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ON THE ESTIMATION OF THE DEFORMABILITY CHARACTERISTICS OF AN ISOTROPIC ELASTIC SOIL MEDIUM BY MEANS OF A VANE TEST

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SUMMARY

The conventional use of the shear vane test is primarily restricted to the *in-situ* measurement of the undrained shear strength characteristics of saturated cohesive soils. Scant attention has been devoted to the use of this test as a means of measuring further properties of geotechnical interest. This paper presents an analytical study which illustrates the possible use of a shear vane test as a technique for the measurement of *in-situ* deformability characteristics of a soil medium. Certain plausible assumptions have been invoked for the analytical treatment of the shear vane problem. The vane blades are represented as elliptical shapes, the soil disturbance associated with the vane penetration is neglected and the soil mass enclosed within the swept boundary of the vane is represented as a rigid region. These, together with assumptions of classical isotropic elastic soil behaviour, enable the development of certain exact solutions for the torque–twist relationships of vanes fully or partially embedded in the soil. The results indicate that the elastic deformability characteristics of a soil medium can be directly recovered from an examination of the initial stages of an experimental torque–twist curve. In particular, the measured parameter would correspond to the linear elastic shear modulus of the soil medium.

INTRODUCTION

The vane shear test is widely used for the determination of undrained shear strength characteristics of undisturbed and remoulded saturated cohesive soils tested under laboratory and field conditions (Skempton,[1] Gray,[2] Eden and Hamilton,[3] Andresen and Bjerrum,[4] Cadling and Odenstad,[5] Aas,[6,7] Helenelund,[8] Richardson *et al.*,[9] Schmertmann,[10] Menzies and Mailey[11]). Shear vanes with rectangular, triangular, circular and polygonal blade shapes have been successfully used to study the sensitivity and anisotropy of the strength characteristics of soft cohesive soils in which the strength parameters can be significantly altered by sampling disturbance (Flaate,[12] Arman *et al.*[13]). Briefly, the shear vane device consists of stainless steel vanes composed of rectangular, triangular, circular or polygonal blades rigidly connected to a steel rod. A torque is gradually applied to the upper end of the rod until the soil yields along the surface swept by the boundary of the vanes. An expression for the undrained shear strength, which is assumed to be mobilized at this boundary, is computed in terms of the external torque and the vane dimensions by purely statical equilibrium considerations. The vane shear tests are usually performed at locations near the surface or at some depth below the bottom of a borehole in the soil medium (Figure 1).

A majority of the investigations to date relating to the use of the shear vane have concentrated upon its primary function as a test for the determination of the strength characteristics of the soil. The accurately measured parameters are usually the maximum applied torque and the minimum torque resulting from the loss of strength due to remoulding of soil along the failure zone. If, on

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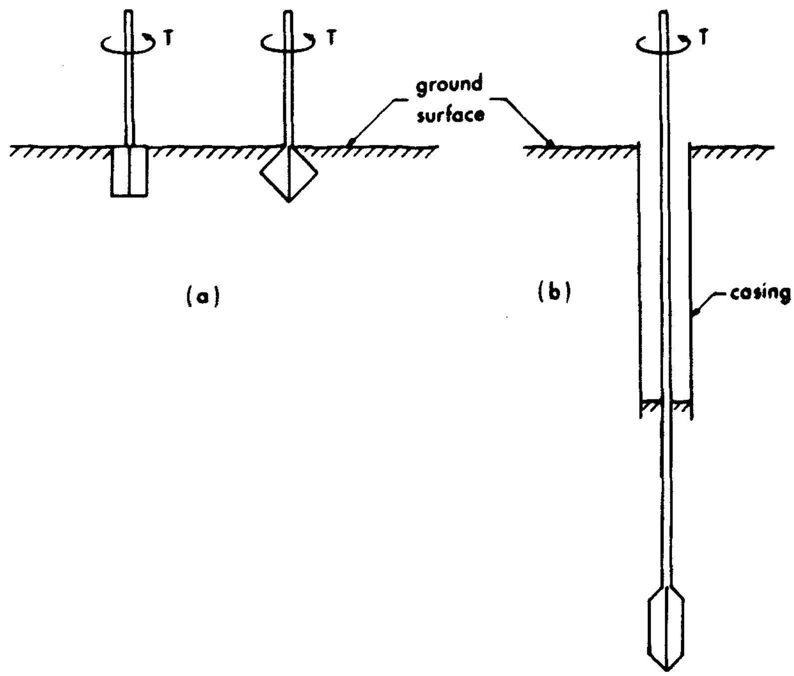

Figure 1. Installation procedures. (a) Shear vane located near the ground surface, (b) Shear van located at a large depth

the other hand, the torque–twist relationship is accurately recorded throughout the test then it seems reasonable to enquire whether the initial portion of such a torque–twist curve is in any way representative of the deformability characteristics of the soil medium in its elastic range. The possibility of such an extension of the shear vane test was first suggested by Cadling and Odenstad[5] and more recently by Madhav and Krishna.[14] The main purpose of this paper is to develop, within the framework of the classical theory of elasticity, certain theoretical results for the torque–twist relationship of a shear vane of prescribed shape embedded in a cohesive soil medium. Such a theoretical result would then provide a means of estimating the deformability characteristics of a soil medium by using the results derived from a shear vane test.

