On the advective-diffusive transport in porous media in the presence of time-dependent velocities

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[1] This paper develops certain *exact closed form solutions* to one-dimensional problems involving both advective and advective-diffusive transport in a porous medium, particularly in the presence of a Darcy velocity field that is time-dependent. Such situations can occur when the boundary potential inducing flow in the porous medium is time-dependent. A particular form chosen is an exponentially decaying time-dependency in the flow velocity, resulting from a non-replenishing source. The value of the one-dimensional solution stems not only from its potential applicability for the calibration of computational schemes used to examine the advection-diffusion equation in general, but also for the study of the purely advective flow problem with time-dependent velocity that requires sophisticated adaptive computational schemes to ensure numerical stability at a leading front in the form a discontinuity. *INDEX TERMS:* 3210 Mathematical Geophysics: Modeling; 1832 Hydrology: Groundwater transport; 1829 Hydrology: Groundwater hydrology; 1842 Hydrology: Irrigation; 9805 General or Miscellaneous: Instruments useful in three or more fields. **Citation:** Selvadurai, A. P. S. (2004), On the advective-diffusive transport in porous media in the presence of time-dependent velocities, *Geophys. Res. Lett.*, *31*, L13505, doi:10.1029/2004GL019646.

1. Introduction

[2] The topic of advective transport of a chemical species in a fluid saturated porous medium has important applications to geophysical problems ranging from water-borne contaminant migration in geoenvironmental engineering to salt movement in oceans. The mathematical similarities in the governing equations also allow extension of the applicability to include creeping movement of glaciers, sediment transport, transport of bacteria in porous media, propagation of diseases, and heat flow in geothermal energy extraction endeavours. In classical analytical treatments of advective-diffusive transport in a porous medium, it is invariably assumed that the advective Darcy flow can exhibit spatial variations but be time-independent [*Bear*, 1972; *Bear and Bachmat*, 1992; *Banks*, 1994; *Charbeneau*, 1999; *Selvadurai*, 2000]. This assumption is a useful first approximation in the mathematical and computational treatment of the advective-diffusive transport problem dealing notably with the linearized problem. The assumption of steady flow velocities is, of course, a simplification, which can be improved. A natural extension to the classical theory is to assume that the flow velocities are both time- and space-dependent. The processes contributing to such variations can include consideration of the hydro-mechanical coupling between the deformations of the porous medium and the compressibility of the pore fluid. The piezo-conduction equation [*Barenblatt et al.*, 1990; *Selvadurai*, 2000] is one such elementary model that introduces time-dependency to the velocity field. Another class of time-dependency can result from a time-dependent change in the boundary potential associated with Darcy flow in the porous medium. There are practical situations where the time-dependency in the boundary potential is of interest when examining transport problems. The most elementary of these is the advective-diffusive flow in a one-dimensional experimental column where the chemical occupies a part of the column and the boundary potential inducing Darcy flow in the one-dimensional column varies exponentially with time (Figure 1). In this paper, we restrict attention to a class of one-dimensional advective and advective-diffusive flows in a porous medium and present *exact closed form solutions* to situations where the Darcy flow velocity has an exponentially decaying time-dependency.

2. The Governing Equation

[3] The development of the partial differential equation governing advective-diffusive transport of a chemical species in a porous medium is described by *Bear* [1972], *Selvadurai* [2000] and others. In the case where the flow velocities are time-dependent and in the absence of any natural attenuation, the partial differential equation governing the one-dimensional movement of a chemical, with concentration $C(x, t)$, reduces to

$$\frac{\partial C}{\partial t} + v(t)\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} \tag{1}$$

where, $v(t)$ is the time-dependent velocity of the fluid in the pore space, D is the classical Fickian diffusion coefficient, x is the spatial variable and t is time. The solutions for initial boundary value problems and initial value problems governing the partial differential equation (1) have been presented by a number of investigators including *Lightfoot* [1929], *Danckwerts* [1953], *Taylor* [1953] and *Ogata and Banks* [1961]. References to further studies in the area are also given by *Lindstrom et al.* [1967], and *Charbeneau* [1999]. The recent contributions by *Selvadurai* [2002, 2003, 2004a] extend the study of the advective transport problem to include three-dimensional problems involving cavities located in unbounded media. The one-dimensional studies in this area deal invariably with situations where the flow velocity is constant. An adaptation of the advective

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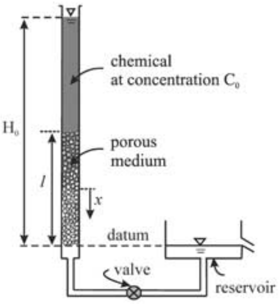

Figure 1. One-dimensional advective flow in a porous column.

