Intake Shape Factors for Entry Points in Porous Media with Transversely Isotropic Hydraulic Conductivity

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Abstract: The intake shape factors for spherical and disc-shaped entry points located within a hydraulically transversely isotropic porous medium of infinite extent are presented in *exact closed form*. Implications of these results on a methodology for the in situ evaluation of the principal hydraulic conductivities are discussed.

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Introduction

The determination of in situ hydraulic conductivity characteristics of porous geomaterials is an important aspect of geotechnical and geoenvironmental investigations. For ease of both operation and analysis, such in situ evaluations are usually done by conducting transient tests involving cased boreholes subjected to either groundwater recharge or groundwater extraction. In theoretical developments of such borehole methods for the measurement of hydraulic conductivity, an important feature is the water entry characteristics of the borehole entry point. For a hydraulically isotropic porous medium, only the geometrical features of the surface through which water can either enter or leave the borehole characterize the intake shape factor. The estimates for the intake shape factor are obtained by examining the groundwater flow pattern in the vicinity of the entry region. The analysis invariably involves the solution of the potential equation that governs Darcy flow through the porous medium. In the context of in situ investigations, Hvorslev (1951) presented a comprehensive list of intake shape factors applicable to borehole entry points located in hydraulically isotropic soil media. The purpose of this paper is to outline some elementary procedures for extending the discussion to include hydraulic transverse isotropy of the porous medium. This extension, as opposed to either general hydraulic anisotropy or hydraulic orthotropy, is largely based on observations of micro- and macrostratifications in many sedimentary geological materials, including varved clays and silty deposits in fluvial and lacustrine environments that tend to display both transversely isotropic mechanical and hydraulic properties (see Mitchell 1956; Maasland 1957; Johnson and Morris 1962; Davis 1969; Bouwer 1978). Although these materials may, in general, display characteristics of hydraulic anisotropy, the transversely isotropic assumption is a useful first approximation for the characterization of hydraulic behavior, particularly in view of the depositional nature of such materials where the hydraulic properties in the plane of deposition are, by and large, expected to be isotropic. Most importantly, the conventional borehole recharge or withdrawal tests cannot be used for the in situ measurement of general hydraulic anisotropy of a porous medium. Such parameter characterizations can be achieved only through careful sample recovery and laboratory testing.

A comprehensive account of literature relating to the measurement of hydraulic conductivity characteristic of both anisotropic and transversely isotropic porous media will be given elsewhere (Selvadurai 2003). While the topic of flow into 3D intakes located in hydraulically transversely isotropic porous media can be deduced from results of the classical mathematical formulations of potential theory, it is shown that the results for the spherical and disc-shaped entry points can be obtained in a very straightforward manner. The approach adopted has some pedagogical merit in the sense that the mathematical manipulations are kept to a minimum. The first problem examined relates to the case of a spherical intake located in a hydraulically transversely isotropic porous medium. The second deals with a disc-shaped entry point that is located in a hydraulically transversely isotropic porous medium, where the plane of the disc region is coincident with the plane of isotropy. The solution to the first problem is obtained by considering a mapping, which transforms the governing problem from a spherical to a spheroidal coordinate system. The second problem is solved by an appeal to the theory of dual integral equations. In both cases, the solution to the intake shape factor for the entry point located in a transversely isotropic medium is obtained in exact closed form, which depends on the characteristic geometric dimension and the ratio of the two principal hydraulic conductivities governing hydraulic transverse isotropy.

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Theoretical Developments

The general theory of steady-state groundwater flow in a porous medium leading to the relevant potential equation can be found in the classical treatises by Muskat (1937), Polubarinova-Kochina (1962), Harr (1962), Bear (1972), Scheidegger (1974), Verruijt (1982), Philips (1991), Freeze and Cherry (1979), Strack (1989), Hermance (1999), and Nield and Bejan (1999). The geoenvironmental aspects of groundwater flow in particular are discussed in texts by Bear and Verruijt (1987), Ingebritsen and Sanford (1998),

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and Zijl and Nawalany (2000). A recent volume by Selvadurai (2000a) also gives the development of the potential equation governing Darcy flow in porous media and the associated general theorems applicable to Dirichlet and Neumann boundary value problems for potential problems, maximum principles, and the relevant uniqueness theorems.

We consider the problem of groundwater flow in a hydraulically transversely isotropic porous medium, which is saturated with an incompressible pore fluid. Although it is possible to adopt a generalized formulation of the problem, in view of the axial symmetry of the problems examined, it is convenient to develop the governing equations in relation to a system of cylindrical polar coordinates (r,θ,z). The fluid velocity components in the porous medium referring to these coordinates are denoted by (v_r,v_θ,v_z). The potential that induces fluid flow in the hydraulically transversely isotropic porous medium is the Bernoulli potential, consisting of the datum head and pressure head components. Because the ensuing studies relate to considerations of fluid flow behavior in the neighborhood of the entry point, we can, without loss of generality, assume that the pressure head $\varphi(r,\theta,z)$ is much greater than both the datum potential and the dimensions of the entry point. In other words, the entry point is assumed to be located in a porous medium of infinite extent. This invariably allows for the development of rather convenient analytical solutions, where the potential is assumed to a constant value in regions remote from the entry point. Also, because the problems examined will be such that the axis of symmetry associated with the problems will be normal to the plane of transverse isotropy, one can also assume that the pressure head is $\varphi(r,z)$. We also note that the flow problem is 3D.