To enable the derivation of the theoretical results it becomes necessary to introduce further plausible assumptions, in addition to the representation of the cohesive soil as a linearly deformable isotropic homogeneous medium.

(i) Firstly, it is assumed that the shear vane can be introduced into the soil medium of soil disturbance; i.e., the torque–twist response of a vane located in the soil is not significantly altered by the soil disturbance associated with the vane penetration. In any event, this assumption has to be effectively realized in practice if reliable results for strength and deformability characteristics are to be determined from *in-situ* tests. At the same time it would be impractical to totally eliminate the soil disturbance associated with the vane penetration. The soil disturbance associated with near-surface vane tests is generally much less than that associated with deep vane tests. For the purpose of the theoretical derivations, we shall neglect the effects of the soil disturbance and represent the soil medium in the vicinity of the vane as a homogeneous continuum.

(ii) The second assumption pertains to the modification of the shape of the shear vane blades. Since the deformability characteristics of the soil medium are being investigated it is advantageous to induce a state of deformation, or stress, in the soil medium which will minimize the zones of premature failure associated with highly stressed locations. This could be achieved to some degreee by representing the vane blades by a regular geometric shape free of sharp edges; a vane with a circular blade shape achieves this to a large extent. However, in order to approximately represent some shear vanes that are currently used in engineering practice, the analytical treatment of a shear vane with blades corresponding to an elliptical shape will be considered. It should, however, be noted that the use of shear vanes with rectilinear shapes (similar to those shown in Figure 1) results in rather simple expressions relating the undrained shear strength and the applied torque.

(iii) Thirdly, it is assumed that the soil contained within the volume swept by the cross-section of the shear vane remains undeformed throughout the initial range of the torque–twist curve. According to this assumption, the entire swept volume can be approximately represented by a rigid inclusion. In the particular case of a vane blade shape in the form of an ellipse, this swept volume will correspond either to a prolate spheroidal or oblate spheroidal rigid region. This particular assumption is central to the development of relatively straightforward analytical solutions to the torque–twist relationship for the posed shear vane problem. The influence of the deformability of the soil region contained within the swept boundary on the accuracy of this solution will be further examined in a subsequent section (see also Selvadurai and Osler[15]). For purposes of reference throughout the paper we shall adopt the following nomenclature: (1) a '*prolate vane*' is a shear vane with an elliptical blade shape in which the length of the semi-major axis (a_p) is greater than the equatorial radius (b_p); (2) an '*oblate vane*' is a shear vane with an elliptical blade shape in which the length of the semi-minor axis (a_0) is less than the equatorial radius (b_0). Prior to the initiation of yield in the soil medium there exists complete continuity of displacements at the interface of this inclusion and the surrounding soil medium; i.e., the inclusion is in bonded contact with the rest of the soil medium.

(iv) Lastly, the dimensions of the vane are assumed to be small in comparison to the dimensions of the soil stratum in which the tests are carried out. It should be appreciated that the majority of the assumptions invoked above aid the development of fairly straightforward solutions for the torque–twist relationships for the shear vane, from which the deformability characteristics can be readily estimated. The general approach propounded in this paper can be further extended to minimize these simplifying assumptions.

The elastic analysis of the shear vane problem is thus reduced to the determination of the torque–twist relationship for (i) a rigid prolate spheroidal inclusion embedded in bonded contract with an infinite isotropic elastic medium or (ii) a rigid prolate spheroidal inclusion which is partially embedded in bonded contact with a semi-infinite isotropic elastic medium. These two categories are assumed to represent shear vane tests which may be conducted at the surface, or at a large depth within the soil respectively. The mathematical formulation of the problem is referred to a system of prolate spheroidal coordinates (α, β, γ). It is found that the shear vane problem as formulated above falls into the general category of rotationally symmetric torsion problems in which the stress and deformation fields are independent of the longitude (γ). The analysis is carried out by making use of the displacement function technique similar to that proposed by Selvadurai and Spencer[16] and Selvadurai.[17] The formal similarity between the displacement function and the associated Stokes' stream function used in the analysis of purely rotary flow in Newtonian viscous fluids (Jeffery,[18] Lamb,[19] Langlois,[20] Happel and Brenner[21]) is fully recognized. The restriction of incompressibility, implicit in the treatment of the viscous flow problem, is, however, not extended to the analysis of the elasticity problem. Using the displacement function technique, an exact *closed form* solution is developed for the torque–twist relationship for a deep shear vane with an elliptical blade shape. Also, the torque–twist relationship for a shallow shear vane can be directly recovered from the above result. The corresponding solutions to the circular shear vane problem occur as limiting cases of these results. Owing to the symmetry of these torsion problems, the torque–twist relationships thus developed are valid for moderately large deformations of the surrounding soil medium. The stress analysis of elastic media subjected to moderately large deformations is carried out by taking into consideration effects of both linear and quadratic terms in the displacement gradients (Rivlin,[22] Spencer,[23] Selvadurai and Spencer[16]). Alternatively, it is clear that in the context of the shear vane test, the torque–twist relationship generated will exhibit a linear relationship significantly beyond the range of applicability of the linear theory of elasticity.