transport equation to include variable velocity fields has also been discussed in connection with the stochastic equations for transport in the presence of a random velocity field. More recently, *Shvidler and Karasaki* [2003a, 2003b] and *Dagan* [2004] have examined the advective transport problem where the velocity field is stochastic. It is perhaps worth noting two essential points concerning the formulation of the one-dimensional initial boundary value problem in which the flow velocity is the time-dependent. If Darcy flow exists in a porous medium that is rigid and the permeating fluid is incompressible, then for the flow velocities to be time-dependent the one-dimensional domain must be finite and the boundary potential must vary with time. If either the porous medium is deformable or the pore fluid is compressible, then there are no restrictions on the spatial extent of the one-dimensional domain and the flow velocities can be both time- and space-dependent. In many of the articles on the one-dimensional problem involving time-dependent flow velocities, the mechanism contributing to the time-dependency is not clearly identified. We shall consider the advective and advective-diffusive transport problems separately, in order to present the solutions in their canonical forms. To the author's knowledge, these solutions are not available in classical or current literature dealing with the general transport problem.

3. The Advective Transport Problem

[4] We consider the one-dimensional advective transport problem for a porous medium governed by the first-order partial differential equation

$$\frac{\partial C}{\partial t} + v_0 \exp(-\lambda t)\frac{\partial C}{\partial x} = 0; x \in (-\infty, \infty); t > 0 \tag{2}$$

As is evident from equation (2) we restrict attention to an advective flow domain $x \in (-\infty, \infty)$, with the understanding that in the calculation of the advective flow velocities, the domain is considered to be of finite extent and the boundary potential has an exponential variation with time. The parameter λ occurring in equation (2) can be interpreted by appeal to Darcy flow in a one-dimensional column of length l. In this case $\lambda = k/l$, since the fluid velocity $v(t)$ is defined in relation to the pore space, k is the Dupuit-Forchheimer measure of hydraulic conductivity of the porous medium. This is also related to the conventional Darcy value applicable to the area averaged hydraulic conductivity $\tilde{k}$, through the relationship $k = \tilde{k}/n^*$, where n^* is the porosity of the porous medium. Also, we denote the velocity $v_0 = kH_0/l$, where H_0 can be identified as the height of the chemical fluid column at the start of the advective flow process and the chemical is also assumed to occupy a part of the column at the start of the advective flow process (Figure 1). The fixed spatial coordinate is x is chosen in relation to the position of the chemical concentration front. Since domain for the advective transport modelling occupies the range $x \in (-\infty, \infty)$, there are no boundary conditions applicable to the problem, except for the requirement that the concentration $C(x, t)$ should be finite and bounded as $\|x\| \to \infty$. The initial condition for the initial value problem takes the form

$$C(x, 0) = C_0[1 - \mathsf{H}(x)]; \quad x \in (-\infty, \infty) \tag{3}$$

where C_0 is a constant concentration and $H(x)$ is the Heaviside Step Function. The limit of $\lambda \to 0$ is mathematically admissible but physically unrealistic, since for $\lambda \to 0$, either $l \to \infty$ or $k \equiv 0$, both of which are inadmissible constraints for the one-dimensional problem. This limit, however, transforms the advective flow problem to its classical equivalent involving constant flow velocity, which becomes an important limit in examining solutions to the cases with unsteady flow velocities.

[5] There are many approaches for the solution of the initial value problem defined by equations (2) and (3), including the method of characteristics and integral transform techniques. Here, we take full advantage of the classical solution to the problem involving constant advective flow velocity and generalize the result to account for the time-dependency in the flow velocity with an exponential form. Considering the initial boundary value problem with $\lambda = 0$, the exact solution for the time-dependent chemical concentration takes the form

$$C(x, t) = C_0\,\mathsf{H}\left[t - \frac{x}{v_0}\right] \tag{4}$$

To retain the requirement that in the limit as $\lambda \to 0$, the solution to the problem involving an unsteady, exponentially decaying boundary potential must reduce to equation (4), we seek a solution to the initial boundary value problem in the form

$$C(x, t) = C_0\,\mathsf{H}\left[f(t) - \frac{x}{v_0}\right] \tag{5}$$

where $f(t)$ is an arbitrary function of time. Substituting equation (5) into the governing partial differential equation (2) we note that the resulting equation can be satisfied by solving a first-order ordinary differential equations for $f(t)$. In addition, the arbitrary constant arising from the integration of the ordinary differential equation is determined using the constraint that in the limit as $\lambda \to 0$, we recover (4). The complete solution to the advective transport problem can be expressed in the form

$$\frac{C(x, t)}{C_0} = \mathsf{H}\left[\frac{1}{\lambda}\{1 - \exp(-\lambda t)\} - \frac{x}{v_0}\right] \tag{6}$$

The validity of the solution (6) is assured through the uniqueness theorem applicable to the classical advective transport equation [see, e.g., *Selvadurai*, 2003, 2004a] and the convergence of the result (6) to the solution (4) as $\lambda \to 0$.