We restrict attention to the flow of an incompressible fluid through porous medium; this requires the velocity field to satisfy the divergence free requirement for the fluid velocities, i.e.

$$\nabla.\mathbf{v}=\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0 \tag{1}$$

By considering Darcy's law for fluid flow through porous medium

$$\mathbf{v}=-\mathbf{K}\nabla\varphi \tag{2}$$

where $\mathbf{K}$=hydraulic conductivity matrix; and $\nabla\varphi$=gradient operator applied to the pressure potential. For a hydraulically transversely isotropic porous medium where the principal axes of hydraulic conductivity are aligned with the coordinate axes r and z, Darcy's law gives

$$v_r=-k_{rr}\frac{\partial\varphi}{\partial r};\qquad v_\theta=0;\qquad v_z=-k_{zz}\frac{\partial\varphi}{\partial z} \tag{3}$$

If we identify the z-axis as the vertical direction, then the hydraulic conductivity in the radial direction corresponds to the conventional hydraulic conductivity in the horizontal direction k_h, and the hydraulic conductivity in the axial direction corresponds to k_v, the equivalent property in the vertical direction. By combining Eq. (3) with the fluid incompressibility condition Eq. (1), we obtain the partial differential equation for the flow of an ideal incompressible fluid in a hydraulically transversely isotropic porous medium as follows:

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{k_{zz}}{k_{rr}}\frac{\partial^2\varphi}{\partial r^2}=0 \tag{4}$$

where k_{rr} is assumed to be nonzero. A coordinate transformation is introduced such that

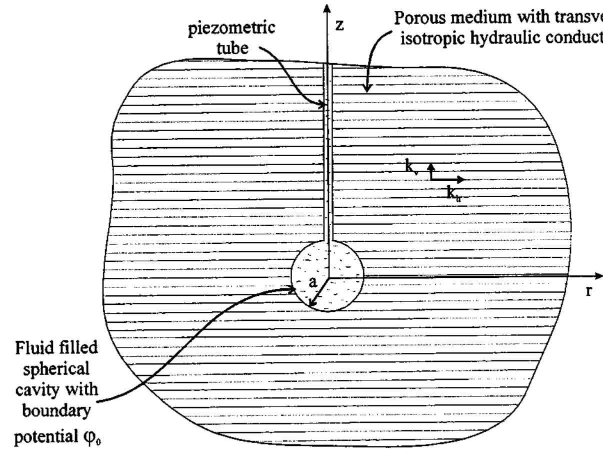

Fig. 1. Spherical entry point located in porous medium with transversely isotropic hydraulic conductivity

$$R=r;\qquad Z=\sqrt{\frac{k_{rr}}{k_{zz}}}z \tag{5}$$

such that the governing partial differential Eq. (3) can now be rewritten as

$$\frac{\partial^2\varphi}{\partial R^2}+\frac{1}{R}\frac{\partial\varphi}{\partial R}+\frac{\partial^2\varphi}{\partial Z^2}=0 \tag{6}$$

In the sections that follow we shall develop solutions of either Eq. (4) or Eq. (6), subject to certain boundary conditions representing the local flow patterns associated with the spherical and disc-shaped intakes at the base of cased boreholes.

Spherical Intake

The problem of a fluid saturated porous medium of infinite extent with a rigid fabric that is hydraulically transversely isotropic is considered. The infinite medium is bounded internally by a spherical cavity of radius a. The boundary of the cavity is maintained at a constant potential φ_0. From a practical point of view, if steady flow is to be maintained, fluid must be supplied to the boundary of the cavity. It is assumed that this is done using a piezometric tube of cross-sectional diameter significantly smaller than the diameter of the spherical entry point (Fig. 1). Because the problem is formulated in relation to the pressure potential, which is substantially greater than the datum and the dimensions of the sphere, the orientation of the plane of transverse isotropy is not central to the problem, and we can always select the z-axis to be oriented normal to the plane of isotropy (Fig. 1). The boundary value problem requires the solution of Eq. (3) subject to the boundary condition

$$\varphi(r,z)=\varphi_0;\qquad r^2+z^2=a^2 \tag{7}$$

In addition, because the domain is of infinite extent and the boundary value problem is 3D, the potential should decay to zero as $\sqrt{r^2+z^2}\to\infty$. For the solution of the boundary value problem, consider the transformed version of the partial differential equation, which is harmonic in the region $R\in(a,\infty)$ and $Z\in(a\sqrt{k_{rr}/k_{zz}},\infty)$. It is evident that by introducing the spatial transformations given by Eq. (5) it has also transformed the boundary from a sphere to a spheroid. In view of this observation, it is convenient to introduce a system of either prolate or oblate