An expression for the limiting torque mobilized due to shear failure along the swept boundary ($\alpha = \alpha_0$) is evaluated for the special case where the undrained shear strength is fully mobilized along that boundary. Finally, equivalent results for the shear vane with a swept volume in the shape of an oblate spheroid are presented for completeness.

GOVERNING EQUATIONS

The analysis of the shear vane problem posed here is referred to a system of prolate spheroidal coordinates (α, β, γ) defined by the transformation

$$[x; y; z] = c_p[\sinh (\alpha \sin \beta \cos \gamma; \sinh \alpha \sin \beta \sin \gamma; \cosh \alpha \cos \beta] \tag{1}$$

where c_p is a positive constant. The parametric surfaces α = constant, say α_0, $\beta = \beta_0$, $\gamma = \gamma_0$ form a triple orthogonal confocal family of prolate spheroids, hyperboloids of two sheets and meridional half planes respectively. The metric or local scale coefficients are given by

$$\begin{aligned} h_1 = h_2 &= [c_p^2(\sinh^2 \alpha + \sin^2 \beta)]^{-1/2} = h \\ h_3 &= [c_p \sinh \alpha \sin \beta]^{-1} \end{aligned} \tag{2}$$

The curvilinear components of the displacement vector are denoted by (u_α, u_β, u_γ) and attention is restricted to the particular class of rotationally symmetric torsion problems characterized by the displacement field

$$u_\alpha = 0; \qquad u_\beta = 0; \qquad u_\gamma = u_\gamma(\alpha, \beta) \tag{3}$$

For this deformation, the none-zero curvilinear components of the Cauchy stress tensor $\boldsymbol{\sigma}$ are given by

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & \sigma_{\alpha\gamma} & 0 \\ \sigma_{\alpha\gamma} & 0 & \sigma_{\beta\gamma} \\ 0 & \sigma_{\beta\gamma} & 0 \end{bmatrix} \tag{4}$$

The corresponding linear elastic constitutive relations are given by

$$[\sigma_{\alpha\gamma}; \sigma_{\beta\gamma}] = \frac{Gh}{h_3}\left[\frac{\partial}{\partial \alpha}(h_3 u_\gamma); \frac{\partial}{\partial \beta}(h_3 u_\gamma)\right] \tag{5}$$

where G is the linear elastic shear modulus. A displacement function $\Omega(\alpha, \beta)$ is now introduced such that

$$u_\gamma = h_3 \Omega \tag{6}$$

Using the above representation and the constitutive relations (6), the non-trivial equation of equilibrium is reduced to the form

$$D^2\Omega(\alpha, \beta) = 0 \tag{7}$$

where D^2 is Stokes' differential operator, which has the form

$$D^2 = h^2 h_3 \left\{ \frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} - \coth \alpha \frac{\partial}{\partial \alpha} - \cot \beta \frac{\partial}{\partial \beta} \right\} \tag{8}$$

The analysis of the elasticity problem in rotational symmetry is thus reduced to the determination of the function $\Omega(\alpha, \beta)$ which satisfies the appropriate boundary conditions. The uniqueness of the assumed displacement field (3) is established by virtue of Kirchhoff's uniqueness theorem (Green and Zerna[24]) in classical elasticity provided $G > 0$ (and $-1 < \nu < \frac{1}{2}$). For this particular class of deformations, the elastic medium is subjected to isochoric (volume preserving) deformations. Thus, the only material property that can be determined from the shear vane problem is the linear elastic shear modulus G. It should, however, be appreciated that although the elastic medium experiences isochoric motion, it does not necessarily imply that the medium itself is incompressible.

ELASTIC ANALYSIS OF THE PROLATE SHEAR VANE

Deep vane problem

Firstly, the problem of a homogeneous isotropic linear elastic medium which is bounded internally by a prolate spheroidal rigid inclusion is considered. The inclusion is assumed to be in bonded contact with the elastic infinite space at the boundary $\alpha = \alpha_0$. To reproduce the action of the shear vane problem, the inclusion is subjected to a torque T which causes a rigid body rotation ω about the axis of symmetry $\beta = 0$ (Figure 2(a)). The displacement boundary condition at the interface is

$$u_\gamma(\alpha_0, \beta) = \omega c_p \sinh \alpha_0 \sin \beta \tag{9}$$

Furthermore, since the elastic medium is of infinite extent the displacement and stress components should (i) tend to zero as $\alpha \to \infty$ and (ii) be single valued in the domain $\alpha_0 < \alpha < \infty$ and $0 < \beta < \pi$. It can be shown that the boundary and regularity conditions are satisfied by the displacement function

$$\Omega(\alpha, \beta) = \frac{\omega c_p^2}{\Phi_0} \sinh^2 \alpha \sin^2 \beta \Phi(\alpha) \tag{10}$$

where

$$\Phi(\alpha) = \left(\tfrac{1}{2} \ln \xi - \coth \alpha \operatorname{csch} \alpha\right); \quad \xi = \frac{\cosh \alpha + 1}{\cosh \alpha - 1} \tag{11a}$$

and

$$\Phi_0 = \Phi(\alpha_0) \tag{11b}$$

The displacement and stress components derived from (10) are

$$u_\gamma = \frac{\omega c_p}{\Phi_0} \sinh \alpha \sin \beta \Phi(\alpha) \tag{12}$$