4. The Advective-Diffusive Transport Problem

[6] We consider the problem of one-dimensional advective-diffusive transport of a chemical species in a fluid saturated porous medium for which Darcy flow is applicable. In one-dimensional conditions and under a flow velocity that is spatially independent and has a time-dependency with an exponential form, the partial differential equation governing advective-diffusive flow takes the form

$$\frac{\partial C}{\partial t} + v_0 \exp(-\lambda t)\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2}; \quad x \in (-\infty, \infty); t > 0 \tag{7}$$

Here again we assume that the advective diffusive transport process commences when the chemical is already present in a section of the column (Figure 1), and as far as the advective-diffusive transport process in concerned the length of the porous column is considered infinite. The initial condition governing the advective- diffusive transport problem is identical to equation (3). We note that for the case of advective flow with constant velocity v_0, the solution to the advective-diffusive transport problem takes the form

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\begin{cases} \left[1 + \text{erf}\left(\dfrac{-x + v_0 t}{2\sqrt{Dt}}\right)\right]; & x < v_0 t \\ \text{erfc}\left(\dfrac{x - v_0 t}{2\sqrt{Dt}}\right); & x \geq v_0 t \end{cases} \tag{8}$$

where erfc(α) is the complimentary error function defined in terms of the error function erf(α) through the relation

$$\text{erf}(\alpha) = 1 - \text{erfc}(\alpha) = \frac{2}{\sqrt{\pi}}\int_0^\alpha \exp(-\varsigma^2)\,d\varsigma \tag{9}$$

For the solution of the partial differential equation (7) governing advective-diffusive transport with an exponentially time-dependent flow velocity we seek a trial solution of the form

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\begin{cases} \left[1 + \text{erf}\left(\dfrac{-x + g(t)}{2\sqrt{Dt}}\right)\right]; & x < v_0 t \\ \text{erfc}\left(\dfrac{x - h(t)}{2\sqrt{Dt}}\right); & x \geq v_0 t \end{cases} \tag{10}$$

where $g(t)$ and $h(t)$ are arbitrary functions chosen such that the respective components of equation (10) reduce to equation (8) as $\lambda \to 0$. Substituting the respective components of equation (10) into the governing partial differential equation, we obtain two first-order ordinary differential equation for $g(t)$ and $h(t)$; these can be integrated and the constants of integration determined by applying the constraint, that the solution should yield the respective terms in equation (8) as $\lambda \to 0$. The algebraic manipulations associated with these trial solutions are straightforward and can been performed using the Symbolic Mathematical software MATHEMATICA®. The complete exact solution to the one-dimensional advective-diffusive transport processes in a porous medium with a flow velocity that has an exponentially decaying variation with time is

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\begin{cases} 1 + \text{erf}\left(\dfrac{-x + v_0\dfrac{\{1-\exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right); & x < v_0 t \\ \text{erfc}\left(\dfrac{x - v_0\dfrac{\{1-\exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right); & x \geq v_0 t \end{cases} \tag{11}$$

The uniqueness theorem [*Selvadurai*, 2004b] for the classical advection-diffusion equation ensures the validity of the result (11). It may also be noted that by introducing the new independent variables

$$\xi = x - v_0\frac{\{1-\exp(-\lambda t)\}}{\lambda} \quad ; \quad \tau = t \tag{12}$$

the partial differential equation governing the advective-diffusive transport problem defined by equation (7) can be reduced to the form

$$\frac{\partial C}{\partial \tau} = D\frac{\partial^2 C}{\partial \xi^2} \quad ; \quad \xi \in (-\infty, \infty); \quad \tau > 0 \tag{13}$$

which is the classical diffusion equation for the transformed problem. Solutions to certain advective-diffusive transport problems with variable advective velocities can therefore be obtained from consideration of the appropriate initial value problems applicable to the diffusion equation. The initial condition corresponding to equation (3) takes the form

$$C(\xi, 0) = C_0[1 - \mathrm{H}(\xi)] \quad ; \quad \xi \in (-\infty, \infty) \tag{14}$$

The solution to the initial value problem defined by equations (13) and (14) can be obtained in a straightforward manner. The extension of the procedures to the development of analytical solutions to initial boundary value problems, however, can be accomplished in only for special cases, involving a time-dependent decrease in the boundary concentration in an artificially prescribed form.