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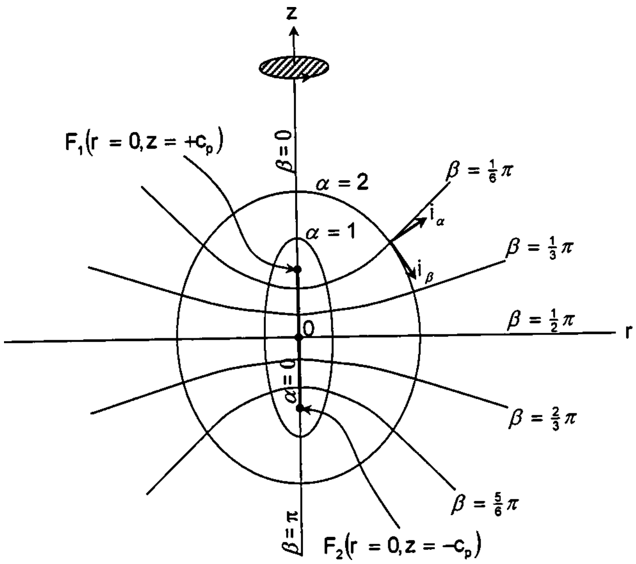

Fig. 2. Prolate spheroidal coordinate system

spheroidal curvilinear coordinates (α,γ,β) to examine the problem. First, consider the system of prolate spheroidal coordinates defined by

$$R=c_p \sinh\alpha \sin\beta; \qquad Z=c_p \cosh\alpha \cos\beta \tag{8}$$

such that the parametric surfaces $\alpha=$const. (say, α_0); $\beta=\beta_0$; and $\gamma=\gamma_0$ form a triple orthogonal confocal family of prolate spheroids, hyperboloids of two sheets and meridional half-planes, respectively (Fig. 2). By considering the expression for a differential arc length (ds) given by

$$(ds)^2=\left(\frac{d\alpha}{h_1}\right)^2+\left(\frac{d\beta}{h_2}\right)^2+\left(\frac{d\gamma}{h_3}\right)^2 \tag{9}$$

where the metric or local scale coefficients are given by (see Selvadurai 2000a)

$$h_1^p=h_2^p=[c_p^2(\sinh^2\alpha+\sin^2\beta)]^{-1/2}=h_p \tag{10}$$

$$h_3^p=(c_p \sinh\alpha \sin\beta)^{-1} \tag{11}$$

The focal distance c_p can be expressed in terms of the dimensions of the semimajor axis and the equatorial radius of the prolate spheroid conforming to the internal transformed boundary of the porous medium that is assumed to be defined by $\alpha=\alpha_0$, such that

$$a^2=c_p^2\sinh^2\alpha_0; \qquad \frac{k_{rr}}{k_{zz}}a^2=c_p^2\cosh^2\alpha_0 \tag{12}$$

and

$$c_p=a\sqrt{\lambda-1}; \qquad \lambda=\frac{k_{rr}}{k_{zz}}>1 \tag{13}$$

Consider the class of flow problems where the axis $\beta=0$ coincides with the z-axis, the axis of hydraulic symmetry of the porous medium. For such problems, the hydraulic potential is independent of the azimuthal coordinate γ, and, in terms of the curvilinear coordinates (α,β) and Laplace's Eq. (6), takes the form

$$\nabla^2\varphi(\alpha,\beta)=h_p^2\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}+\coth\alpha\frac{\partial}{\partial\alpha}+\cot\beta\frac{\partial}{\partial\beta}\right)\varphi(\alpha,\beta)=0 \tag{14}$$

subject to the boundary conditions

$$\varphi(\alpha,\beta)=\varphi_0 \quad \text{on} \quad \alpha=\alpha_0 \tag{15}$$

where $\alpha=\alpha_0$ corresponds to the boundary of the cavity and

$$\varphi(\alpha,\beta)\to 0 \quad \text{as} \quad \alpha\to\infty \tag{16}$$

For the solution of the boundary value problem we seek Lame' products associated with spheroidal coordinate systems (see Hobson 1931; Sadowsky and Sternberg 1947; Selvadurai 1976; Moon and Spencer 1988), where the general expression can be obtained in the form

$$\varphi(\alpha,\beta)=[P_n^{(m)}(\cos\beta) \;\text{ or }\; Q_n^{(m)}(\cos\beta)] \times[P_n^{(m)}(\cosh\alpha) \;\text{ or }\; Q_n^{(m)}(\cosh\alpha)] \tag{17}$$