and

$$\sigma_{\alpha\gamma} = \frac{2G\omega \sin \beta}{\sinh^2 \alpha \Phi_0 [\cosh^2 \alpha - \cos^2 \beta]^{1/2}}$$

$$\sigma_{\beta\gamma} = 0 \tag{13}$$

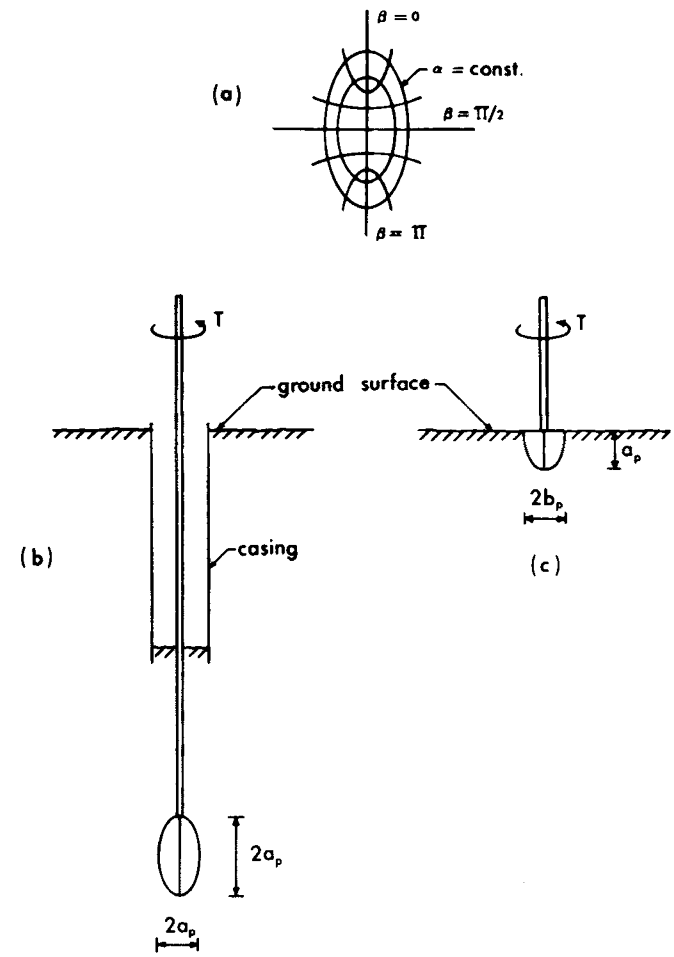

Figure 2. The prolate shear vane problem. (a) The spheroidal coordinate system, (b) Shear vane located at a large depth, (c) Shear vane located at the ground surface

respectively. The torque–twist relationship for the prolate spheroidal inclusion can be obtained by considering the resultant of moments induced about the axis $\beta = 0$ by the shear traction acting on the boundary $\alpha = \alpha_0$, i.e.,

$$T = \int_0^{\pi} \int_0^{2\pi} \left[\sigma_{\alpha\gamma} c_p \frac{\sinh \alpha \sin \beta}{h h_3} \right]_{\alpha=\alpha_0} d\gamma \, d\beta \tag{14}$$

With the understanding that the rotation of the inclusion occurs in the direction of the applied torque, (14) yields

$$T = \frac{32\pi c_p^3 G\omega}{3\{2 \coth \alpha_0 \operatorname{csch} \alpha_0 - \ln \xi_0\}} \tag{15}$$

where $\xi_0 = \xi(\alpha_0)$. From the geometry of the prolate spheroidal inclusion, the dimensions of the semi-major axis (a_p) and the equatorial radius (b_p) are given by

$$a_p = c_p \cosh \alpha_0; \qquad b_p = c_p \sinh \alpha_0 \tag{16}$$

Using the above equations, the torque–twist relationship (15) can be reduced to the form

$$T=\frac{32\pi a_p^3 G\omega[1-\lambda^2]^{3/2}\lambda^2}{3\left[2(1-\lambda^2)^{1/2}-\lambda^2\ln\left\{\dfrac{1+\sqrt{(1-\lambda^2)}}{1-\sqrt{(1-\lambda^2)}}\right\}\right]} \tag{17}$$

where

$$\lambda=\frac{b_p}{a_p}\quad\text{and}\quad \lambda\leq 1.$$

The result (17) is applicable to an elliptical shear vane or a prolate vane located at a large depth (Figure 2(b)).

Circular vane problem

In the particular case when $\lambda\to 1$, the elastic torque–twist relationship for a shear vane with a circular blade shape can be obtained from (17). Taking the appropriate limit, (17) gives

$$T=8\pi a_p^3 G\omega \tag{18}$$

where a_p is the radius of the vane.

