqrt{1-\lambda}\right)}$$

$$\lambda = \frac{k_{rr}}{k_{zz}} \leq 1 \tag{39}$$

Similarly, as $\lambda \to 1$, the result (39) corresponds to the result for the intake shape factor for the spherical cavity located in a hydraulically isotropic porous medium, given by (34).

In the particular case when $\eta - \infty$, the oblate spheroidal cavity degenerates to a flat disc-shaped cavity which is located in a hydraulically transversely isotropic porous medium. Taking the limit of (38) as $\eta - \infty$, we obtain

$$[F]_{\substack{\text{disc-shaped cavity} \\ \text{trans. isotropic}}} = 4D\sqrt{\frac{k_{zz}}{k_{rr}}} \tag{40}$$

The result confirms the observation that in the case of the disc-shaped cavity which is located in hydraulically transversely isotropic porous medium, for flow to take place $k_{zz} > 0$ and $k_{rr} > 0$ in view of the assumption invoked prior to (5). As $\lambda \to 1$, the result (40) gives the intake shape factor for a disc-shaped cavity located in a porous medium of infinite extent, i.e.,

$$[F]_{\substack{\text{disc-shaped cavity} \\ \text{isotropic}}} = 4D \tag{41}$$

We note that in this instance, flow into the cavity region takes place from both faces of the disc-shaped entry region.

4. SUMMARY OF SOLUTIONS

In Section 3 we examined the development of intake shape factors for situations where the measure of hydraulic anisotropy was such that the cavity with a prolate spheroidal shape transformed into a prolate spheroidal shape, albeit with different dimensions, and the cavity with an oblate spheroidal shape similarly transformed to an oblate shape. There can, however, be situations where a cavity with an initially prolate spheroidal shape can transform to an oblate form because of the nature of the anisotropy, and vice versa. The solution for these situations can be approached along the lines outlined in Section 3 and the details will not be repeated here. It is sufficient to record here the relevant final results.

Consider the problem of a *prolate spheroidal cavity* with $\eta(= b^*/a^*) \le 1$, situated in a hydraulically transversely isotropic elastic porous medium where $\lambda(= k_{rr}/k_{zz}) \le 1$. The appropriate form of the intake shape factor F_{oo} can be evaluated in the form

$$F_{po} = \frac{2\pi D\sqrt{\eta^2 - \lambda}}{\eta\sqrt{\lambda}\,\cot^{-1}\left(\sqrt{\lambda}\big/\sqrt{\eta^2 - \lambda}\right)}$$

$$\lambda = \frac{k_{rr}}{k_{zz}} \le 1 \qquad \eta = \frac{b^*}{a^*} \le 1 \tag{42}$$

We now consider the problem of an oblate spheroidal cavity with $\eta(= b^*/a^*) \ge 1$, which is situated in a hydraulically transversely isotropic porous medium where $\lambda(= k_{rr}/k_{zz}) \ge 1$. The appropriate form of the intake shape factor F_{oo} can be evaluated in the form

$$F_{op} = \frac{4\pi D\sqrt{\lambda - \eta^2}}{\eta\sqrt{\lambda}\,\ln\left[\left(\sqrt{\lambda} + \sqrt{\lambda - \eta^2}\right)\Big/\left(\sqrt{\lambda} - \sqrt{\lambda - \eta^2}\right)\right]}$$

$$\lambda \ge 1 \qquad \eta \ge 1 \tag{43}$$

In summary, the complete range of the solutions for the intake shape factors, applicable to both *prolate* and *oblate* spheroidal cavities in hydraulically transversely isotropic porous media with either $k_{rr} \ge k_{zz}$ or $k_{rr} \le k_{zz}$, can be stated in the forms

$$\left.\begin{matrix} F_{pp} \\ \\ F_{op} \end{matrix}\right\} = \frac{4\pi D\sqrt{\lambda - \eta^2}}{\eta\sqrt{\lambda}\,\ln\left[\left(\sqrt{\lambda} + \sqrt{\lambda - \eta^2}\right)\Big/\left(\sqrt{\lambda} - \sqrt{\lambda - \eta^2}\right)\right]}$$

$$\lambda = \frac{k_{rr}}{k_{zz}} \ge 1 \qquad \begin{cases} \eta = \dfrac{b^*}{a^*} \ge 1 \\ \\ \eta = \dfrac{b^*}{a^*} \le 1 \end{cases} \tag{44}$$