with $m,n=0,1,2,3,\ldots,$ and where $P_n^{(m)}$ and $Q_n^{(m)}$ are associated Legendre functions of the first and second kind (Hobson 1931; Morse and Feshbach 1953). By considering the boundary conditions and the regularity conditions, Eqs. (15) and (16), respectively, applicable to the potential in the transformed domain, it is necessary to select solutions of Eq. (14) for which $\varphi(\alpha,\beta)=\varphi(\alpha)$. The single solution, which also satisfies the regularity condition Eq. (16), can be obtained by selecting $m=n=0$ and neglecting the remaining terms of the sequence Eq. (17); thus we have

$$\varphi(\alpha)=\frac{C}{2}\ln(\xi) \tag{18}$$

where C is an arbitrary constant and

$$\xi=\left(\frac{\cosh\alpha+1}{\cosh\alpha-1}\right) \tag{19}$$

The arbitrary constant C can be obtained by considering the boundary condition Eq. (15); this gives

$$\varphi(\alpha)=\frac{\varphi_0}{\ln\xi_0}\ln\xi \tag{20}$$

where

$$\xi_0=\xi(\alpha_0) \tag{21}$$

The solution to the problem is now formally complete, in the sense that the coordinates R and Z can be expressed in terms of the spheroidal coordinate ξ in the forms

$$\cosh\xi=\frac{(R_1+R_2)}{2c_p} \tag{22}$$

$$R_1=\sqrt{(Z+c_p)^2+R^2}; \qquad R_2=\sqrt{(Z-c_p)^2+R^2} \tag{23}$$

and the original coordinates r and z can be expressed through the transformation Eq. (4). These expressions can be used to obtain the velocity components v_r and v_z, which in turn can be used to compute the flow from the spherical cavity due the constant potential φ_0 at its surface. It is worth noting that because the fluid is incompressible, the flow through *any closed surface* in the porous medium encompassing the spherical cavity will also represent the flow from the spherical cavity. We can use this conservation principle to calculate the flow at a large distance from the spheroid. Therefore, at distances remote from the spheroid $\alpha=\alpha_0$, the coordinate α becomes equal to $\mathsf{R}/2c_p$, where $\mathsf{R}=\sqrt{R^2+Z^2}$ is the distance from the center of the spheroid. Therefore, the hydraulic potential from the spheroid, at large distances from it, is the inverse first power potential, which takes the form

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$$\varphi(R,Z)=\frac{\varphi_0}{\ln\left[\dfrac{\cosh\alpha_0+1}{\cosh\alpha_0-1}\right]}\left(\frac{2c_p}{(R^2+Z^2)^{1/2}}\right) \tag{24}$$

The velocity components $v_r(r,z)$ and $v_z(r,z)$ can be obtained by using Eq. (24) along with the transformations [Eq. (5)] in Darcy's expressions [Eq. (3)] for the velocity components

$$v_r=\frac{2\varphi_0 c_p k_{rr}}{\ln\left[\dfrac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}}\right]}\frac{r}{\left(r^2+\dfrac{k_{rr}}{k_{zz}}z^2\right)^{3/2}} \tag{25}$$

$$v_z=\frac{2\varphi_0 c_p k_{rr}}{\ln\left[\dfrac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}}\right]}\frac{z}{\left(r^2+\dfrac{k_{rr}}{k_{zz}}z^2\right)^{3/2}} \tag{26}$$

The rate of fluid flow Q through a spherical surface $\mathsf{R}=\text{const}$, or the flux through the surface, located at a large distance from the spherical cavity of radius a located in the transversely isotropic porous medium, can be evaluated in terms of velocity components Eqs. (25) and (26)

$$Q=2\pi\int_0^{\pi}\{v_r\sin\Theta+v_z\cos\Theta\}\mathsf{R}^2\sin\Theta\, d\Theta \tag{27}$$

where the factor 2π in Eq. (27) accounts for the integration with respect to the azimuthal direction γ. Also, using the relations

$$R=\mathsf{R}\sin\Theta;\qquad Z=\mathsf{R}\cos\Theta \tag{28}$$

Eq. (27) can be written in the form

$$Q=\frac{4\pi\varphi_0 a\sqrt{\lambda-1}\,k_{rr}k_{zz}^{3/2}}{\ln\left[\dfrac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}}\right]}\int_0^{\pi}\frac{\sin\Theta\, d\Theta}{[k_{zz}+(k_{rr}-k_{zz})\cos^2\Theta]} \tag{29}$$

By evaluating Eq. (29), one obtains

$$Q=\frac{8\pi\varphi_0 a\sqrt{\lambda-1}\sqrt{k_{rr}k_{zz}}}{\ln\left[\dfrac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}}\right]} \tag{30}$$

If the flow rate is defined to the entry point by a Hvorslev-type expression

$$Q=Fk_{eq}\varphi_0 \tag{31}$$

and the equivalent hydraulic conductivity k_{eq} is interpreted as

$$k_{eq}=\sqrt{k_{rr}k_{zz}} \tag{32}$$

then the intake shape factor F for the spherical entry point located in the hydraulically transversely isotropic medium can be expressed in the form

$$F=\frac{8\pi a\sqrt{\lambda-1}}{\ln\left[\dfrac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}}\right]} \tag{33}$$

Because the theoretical solution has been developed by assuming that the transformation results in a mapping to a prolate spheroid, it is required that $\lambda\geqslant 1$ or $k_{rr}>k_{zz}$.