Surface vane problem

Consideration is now given to the problem where the prolate spheroidal inclusion is partially embedded in bonded contact with a semi-infinite homogeneous isotropic elastic medium (Figure 2(c)). In this particular case the boundary conditions of the problem relate to the continuity of displacements at the interface $\alpha=\alpha_0$ and the traction-free conditions on the plane $\beta=\pi/2$. The displacement boundary conditions are

$$u_\gamma(\alpha_0,\beta)=\omega c_p\sinh\alpha_0\sin\beta \tag{19a}$$

and the traction boundary conditions reduce to

$$\sigma_{\beta\beta}\left(\alpha,\frac{\pi}{2}\right)=\sigma_{\alpha\beta}\left(\alpha,\frac{\pi}{2}\right)=\sigma_{\beta\gamma}\left(\alpha,\frac{\pi}{2}\right)=0\qquad\text{for }\alpha>\alpha_0 \tag{19b}$$

An inspection of the solution developed for the fully embedded inclusion indicates that the displacement component (12) and the non-zero component of $\boldsymbol{\sigma}$ (13) identically satisfy the boundary conditions (19). The torque–twist relationship for the partially embedded inclusion is, however, modified owing to the change in the limits of integration in (14) ($\pi/2<\beta<\pi$). Therefore the torque–twist relationship for the partially embedded prolate spheroidal inclusion, which is assumed to represent the near surface shear vane problem, is given by

$$T=\frac{16\pi a_p^3 G\omega[1-\lambda^2]^{3/2}\lambda^2}{3\left[2(1-\lambda^2)^{1/2}-\lambda^2\ln\left\{\dfrac{1+\sqrt{(1-\lambda^2)}}{1-\sqrt{(1-\lambda^2)}}\right\}\right]} \tag{20}$$

ULTIMATE TORQUE MOBILIZED BY A PROLATE VANE

Deep vane problem

An expression for the ultimate torque (T_y) mobilized by a deep shear vane with an elliptical blade shape, located in a saturated cohesive soil medium can be obtained by assuming that the undrained shear strength (c_u) is fully mobilized along the entire boundary $\alpha = \alpha_0$, i.e.,

$$T_y = \int_0^{\pi} \int_0^{2\pi} \left[c_u c_p \frac{\sinh \alpha \sin \beta}{h h_3} \right]_{\alpha=\alpha_0} d\gamma \, d\beta \tag{21}$$

An evaluation of the above integral leads to

$$T_y = \frac{4\pi a_p^3 c_u \lambda^2}{3} \left[\left(\frac{2-\lambda^2}{1-\lambda^2} \right) E\left(\frac{\pi}{2}, \sqrt{(1-\lambda^2)} \right) - \frac{\lambda^2}{(1-\lambda^2)} F\left(\frac{\pi}{2}, \sqrt{(1-\lambda^2)} \right) \right] \tag{22}$$

where $E(\pi/2, \zeta)$ and $F(\pi/2, \zeta)$ are complete elliptic integrals of the first and second kind respectively, defined by

$$\begin{aligned} E\left(\frac{\pi}{2}, \zeta\right) &= \int_0^{\pi/2} \sqrt{(1-\zeta^2 \sin^2\theta \, d\theta)} \\ F\left(\frac{\pi}{2}, \zeta\right) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{(1-\zeta^2 \sin^2\theta)}} \end{aligned} \tag{23}$$

Tabulated numerical values for these functions are given by Byrd and Friedman.[25]

Circular vane problem

The ultimate torque for the deep circular vane occurs as a limiting case of (22) as $\lambda \to 1$. By expanding E and F as power series in $\zeta (= \sqrt{(1-\lambda^2)})$ and taking the limit as $\zeta \to 0$, the ultimate torque for the spherical vane is obtained as

$$T_y = \pi^2 a_p^3 c_u \tag{24}$$

The above result is in agreement with the expression for a circular vane which can be easily derived from first principles.

Surface vane problems

From symmetry considerations, the ultimate torque for a partially embedded elliptical shear vane (Figure 2(b)) is identically equal to one half of the expression given by the right-hand side of (22). Similar considerations apply for the partially embedded circular vane.

ELASTIC ANALYSIS OF THE OBLATE SHEAR VANE

The analysis presented in the preceding sections can also be extended to the case of an oblate shear vane in which the equatorial radius (b_0) is greater than the semi-minor axis (a_0). Final results for the various cases shall be presented here without a detailed account of the analysis or calculations.

Deep vane problem

The torque–twist relationship for an oblate spheroidal inclusion full embedded in bonded contact with an isotropic elastic infinite medium (Figure 3(a)) is given by

$$T = \frac{16\pi b_0^3 G \omega (1-\kappa^2)^{3/2}}{3\left[\cot^{-1}\left\{ \frac{\kappa}{\sqrt{(1-\kappa^2)}} \right\} - \kappa \sqrt{(1-\kappa^2)} \right]} \tag{25}$$

where $\kappa = a_0/b_0$ and $\kappa \leq 1$.

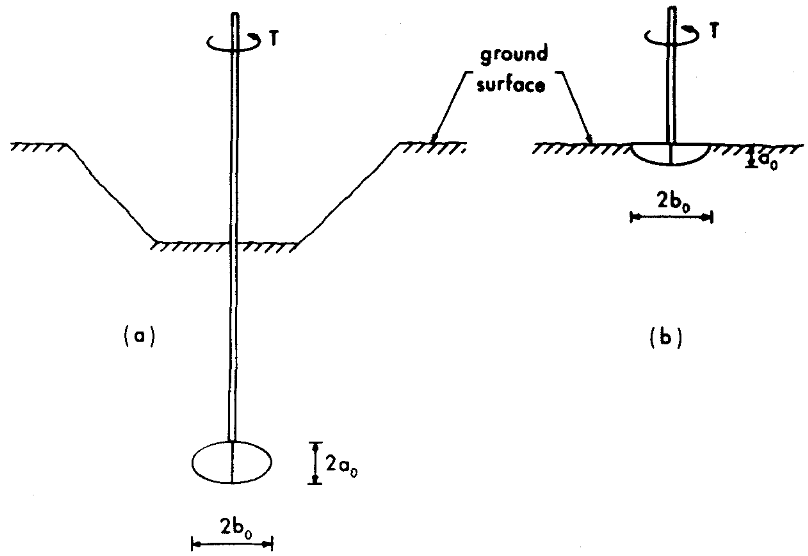

Figure 3. The oblate shear vane problem. (a) Shear vane located at a large depth, (b) Shear vane located at the ground surface