Similarly,

$$\left.\begin{matrix} F_{oo} \\ \\ F_{po} \end{matrix}\right\} = \frac{2\pi D\sqrt{\eta^2 - \lambda}}{\eta\sqrt{\lambda}\,\cot^{-1}\left(\sqrt{\lambda}\big/\sqrt{\eta^2 - \lambda}\right)}$$

$$\lambda = \frac{k_{rr}}{k_{zz}} \le 1 \qquad \begin{cases} \eta = \dfrac{b^*}{a^*} \ge 1 \\ \\ \eta = \dfrac{b^*}{a^*} \le 1 \end{cases} \tag{45}$$

As remarked in the introduction, the condition $\lambda > 1$ or $k_h > k_v$ can be realized even with relatively plausible choices of porous materials with low hydraulic conductivity conforming to a sequence of sedimentary layers. Finally, since the results for the intake shape factors F_{ij}, ($i, j = p, o$) are in *exact closed form*, the presentation of detailed numerical results is perhaps unnecessary. It is also important to note that the expressions developed for the prolate spheroidal intake are strictly applicable to situations where $0 < \eta \le 1$. This eliminates the possibility of the intake region degenerating to a cylindrical cavity of radius b^*, for which the solution can be determined only by specifying

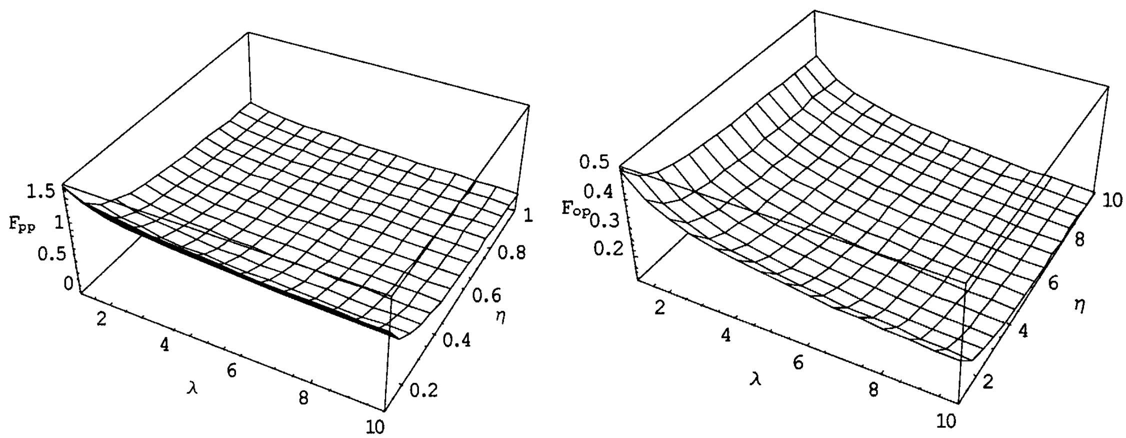

Figure 2. Variation of $F_{pp}/4\pi D$ and $F_{op}/4\pi D$ with $\lambda \in (1, 10)$ and $\eta \in (0.1, 1)$.