In the limit when $\lambda\to 1$, the porous medium is isotropic, and by using L'Hospital's rule, Eq. (33) gives

$$F=4\pi a \tag{34}$$

which is the classical result that can be obtained by elementary considerations of radially symmetric flow from a spherical cavity in an extended porous medium, where the surface is maintained at a potential φ_0.

The analysis can be extended to include situations where $k_{zz}>k_{rr}$; in this case, one needs to formulate the problem in relation to a system of *oblate* spheroidal coordinates defined by

$$R=c_o\cosh\alpha\sin\beta;\qquad Z=c_o\sinh\alpha\cos\beta \tag{35}$$

The metric coefficients are

$$h_1^o=h_2^o=[c_o^2(\cosh^2\alpha-\sin^2\beta)]^{-1/2}=h_o \tag{36}$$

$$h_3^o=(c_o\cosh\alpha\sin\beta)^{-1} \tag{37}$$

The equivalent form of Laplace's Eq. (6), expressed in oblate spheroidal coordinates, takes the form

$$\nabla^2\varphi(\alpha,\beta)=h_o^2\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}+\tanh\alpha\frac{\partial}{\partial\alpha}+\cot\beta\frac{\partial}{\partial\beta}\right)\varphi(\alpha,\beta)=0 \tag{38}$$

Again, the boundary and regularity conditions applicable to the problem of fluid flow in a porous region bounded internally by an oblate spheroidal region are given by Eqs. (15) and (16), respectively. The appropriate solution of Eq. (38), which also satisfies these boundary conditions, takes the form

$$\varphi(\alpha,\beta)=\varphi_0\frac{\cot^{-1}(\sinh\alpha)}{\cot^{-1}(\sinh\alpha_0)} \tag{39}$$

The procedure for the determination of the flow rate from the spherical cavity follows that outlined previously in connection with the transformation of the problem to the prolate spheroidal domain. Avoiding details of calculations, it can be shown that

$$Q=\frac{4\pi\varphi_0 a\sqrt{\mu-1}\sqrt{k_{rr}k_{zz}}}{\sqrt{\mu}\cot^{-1}\left(\dfrac{1}{\sqrt{\mu-1}}\right)};\qquad \mu=\frac{k_{zz}}{k_{rr}}>1 \tag{40}$$

Again, the intake shape factor can be defined by taking into consideration the effective hydraulic conductivity defined by Eq. (32). One can now present a general expression for the intake shape factor covering the entire range, as follows:

$$F=8\pi a\begin{cases}\dfrac{\sqrt{\lambda-1}}{\ln\left[\dfrac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}}\right]} & \text{for}\quad \lambda>1\\[2ex] \dfrac{\sqrt{1-\lambda}}{2\cot^{-1}\left[\dfrac{\sqrt{\lambda}}{\sqrt{1-\lambda}}\right]} & \text{for}\quad \lambda<1\end{cases} \tag{41}$$

In the case of a stratified geological material, with hydraulic conductivities in the horizontal and vertical directions denoted by k_h and k_v, respectively, one can identify the horizontal direction with the radial direction and the vertical direction with the axial direction, which gives

$$\lambda=\frac{k_h}{k_v} \tag{42}$$

It should be noted that the definition of effective hydraulic conductivity also contains the dependency on both hydraulic conductivities. This representation is retained because the ensuing section examines the problem of a disc-shaped entry point located

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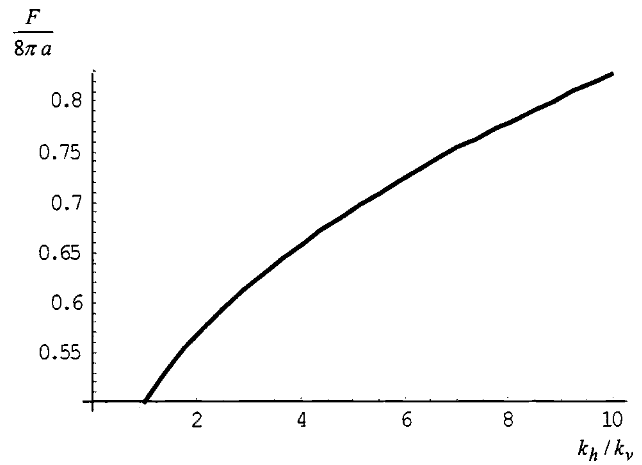