Disc vane problem

In the particular case when $\kappa \to 0$(i.e., $a_0 \to 0$), the oblate spheroid degenerates to a flat disc vane of infinitesimal thickness. This particular category of vane has, admittedly, limited application in geotechnical engineering. The torque–twist relationship is given by

$$T = \tfrac{32}{3} b_0^3 G\omega \tag{26}$$

where b_0 is the radius of the circular flat disc. It may be easily verified that in the limit $\kappa \to 1$, (25) yields the result derived earlier (see equation (18)) for the fully embedded spherical vane.

Surface vane problem

The torque–twist relationship for an oblate spheroidal inclusion partially embedded in bonded contact with an isotropic elastic halfspace (Figure 3(b)) can be directly recovered from (25) by invoking additional boundary conditions similar to those outlined earlier for the prolate vane problem. Therefore, for the partially embedded oblate vane we have

$$T = \frac{8\pi b_0^3 G\omega (1-\kappa^2)^{3/2}}{3\left[\cot^{-1}\left\{\dfrac{\kappa}{\sqrt{(1-\kappa^2)}}\right\} - \kappa\sqrt{(1-\kappa^2)}\right]} \tag{27}$$

Similarly the torque–twist relationship for a circular disc bonded to the surface of a halfspace is given by

$$T = \tfrac{16}{3} b_0^3 G\omega \tag{28}$$

The result (28) is identical to that obtained by Reissner and Sagoci[26] for exactly the same torsion problem.

ULTIMATE TORQUE MOBILIZED BY THE OBLATE VANE

Again, expressions for the ultimate torque mobilized by an oblate vane can be developed using the techniques outlined earlier. It is assumed that the undrained shear strength of the saturated cohesive soil is fully mobilized along the failure plane, which is assumed to be the boundary $\alpha = \alpha_0$.

Deep vane problem

The ultimate torque mobilized by an oblate vane fully embedded in a cohesive soil medium is given by

$$T_y = \frac{4\pi b_0^3 c_u}{3}\left[\frac{\kappa^2}{(1-\kappa^2)}F\left(\frac{\pi}{2}, \sqrt{(1-\kappa^2)}\right) + \left(\frac{1-2\kappa^2}{1-\kappa^2}\right)E\left(\frac{\pi}{2}, \sqrt{(1-\kappa^2)}\right)\right] \tag{29}$$

Disc vane problem

As $\kappa \to 0$ (i.e., $a_0 \to 0$) so $F(\pi/2, 1) \to \infty$; $E(\pi/2, 1) \to 1$. Using these limits in (29) the ultimate torque mobilized by a disc of radius b_0 embedded in a saturated cohesive soil is obtained; i.e.,

$$T_y = \frac{4\pi b_0^3}{3} c_u$$

Similarly, as $\kappa \to 1$, (29) gives the equivalent result for the deep circular vane.

Surface vane problems

Appropriate results for the ultimate torque mobilized by an oblate vane partially embedded in a halfspace or a disc bonded at the surface can be recovered from (29) and (30) by applying the requisite symmetry arguments.

A detailed numerical evaluation of the expressions presented for the various cases is not warranted. Once the aspect ratio (a_p/b_p or b_0/a_0) for a particular elliptical vane is assigned, explicit numerical results can be easily obtained for the elastic torque–twist relationships, for both full embedded and partially embedded shear vanes.

NUMERICAL RESULTS

The assumption of undeformability of the soil region contained within the swept boundary of the vane is an assumption which is vital to the development of the preceding analytical results. It is therefore of interest to further examine the accuracy of this particular assumption. Madhav and Krishna[14] have recently examined the problem relating to the torsional response of a rectangular vane embedded in an isotropic elastic halfspace and an infinite space. Their treatment takes into account the deformability of the cylindrical material region contained within the intersecting blades and the swept boundary. A discretized solution is developed by making use of Mindlin's [27] fundamental results for the internal loading of an isotropic elastic halfspace. The torque–twist relationship for a rectangular vane (diameter D and height H) developed in Reference 14 for the *deeply embedded* vane takes the form

$$T = E\omega D^3 [I_\omega]_{\text{Rect}} \tag{30}$$

where E is the modulus of elasticity of the soil material and the values for $[I_\omega]_{\text{Rect}}$ are shown in Figure 4.