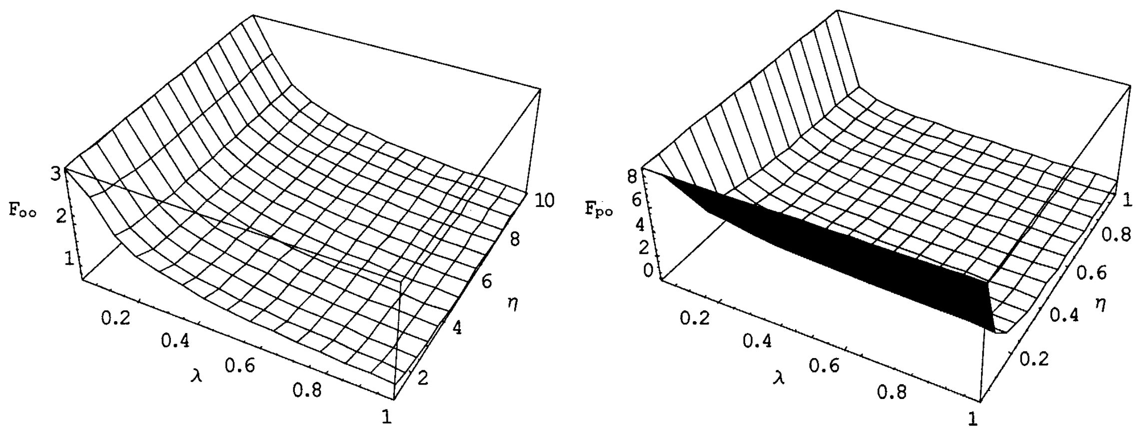

Figure 3. Variation of $F_{oo}/2\pi D$ and $F_{po}/2\pi D$ with $\lambda \in (0, 1)$ and $\eta \in (1, 10)$.

an external boundary at a finite location. Similar constraints apply to situations where there is a sharp contrast in the hydraulic conductivity characteristics that would permit flow either only in the radial direction or in the axial direction [i.e., $(k_v/k_h) \rightarrow 0$ or $(k_h/k_v) \rightarrow 0$, respectively]. In these circumstances the form of the partial differential equation governing flow in the transversely isotropic domain will permit solutions only if the porous domain is finite.

The results (44) and (45) are applicable to the complete range of *entry-point geometries* associated with the spheroidal cavity and the complete range of *hydraulic transverse isotropy* associated with the porous medium. It is noted that the result which is most likely to be applicable to a wide range of stratified sedimentary soils is that given by (44), where $\lambda (= k_{rr}/k_{zz}) \geq 1$. For purposes of completeness, however, Figs. 2 and 3 present the variation of F_{ij} applicable to the various ranges of the independent parameters $\lambda = k_h/k_v$ and $\eta = b^*/a^*$. Numerical values for the intake shape factors can be easily obtained by using symbolic mathematical manipulation routines such as MATHEMATICA™ and MAPLE™.

5. APPLICATIONS TO THE CYLINDRICAL INTAKE

In examining the problem related to flow from both prolate and oblate spheroidal cavities that are located in a porous medium with transversely isotropic hydraulic conductivity, we are primarily interested in applying the results to develop an intake shape factor for a cylindrical entry point with diameter D and length L, also located in a porous medium with transversely isotropic hydraulic conductivity. (Also, it is assumed that flow into such a cylindrical intake occurs over its entire surface, whereas in actual practice the upper plane surface of the cylindrical entry point does not contribute to the incoming flow. The exact formulation of the cylindrical intake region shown in Fig. 1 is a nontrivial problem in potential theory that invariably results in dual series formulations of a complicated mixed-boundary-value problem.) In order to obtain such a correlation we need to establish a relationship between the dimensions of the spheroid and that of the cylindrical shape. One obvious choice is to assume (Kirkham, 1945; Hvorslev, 1951; Maasland, 1957; Maasland and Kirkham, 1955, 1959) that $(L/D) = (a^*/b^*)$. This, however, is a convenient but poor correlation, which does not reflect any attributes of the physical process of fluid flow into the porous region. A similar choice relates to comparison of the volume of the cylindrical region with the volume of the spheroid (Wilkinson, 1968). In order to obtain this comparison we assume that the diameter of the cylindrical shape is the same as the equatorial diameter of the spheroid. (i.e., $D = 2b^*$). This gives

$$L = \frac{2}{3}(2a^*) \qquad \frac{D}{L} = \frac{3}{2}\left(\frac{b^*}{a^*}\right) \tag{46}$$

While the comparison of volumes gives a direct relationship between D/L and b^*/a^*, it should be noted that the comparison of volumes does not incorporate any attributes of the fluid flow process.