Fig. 3. Variation of intake shape factor for spherical intake with hydraulic conductivity ratio $(k_h/k_v)>1$

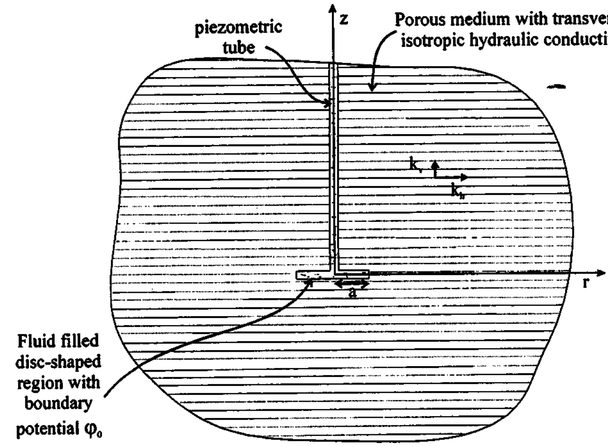

Fig. 5. Disc-shaped entry point located in porous medium with transversely isotropic hydraulic conductivity

in a porous medium with transversely isotropic hydraulic conductivity. If, on the other hand, the intake shape factor is defined in relation to a single hydraulic conductivity, say k_{rr}, then the flow rate can be defined as

$$Q=\tilde{F}k_{rr}\varphi_0 \tag{43}$$

where

$$\tilde{F}=8\pi a\begin{cases}\dfrac{\sqrt{\lambda-1}}{\sqrt{\lambda}\ln\left[\dfrac{\sqrt{\lambda}+\sqrt{\lambda-1}}{\sqrt{\lambda}-\sqrt{\lambda-1}}\right]} & \text{for}\quad \lambda>1\\[2ex] \dfrac{\sqrt{1-\lambda}}{2\sqrt{\lambda}\cot^{-1}\left[\dfrac{\sqrt{\lambda}}{\sqrt{1-\lambda}}\right]} & \text{for}\quad \lambda<1\end{cases} \tag{44}$$

Again, as $\lambda\rightarrow 1$, Eqs. (41) and (44) converge to the result of Eq. (33). Because the results are in exact closed form it is unnecessary to provide extensive numerical data. Figs. 3 and 4, however, illustrate the typical variation in F with λ. In a forthcoming study (Selvadurai 2003) has extended the results presented here to cover the case where the entry point region itself has either a prolate or an oblate form. The forthcoming paper also presents a comprehensive record of both analytical and computational approaches developed in the literature for the determination of the intake shape factors for cylindrical intakes located in hydraulically transversely isotropic porous media.

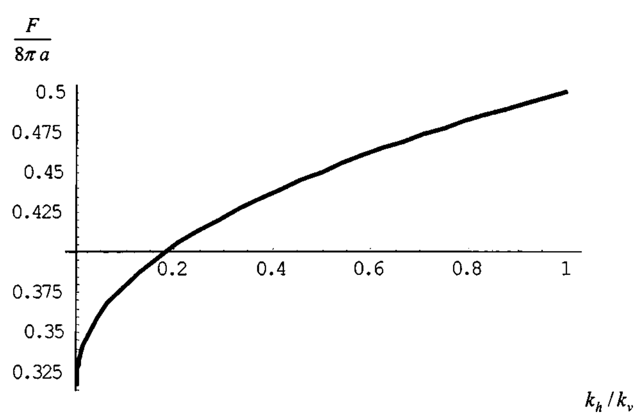

Fig. 4. Variation of intake shape factor for spherical intake with hydraulic conductivity ratio $(k_h/k_v)<1$

Disc-Shaped Intake

Now, the focus is on the problem of a disc shaped entry point, located in a fluid saturated porous medium with transversely isotropic hydraulic conductivity characteristics. It is assumed that the plane of the disc-shaped entry point coincides with the plane of transverse isotropy. Again, it is assumed that for steady flow to take place the disc-shaped cavity region is supplied with fluid through a piezometric pipe, the diameter of which is much smaller than the diameter $2a$ of the disc (Fig. 5). The area of this pipe can be neglected when prescribing the boundary conditions associated with the steady flow problem. The plane boundaries of the disc-shaped region are maintained at a constant potential φ_0. The boundary value problem associated with the disc-shaped entry point located in a porous medium with hydraulic orthotropy requires the solution of the governing transformed partial differential equation [Eq. (6)], subject to appropriate boundary and regularity conditions. It is noted that in this case the state of flow is symmetric about the plane $Z=0$. Therefore, we can restrict attention to a single half-space region, $R\in(0,\infty)$ and $Z\in(0,\infty)$, over which the flow takes place. The boundary of the flow domain now corresponds to $Z=0$, where the appropriate boundary conditions are

$$\varphi(R,Z)=\varphi_0;\qquad R\in(0,a);\ \ Z=0 \tag{45}$$

$$\frac{\partial\varphi}{\partial Z}=0;\qquad R\in(a,\infty);\ \ Z=0 \tag{46}$$

The solutions of Eq. (6) applicable to half-space regions can be obtained by integral transform techniques applicable to semiinfinite domains (see Sneddon 1951). Because the problem of fluid flow in a porous medium is axisymmetric, one can also employ a method of solution based on a Hankel transform development of Eq. (6). The relevant solution of Eq. (6) can be written in the form

$$\varphi(R,Z)=\int_0^{\infty}[A(\xi)e^{-\xi Z}+B(\xi)e^{+\xi Z}]J_0(\xi R)d\xi \tag{47}$$