We now consider the analytical solution (17) developed for the deeply embedded prolate spheroidal vane. Although the geometrical shapes of the elliptical vane and the rectangular vane cannot be compared, it is of interest to examine the correlation between the results given by Madhav and Krishna[14] with equivalent results derived for the elliptical vane possessing the same aspect ratio as the rectangular vane. The result (17) can be rewritten in the form

$$T = E\omega D^3[I_\omega]_{\mathrm{Ellip}} \tag{31}$$

where

$$[I_\omega]_{\mathrm{Ellip}} = \frac{2\pi}{3(1+\nu)}\left(\frac{2a_p}{D}\right)\left(\frac{2b_p}{D}\right)^2 \frac{(1-\lambda^2)^{3/2}}{\left[2\sqrt{(1-\lambda^2)}-\lambda^2 \ln\left\{\dfrac{1+\sqrt{(1-\lambda^2)}}{1-\sqrt{(1-\lambda^2)}}\right\}\right]} \tag{32}$$

For *similar aspect ratios* (i.e., $2a_p = H$; $2b_p = D$) the numerical results derived from (32) for $[I_\omega]_{\mathrm{Ellip}}$ are shown in Figure 4.

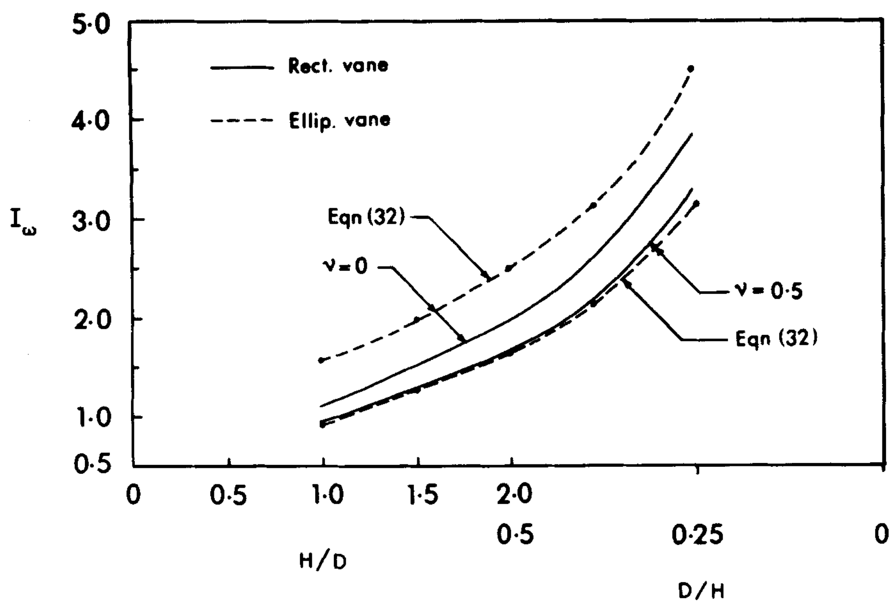

Figure 4. A comparison of solutions developed for the rectangular vane[14] and the elliptical vane: $2a_p = H$; $2b_p = D$; $H/D = 1/\lambda$

We note that for undrained behaviour of the soil (i.e., $\nu = \frac{1}{2}$) the results given in Reference 14 for the rectangular vane compare very favourably with those developed for the elliptical vane. The two sets of results appear to be at variance when $\nu = 0$. It thus appears that when considering the undrained torque–twist curve for a rectangular vane, the region corresponding to the inscribed spheroid remains practically rigid and the deformation of the remainder of the elastic medium (contained within the swept boundary) contributes little to the deformational response of a measured torque–twist curve. The assumption pertaining to the 'rigid behaviour' of the soil region contained within the swept boundary is appropriate in instances where the vane is employed in the determination of mechanical properties of saturated cohesive soils.

CONCLUSIONS

The shear vane tests are extensively used in the determination of the undrained shear strength characteristics of cohesive soils tested under both laboratory and field conditions. This paper examines the possible further use of a shear vane test as a technique for the determination of *in-situ* deformability characteristics of a soil medium. A generalized theoretical basis is provided whereby the linear elastic shear modulus of a cohesive soil medium can be estimated from an examination of the initial stages of a torque–twist relationship. The development of these analytical estimates assumes that the shear vane is composed of blades with an elliptical shape and that the material region enclosed within the swept boundary of the vane remains undeformed throughout the application of the torque. It has been shown that this latter assumption appears to be satisfactory for shear vane tests carried out on soil media which exhibit incompressible elastic characteristics. The soil disturbance associated wihth the vane penetration is assumed to be small and thus neglected in the analytical treatment. Using these assumptions, exact closed form solutions are developed for the torque–twist relationships for elliptical shear vanes either fully embedded in an isotropic infinite elastic medium or partially embedded in an isotropic elastic halfspace. These analyses correspond respectively, to vane tests that are carried out at a large depth from a boundary or at the boundary itself. The torque–twist relationships thus developed can be directly employed to estimate the linear elastic shear modulus of the cohesive soil medium. Expressions have also been derived for the ultimate torque mobilized by these elliptical vanes; these in turn could be used in the estimation of the undrained shear strength of the cohesive soil medium. The basic ideas outlined by this paper can be further extended to provide theoretical estimates for other classes of material behaviour (e.g., a transversely isotropic elastic medium).