A more plausible correlation that would account for the flow process can be established by considering the equivalence of *surface area* between the cylindrical entry point and the prolate spheroidal entry region through which the flow takes place. Again, we assume that $D = 2b^*$ and obtain a relationship between the aspect ratio of the cylindrical cavity, D/L, and the aspect ratio of the spheroidal cavity, b^*/a^*. We have

$$\frac{D}{L} = \frac{2\left(b^*/a^*\right)\sqrt{1-\left(b^*/a^*\right)^2}}{\left(b^*/a^*\right)\sqrt{1-\left(b^*/a^*\right)^2} + \sin^{-1}\left[\sqrt{1-\left(b^*/a^*\right)^2}\right]} \tag{47}$$

Although this representation of the relationship between the two aspect ratios is perhaps the most realistic, it is not possible to invert the relationship to express b^*/a^* explicitly in terms of D/L. From a practical point of view, however, it is possible to find such a relationship through an explicit plot of the relationship. In this sense, once the ratio D/L is specified, the value of b^*/a^* required for the evaluation of (44) and (45) is known explicitly and uniquely. The variation of D/L with $\eta(= b^*/a^*)$ is shown in Fig. 4. It is also useful to establish the correlation

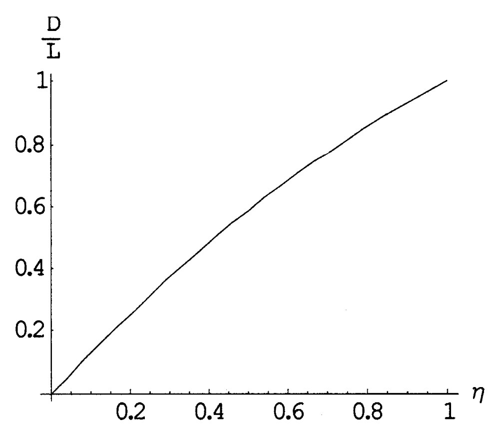

Figure 4. Variation of D/L with η as computed on the basis of equal surface areas between the spheroidal cavity and the cylindrical cavity.

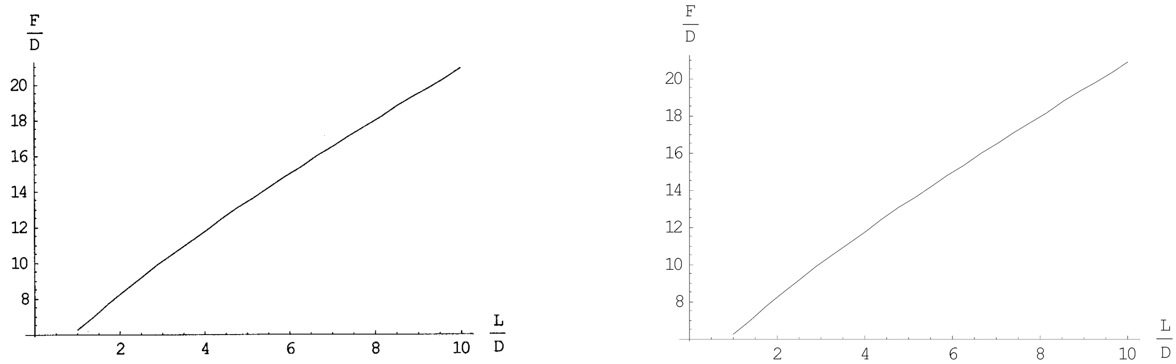

Figure 5. Comparisons between Hvorslev's result for the intake shape factor and that derived from the prolate spheroid analysis, where the dimensions are assigned through equivalence of surface areas

between Hvorslev's result given in the introduction and the result for the prolate spheroid located in an isotropic porous medium, where the result (47) is used to establish the equivalent dimension ratio in Hvorslev's expression. The relevant results are given in Fig. 5. It is evident that there is reasonable correlation between the two approaches, particularly if equivalent surface areas are used. Other comparisons with Hvorslev's result can be found in the literature (Brand and Premchitt, 1980b; Chapuis, 1989; Ratnam et al., 2001).