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where $A(\xi)$ and $B(\xi)$ are arbitrary functions and $J_0(\xi R)$ is the 0th order Bessel function of the first kind. One can now pose the flow problem in the porous half-space region as a mixed boundary value problem, where the Dirichlet boundary condition [Eq. (45)] is prescribed over $R\in(0,a)$ of the surface $Z=0$ and the Neumann boundary condition [Eq. (46)] is prescribed on $R\in(a,\infty)$, the remaining portion. In addition to these mixed boundary conditions, one imposes the regularity requirement for potential problems involving 3D regions; the potential should either vanish or at most be a prescribed constant value as $(R,Z)\rightarrow\infty$. Considering this regularity condition first, it requires that

$$B(\xi)\equiv 0 \tag{48}$$

Next, considering the remaining solution of Eq. (47) and the mixed boundary conditions Eqs. (45) and (46), the following system of dual integral equations is obtained

$$\int_0^\infty A(\xi)J_0(\xi R)d\xi=\varphi_0;\qquad R\in(0,a) \tag{49}$$

$$\int_0^\infty \xi A(\xi)J_0(\xi R)d\xi=0;\qquad R\in(a,\infty) \tag{50}$$

The solution of this system of dual integral equations is discussed in standard and recent books on mathematical methods (see Sneddon 1951; Selvadurai 2000b). The methodology involves the introduction of a finite Fourier transform representation of $A(\xi)$ in terms of a new function $\Omega(t)$ in the form

$$A(\xi)=\int_0^a \Omega(t)\cos(\xi t)dt \tag{51}$$

such that the integral Eq. (50) is identically satisfied, and the integral Eq. (49) is equivalent to an Abel integral equation for the unknown function $\Omega(t)$ of the form

$$\int_0^R \frac{\Omega(t)dt}{\sqrt{R^2-t^2}}=\varphi_0 \tag{52}$$

which has a solution of the form

$$\Omega(t)=\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{\varphi_0 R dR}{\sqrt{t^2-R^2}}=\frac{2\varphi_0}{\pi} \tag{53}$$

This completes the formal solution of the mixed boundary value problem. The complete integral expression for the potential can be written in terms of the spatial variables (r,z) in the form

$$\varphi(r,z)=\frac{2\varphi_0}{\pi}\int_0^\infty \frac{\sin(\xi a)}{\xi}\exp\left(-\xi z\sqrt{\frac{k_{rr}}{k_{zz}}}\right)J_0(\xi r)d\xi \tag{54}$$

The fluid velocity components can be obtained by making use of the result [Eq. (3)]. The velocity component of primary interest for calculating the flow into the half-space region from the circular area involves the velocity component $v_z(r,z)$, which is given by the integral

$$v_z(r,z)=\frac{2\varphi_0\sqrt{k_{rr}k_{zz}}}{\pi}\int_0^\infty \sin(\xi a)\exp\left(-\xi z\sqrt{\frac{k_{rr}}{k_{zz}}}\right)J_0(\xi r)d\xi \tag{55}$$

Although it is not possible to obtain a simple expression for the general form of this velocity component, one can evaluate this integral on the boundary of the half-plane $z=0$. From Erdelyi et al. (1954)

$$v_z(r,0)=\frac{2\varphi_0\sqrt{k_{rr}k_{zz}}}{\pi\sqrt{a^2-r^2}};\qquad r\in(0,a) \tag{56}$$

$$v_z(r,0)=0;\qquad r\in(a,\infty) \tag{57}$$

As is evident, the velocity field is *singular* at the boundary of the circular region with constant potential. This is primarily due to the mixed nature of the boundary conditions on the plane $z=0$. The quantity of flow into the region is, however, *finite*. We can evaluate this by considering the total flux, or the volume rate per unit time, from the result

$$Q=2\int\int_S \mathbf{v}\cdot\mathbf{n}dS \tag{58}$$

where $\mathbf{v}$=velocity vector; and $\mathbf{n}$=unit normal to the surface S over which the flow takes place. For the surface of the half-space region, $S=S_D\cup S_N$ where the subregions apply to the surface of the half-space on which Dirichlet and Neumann boundary conditions are prescribed. The factor of *two* in Eq. (58) accounts for fluid flow through the regions S; $r\in(0,a)$; and $z=0^+,0^-$, which correspond to both surfaces of the disc-shaped entry point.