This paper, admittedly, examines only the theoretical aspects of the title problem; nonetheless, it provides a basis for future detailed experimental investigations. With regard to *in-situ* shear vane testing, the soil disturbance associated with the vane penetration is a factor which defies quantitative treatment. The manner in which such soil disturbance may influence the measured deformability properties of the soil can only be resolved by comparison of vane test data with results derived from both *in-situ* tests, such a pressuremeter, screw-plate and plate load tests, and triaxial tests carried out on relatively 'undisturbed' soil samples. Such investigations are necessary to examine the extent of validity of the theoretical modelling and to establish clearly the use of the shear vane as a further device for the assessment of the *in-situ* deformability characteristics of cohesive soil media.

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REFERENCES

1. A. W. Skempton, 'Vane tests in alluvial plain of River Forth near Grangemouth', *Geotechnique*, **1**, 111–124 (1948).
2. H. Gray, 'Field vane shear tests of sensitive cohesive soils', *Proc. Am. Soc. Civ. Eng.* **81**, Paper 755; in *ASCE Trans.*, 844–857 (1957).
3. W. J. Eden and J. J. Hamilton, 'The use of a field vane in sensitive clay', *American Society for Testing and Materials: Special Technical Publication 193*, 41–53 (1956).
4. A. Andresen and L. Bjerrum, 'Vane Testing in Norway: Discussion', *American Society for Testing and Materials: Special Technical Publication 193*, 54–60. Also published in *Norwegian Geotechnical Institute Publication 28* (1957).

5. L. Cadling and S. Odenstad, 'The vane borer: An apparatus for determining the shear strength of clay soils directly from ground', *Royal Swedish Geotechnical Institute Proc.* 7–88 (1950).
6. G. Aas, 'A study of the effect of vane shape and rate of strain on the measured values of in-situ shear strength of clays', *Proc. 6th Int. Conf. Soil Mech. Fdn. Engng.*, **1**, 141–145 (1965).
7. G. Aas, 'Vane tests for investigation of anisotropy of undrained shear strength of clays', *Proc. Geotech. Conf., Oslo*, **1**, 3–8 (1967).
8. K. V. Helenelund, 'Torsional field shear tests', *Proc. 6th Int. Conf. Soil Mech. Fdn. Engng.*, **1**, 240–243 (1965).
9. A. M. Richardson, E. W. Brand and A. Memon, 'In-situ determination of anisotropy of a soft clay', *Proc. A.S.C.E. Specialty Conf. on In-Situ Measurement of Soil Properties*, Raleigh, N.C. **1**, 336–349(1975).
10. J. H. Schmertmann, 'Measurement of in-situ shear strength', *Proc. A.S.C.E. Specialty Conf. on In-Situ Measurement of Soil Properties*, Raleigh, N.C. **2**, 57–138 (1975).
11. B. K. Menzies and L. K. Mailey, 'Some measurements of strength anisotropy in soft clays using diamond-shaped shear vanes', *Géotechnique*, **26**, 535–538 (1976).
12. K. Flaate, 'Factors influencing the results of vane tests', *Can. Geotech. J.* **3**, 18–31 (1966).
13. A. Arman, J. K. Poplin and N. Ahmad, 'Study of the shear vane', *Proc. A.S.C.E. Specialty Conf. on In-Situ Measurement of Soil Properties*, Raleigh N.C. **1**, 93–120 (1975).
14. M. R. Madhav and K. S. R. Krishna, 'Undrained modulus from vane shear tests', *J. Geotech. Engng. Div., Proc. Am. Soc. Civ. Eng.*, **103**, 1337–1340 (1977).
15. A. P. S. Selvadurai and J. C. Osler, 'Discussion of "Undrained modulus from vane shear test" by M. R. Madhav and K. S. R. Krishna', *J. Geotech. Engng Div., Proc. Am. Soc. Civ. Eng.*, **103**, 1337–1340 (1977).
16. A. P. S. Selvadurai and A. J. M. Spencer, 'Second-order elasticity with axial symmetry. I. General theory', *Int. J. Engng Sci.* **10**, 97–114 (1972).
17. A. P. S. Selvadurai, 'Second-order elastic effects in the torsion of a spherical annular region', *Int. J. Engng Sci.* **12**, 295–310 (1974).
18. G. B. Jeffery, 'On the steady rotation of a solid of revolution in a viscous fluid', *Proc. Lond. Math. Soc. (2)*, **14**, 327–338 (1914).
19. H. Lamb, *Hydrodynamics*, 2nd edition, Dover Publications, New York, 1945.
20. W. E. Langlois, *Slow Viscous Flow*, The Macmillan Company, New York, 1964.
21. J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Prentice-Hall, Englewood Cliffs, N.J., 1965.
22. R. S. Rivlin, 'The solution of problems in second-order elasticity theory', *J. Rational Mech. Anal.* **2**, 53–81 (1953).
23. A. J. M. Spencer, 'The static theory of finite elasticity', *J. Inst. Math. Appl.* **6**, 164–200 (1970).
24. A. E. Green and W. Zerna, *Theoretical Elasticity*, 2nd edition, Oxford University Press, London, 1968.
25. P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Springer-Verlag, Berlin, 1971.
26. E. Reissner and H. F. Sagoci, 'Forced torsional oscillations of an elastic halfspace', *J. Appl. Phys.* **15**, 652–662 (1944).
27. R. D. Mindlin, 'Force at a point in the interior of a semi-infinite solid', *Physics*, **7**, 195–202 (1936).