6. CONCLUDING REMARKS

The simplest of the *in-situ* tests used for the measurement of hydraulic conductivity is the cased borehole test, in which either water recharging or water extraction is used to create the potential difference required for the transient flow. With geomaterials that are hydraulically isotropic, the rate at which the water enters the cased borehole is governed primarily by the *geometric characteristics* of the intake region. One of the key results applicable to a cylindrical intake region located at the base of an impervious casing installed in an isotropic porous medium was proposed by Hvorslev (1951). It is shown that this result is applicable only to situations where the length of the cylindrical intake must be greater than the diameter of the region. Its application to the limiting case of a disc-shaped entry region at the base of a casing is therefore unwarranted. Also, when considering intake regions located at the base of cylindrical casings, it is shown that for situations where the boundary conditions at the entry point changes from a potential prescribed to an impervious condition, the flow velocities in particular can exhibit singular behavior at the demarcation point. Unless such singular behavior is either accounted for in the computational formulation or the influences of the singular behavior are circumvented, the flow rates calculated from computational methods can be open to error. Procedures for avoiding the influences of the singularities in the computation of the flow rates to the intake cavity are also suggested. When the intake region is located in a porous medium with directional hydraulic properties, the flow rate will be influenced both by the geometric shape of the intake region and the directional hydraulic conductivity properties of the porous medium. This article discusses the problem of the characterization of the intake shape factor for a cavity region located in a porous medium with transversely isotropic hydraulic conductivity. A transversely isotropic medium could be regarded as either a layered or a stratified geomaterial where the thickness of the stratifications are small enough, in comparison with the dimensions of the entry point, to warrant the application of the continuum theory for flow through a porous medium by appeal to a transversely isotropic equivalent of Darcy's law. When the intake region is modeled as a spheroidal cavity with either a prolate or an oblate form, this enables the development of a range of exact closed-form solutions to model the intake behavior. Some of these solutions can be deduced from results available in historical literature. The presentation here, however, is geared specifically to the discussion of fluid intakes in porous media, and the derivations use a systematic exposition. The results for a prolate spheroidal intake region, which usually represents the conventional cylindrical intake, are then used to obtain to the dimensions of the representative cylindrical intake region, by establishing the *equivalence of the surface areas* between the two geometric shapes. The development of exact closed-form results for the spheroidal intakes makes the procedure quite straightforward. The direct analysis of the flow into an extended entry point located at the base of a casing, however, entails the solution of a mixed-boundary-value problem which results in a set of dual series integral equations, where the solution must also account for the singular behavior of the gradient of the potential at the edges of the entry point where the boundary conditions change from a Dirichlet to a Neumann type. An elaborate mathematical procedure of this nature is perhaps not justified in view of the other complexities and uncertainties associated with the *in-situ* determination of a parameter such as hydraulic conductivity. As noted in the literature, the hydraulic conductivity is the single geotechnical parameter that exhibits the greatest degree of variability (Harr, 1987).

The general solutions presented in the article prompts the following observations. When the intake region corresponds to a flattened oblate spheroid in the form of a disc-shaped entry point, the flow rate depends on the "effective hydraulic conductivity," $\sqrt{k_h k_v}$, and the flow rate is given by $Q = 4D\varphi_0 \sqrt{k_h k_v}$, where k_h and k_v are the hydraulic conductivities in the horizontal (radial) and vertical (axial) directions, respectively. Therefore the arrangement of an entry point in the form of a disc, with admittedly a diameter considerably greater than that of the casing, would allow the direct determination of the effective hydraulic conductivity $\sqrt{k_h k_v}$. The solution to the analogous problem for the case where the elongated entry point is of infinite length is a two-dimensional problem, which has no solution for a medium of infinite extent. This is in view of the fact that the flow potential for the two-dimensional problem varies as $\ln(r)$, where r is the radial coordinate for the two-dimensional problem, as opposed to the variation proportional to $(1/R)$, applicable to the three-dimensional problem, where R is the spherical radius. Any finite value of the aspect ratio D/L [which is related to b^*/a^* through (47)] can be used in conjunction with (44) and (45) to determine the intake shape factor applicable to the particular aspect ratio. The flow rate is now determined by expressions of the type (30) and (36), where the geometry of the entry point is specified. Alternatively, from a practical perspective, the geometric aspect ratio of the intake can be altered to determine the different flow rates that are associated with the different intakes. These expressions for the flow rate, which incorporate the hydraulic transverse isotropy measure, can then be inverted to determine the hydraulic transverse isotropy ratio k_h/k_v in a unique manner. Admittedly, this requires some knowledge of whether $k_h/k_v < 1$ or whether $k_h/k_v > 1$. Site investigations that involve core recovery from the stratified geologic medium will naturally indicate the plausible choice. Also, for most sedimentary geologic media, the sequential deposition of relatively impervious layers will result in the condition where, invariably, $k_h/k_v > 1$.

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