By considering the results of Eqs. (56) and (57), and noting that on the boundary plane $\mathbf{n}=(0\mathbf{i}_r+0\mathbf{i}_\theta+1\mathbf{i}_z)$, one obtains from Eq. (58)

$$Q=\frac{4\varphi_0\sqrt{k_{rr}k_{zz}}}{\pi}\int_0^{2\pi}\int_0^a \frac{r dr d\theta}{\sqrt{a^2-r^2}} \tag{59}$$

By evaluating Eq. (59), one obtains

$$Q=8a\varphi_0\sqrt{k_{rr}k_{zz}} \tag{60}$$

Again, one can define an intake shape factor F for the disc-shaped entry point located in a hydraulically transversely isotropic porous medium such that

$$Q=\hat{F}k_{eq}\varphi_0 \tag{61}$$

where k_{eq} is an implied "effective hydraulic conductivity" for the hydraulically transversely isotropic porous medium defined previously by Eq. (32). Then

$$\hat{F}=8a \tag{62}$$

Alternatively, if the porous medium is hydraulically isotropic, $k_{rr}=k_{zz}=k$, then one recovers the result given in the compilation by Hvorslev (1951), which can also be deduced by mathematical analogy from results of electrostatic potential theory (Jeans 1925).

Concluding Remarks

The transient process of either water level rise or fall in a cased borehole is an efficient method for the in situ determination of the hydraulic conductivity characteristics of geomaterials with relatively low hydraulic conductivity (in comparison to coarse grained materials such as sands and gravels). The rate of water level change in the casing is governed by the entry point geometry at the base of the casing. When the geomaterials being tested have isotropic hydraulic conductivity characteristics, the intake shape factor depends only on the dimensions of the intake. This feature lends itself to the development of a number of specific results for intake shape factors of practical interest. The situation

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changes when casing tests are conducted in geomaterials with directional hydraulic conductivity characteristics. First, casing tests cannot provide sufficient information to conveniently determine all six components of the hydraulic conductivity tensor. The cased borehole method can, however, provide a convenient technique for the determination of hydraulic conductivity characteristics of geomaterials that are considered to be transversely isotropic. Many geomaterials that are formed as a result of a layering process can be expected to display such transversely isotropic characteristics in their mechanical and hydraulic properties. In conventional terms, the hydraulic conductivities of the transversely isotropic material can be identified as those in the horizontal (k_h) and vertical (k_v) directions. For example, the elementary calculations for the equivalent vertical hydraulic conductivity k_v of n layers is given by the weighted harmonic mean of the hydraulic conductivities of the individual layers

$$k_v=\frac{\Sigma_{i=1}^{n} t_i}{\Sigma_{i=1}^{n}(t_i/k_i)} \tag{63}$$

where t_i and k_i=thickness and hydraulic conductivity of the ith layer. Similarly, the equivalent horizontal hydraulic conductivity k_h is given by

$$k_h=\frac{\Sigma_{i=1}^{n} k_i t_i}{\Sigma_{i=1}^{n} t_i} \tag{64}$$

Consider a sedimentary sequence of interbedded silty sand and unweathered marine clay of equal thickness and with isotropic individual hydraulic conductivities of 10^{-6} m/s and 10^{-12} m/s, respectively. The equivalent vertical hydraulic conductivity of the entire sequence will be approximately 2×10^{-12} m/s, and the equivalent hydraulic conductivity in the horizontal plane will be approximately 0.5×10^{-6} m/s. As is evident, significant anisotropy in the scale of a representative volume element (RVE), with dimensions significantly larger than the layer thickness, can readily materialize even with plausible choices of hydraulic conductivities of the individual layers.

This paper discusses the procedure for determining the intake shape factors for flat disc-shaped and spherical entry points located in a porous medium with transversely isotropic hydraulic conductivity characteristics. It is found that these intake shape factors can be evaluated in exact closed forms. It is also noted that the effective hydraulic conductivity of the transversely isotropic medium, defined by $\sqrt{k_h k_v}$ can be directly determined from casing tests where the intake can be formed to correspond to the shape of a disc. The hydraulic conductivity estimated by performing a casing test in a transversely isotropic porous medium, where the entry point has a spherical shape, accounts for both the effective hydraulic conductivity and the conductivity ratio k_h/k_v. Therefore, in order to determine the hydraulic conductivities separately, two casing experiments need to be performed with different intake shape characteristics. The mathematical solutions presented in this paper can be conveniently used for this purpose. Mathematics and modelling aside, it is important to recognize that the hydraulic conductivities of porous geomaterials are difficult to determine even under highly controlled laboratory conditions. Field estimation of directionally dependent hydraulic conductivities also requires careful subsurface investigations to ascertain the uniformity and homogeneity of the region in which the entry point is located.

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