[7] Since the initial value problem described by equations (7) and (3) is linear, the solution (11) can be quite easily adopted to examine the advective-diffusive transport resulting from other forms of uniform or non-uniform initial concentration profiles. As an example, consider the problem of the advective–diffusive transport of a "plug" of the chemical of uniform concentration located within the porous column. The chemically dosed plug region is assumed to occupy the interval $x \in (-a, a)$, such that the initial condition can be written as

$$C(x, 0) = C_0[\mathsf{H}(x + a) - \mathsf{H}(x - a)]; x \in (-\infty, \infty) \tag{15}$$

The solution to the resulting advective-diffusive flow problem can be expressed in the form

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\text{erf}\left(\frac{-x + a + v_0\dfrac{\{1-\exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right) - \frac{1}{2}\text{erf}\left(\frac{-x - a + v_0\dfrac{\{1-\exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right); x < v_0 t \tag{16a}$$

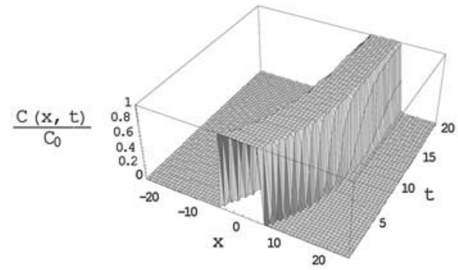

Figure 2. Advective-diffusive transport of a plug of chemical material in a porous medium, due to a time-dependent velocity [D = 0.005 m^2/day].

$$\frac{C(x,t)}{C_0} = \frac{1}{2}\operatorname{erfc}\left(\frac{x - a - v_0\dfrac{\{1-\exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right) - \frac{1}{2}\operatorname{erfc}\left(\frac{x + a - v_0\dfrac{\{1-\exp(-\lambda t)\}}{\lambda}}{2\sqrt{Dt}}\right);\, x \geq v_0 t \quad (16b)$$

Figures 2, 3, and 4 present typical numerical results for the time-dependent progression of the chemical plug within the porous medium for a typical situation where $v_0 \approx 2$ m/day; $\lambda \approx 0.2$ (days)$^{-1}$ and a = 5 m. The dimensions of the spatial coordinate x and the time t in Figures 2–4 are therefore in consistent units. The diffusion coefficient D is altered to illustrate its influence of the spatial distribution of the chemical profile.

5. Conclusions

[8] The classical problems of advective and advective-diffusive transport of a chemical species in a porous medium are based on the assumption of constant advective velocities in the porous medium, usually induced by Darcy flow. The concept of a time-dependent advective velocity becomes meaningful only in situations where transient flow is induced in the porous medium either because of a piezo-elastic drive or as a result of time variations of a non-replenishing boundary potential. The exponential decay of the velocity with time is a useful approximation for examining one-dimensional problems of advective-diffusive flow, similar to those encountered in column flow. The basic concepts have potential applications to practical problems involving deep geological disposal of chemical species. The paper presents *exact closed form solutions* to the one-dimensional problems of advective and advective-diffusive flow in porous media, where the movement of a chemical front located within the porous medium is induced by an exponentially decaying boundary potential. It is shown that these exact solutions can be obtained relatively easily by observing the structure of the solutions that involve advective or advective-diffusive flow in the presence of a constant flow velocity. The exact nature of the analytical results is of considerable benefit to computational modelling of this class of problems where the computational schemes can experience numerical instabilities, in the form of oscillations, negative concentrations, etc., at sharp discontinuities in the concentration profiles.

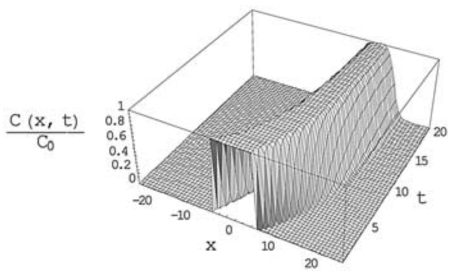

Figure 3. Advective-diffusive transport of a plug of chemical material in a porous medium, due to a time-dependent velocity [D = 0.05 m^2/day].

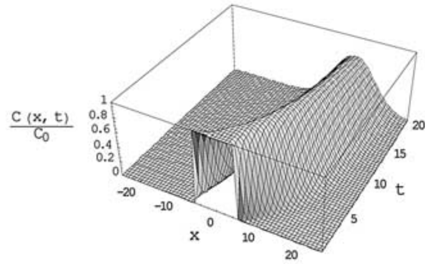

Figure 4. Advective-diffusive transport of a plug of chemical material in a porous medium, due to a time-dependent velocity [D = 0.50 m^2/day].

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