Contaminant migration from an axisymmetric source in a porous medium

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[1] This paper examines the problem of the nonreactive advective transport of a contaminant that is introduced at the boundary of a three-dimensional cavity contained in a fluid-saturated nondeformable porous medium of infinite extent. The advective Darcy flow is caused by a hydraulic potential maintained at a constant value at the boundary of the three-dimensional cavity. In order to develop a generalized solution to the problem the three-dimensional cavity region is modeled as having either a prolate or an oblate shape. Analytical results are developed for the time- and space-dependent distribution of contaminant concentration in the porous medium, which can also exhibit natural attenuation. The exact closed-form analytical results are also capable of providing solutions to advective transport problems related to spherical, flat disc-shaped and elongated needle-shaped cavities. *INDEX TERMS:* 1719 History of Geophysics: Hydrology; 1829 Hydrology: Groundwater hydrology; 1832 Hydrology: Groundwater transport; *KEYWORDS:* advective transport, transport from a cavity, spheroidal cavity, analytical results, contaminant migration from cavity, closed form solutions

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1. Introduction

[2] The transport of chemicals and other contaminants in porous geological media is a topic of fundamental importance in the general area of earth sciences and of particular interest to geoenvironmental engineering. The basic mechanisms of transport range from advective transport, which depends on an advective flow velocity, to diffusive transport that depends on a concentration gradient. Although the fundamental processes governing these basic modes of transport can be highly nonlinear and dependent on the microstructural morphology and chemistry of both the contaminant and the porous medium, the linear theories associated with these basic transport processes provide useful first approximations for the study of both advective and diffusive processes. The advective transport is related to the flow velocity, which is governed by Darcy flow whereas the diffusive transport processes are governed by Fick's law. The extent to which one process or the other dominates depends primarily on the flow characteristics as opposed to the diffusive transport characteristics of the system, which consists of the porous medium, the pore fluid and the contaminant that is being transported. There are, however, situations, characterized by a Peclet number greater than unity, where the advective flow velocities are sufficiently large enough to transport the contaminant solely by advective means (The Peclet number, $Pe = VL/D$, where V is a characteristic velocity, L is a characteristic length and D is a diffusion coefficient associated with the problem). This paper investigates the three-dimensional advective transport problems resulting from such a situation. The three-dimensional nature of the problem stems from the spheroidal cavities with prolate and oblate shapes where the boundaries are subjected to a hydraulic potential to induce steady flow through the porous medium; with this steady flow in place, the boundaries of the spheroidal cavities are maintained at a constant concentration to induce the advective transport of the contaminant. It is shown that the potential flow resulting from the spheroidal cavities that are maintained at a constant potential can be evaluated in exact closed form by adopting solutions of Laplace's equation referred to a system of spheroidal curvilinear coordinates. The advective transport problem is then solved using a Laplace transform technique. It is shown that owing to the closed form nature of the potential flow problem, the solutions for the advective transport problem can also be obtained in explicit form. The particular advantage of the results related to this study are that specific solutions to advective transport problems dealing with elongated needle-shaped cavities, spherical cavities and flat disc-shaped cavities can be obtained simply as limiting cases of the generalized solutions. The analytical solutions have a further important function as benchmarks for the calibration of computational procedures that model advective transport in porous media.

2. Governing Equations

[3] The equation governing advective transport of either contaminants or chemical species in a porous medium has been presented in a number of textbooks dealing with both mathematical modeling and engineering applications of geoenvironmental transport processes [*Bear*, 1972; *Greenkorn*, 1983; *Bear and Verruijt*, 1990; *Philips*, 1991; *Bear and Bachmat*, 1992; *Appelo and Postma*, 1993; *Banks*, 1994; *Vukovich*, 1997; *Domenico and Schwartz*, 1998; *Nield and*

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Bejan, 1999; *Zijl and Nawalany*, 2000]. Several studies also point to the similarity between the partial differential equation governing advective transport of a chemical in a porous medium and partial differential equations governing a host of other problems, including flow of vehicular traffic, movement of waves in shallow water, movement of charged particles such as electrons, gas dynamics, biological processes and marine ecology, salt movement in oceans, mechanics of surging glaciers, meteorology, migration of fine particles, bacteria and viruses in porous media, resin migration during injection molding and the physics of heat exchangers [*Haight*, 1963; *Whitham*, 1976; *Gill*, 1982; *Bennett and Kloeden*, 1982; *Carroll*, 1985; *McDowell-Boyer et al.*, 1986; *Edelstein-Keshet*, 1988; *Shutie*, 1992; *Greenberg and Shyong*, 1993; *Chung*, 1993; *Segol*, 1994; *Sun*, 1996; *Khilar and Folger*, 1998; *Ingham and Pop*, 1998; *David*, 1998; *Massel*, 1999; *Panfilov*, 2000; *Billi and Farina*, 2000; *Selvadurai*, 2000a; *Bird et al.*, 2002] (the work by *McDowell-Boyer et al.* [1986] also contains an excellent review of particle transport in porous media). It is desirable, for completeness, to briefly document the derivation of the basic partial differential equation governing advective flow of a contaminant through a nondeformable porous medium [see, e.g., *Selvadurai*, 2000a]. Consider an arbitrary region V of a porous medium with surface S. The porosity of the medium, defined by the ratio of the volume of pore space to the total volume, is denoted by n^*. The concentration of the contaminant per unit volume of the fluid contained in the pore space is defined by $\widetilde{\mathbf{C}}(\mathbf{x}, t)$, where $\mathbf{x}$ is a position vector and t is time. We can also define a contaminant concentration $C(\mathbf{x}, t)$, which is measured per unit total volume of the porous medium. At any given location at any time

$$C(\mathbf{x}, t) = n^* \widetilde{C}(\mathbf{x}, t) \tag{1}$$

The velocity vector $\widetilde{\mathbf{v}}(\mathbf{x}, t)$ defines the advective velocity in the pore space. We can also define an averaged advective velocity $\mathbf{v}(\mathbf{x}, t)$, taken over the entire cross section over which flow takes place, such that

$$\mathbf{v}(\mathbf{x}, t) = n^* \widetilde{\mathbf{v}}(\mathbf{x}, t) \tag{2}$$

The assumption here is that the area porosity is identical to the volume porosity. There are no assurances that this is indeed the case for all porous media [*Drew and Passman*, 1999]. For a porous medium, which displays a relatively isotropic pore structure, the two measures of porosity are assumed to be approximately the same. The flux is defined as the mass of the contaminant either entering or leaving a unit total area in unit time. The flux vector, or mass being transported by advection, per unit total area per unit time is given by

$$\mathbf{F_a} = n^* \mathbf{v} \widetilde{C}(\mathbf{x}, t) = \mathbf{v} C(\mathbf{x}, t) \tag{3}$$

The total mass of the contaminant transported into the volume V of surface S is given by

$$m_i = -\int\!\!\int_S \mathbf{F}_a \mathbf{n}\, dS \tag{4}$$

where $\mathbf{n}$ is the outward unit normal to the elemental surface area dS. Applying the divergence theorem, we can rewrite (4) as

$$m_i = -\int\!\!\int\!\!\int_V \nabla.\{\mathbf{v}C\} dV \tag{5}$$

The contaminant is assumed to accumulate in the fluid within the void space of the porous medium. The rate of accumulation of the contaminant is given by

$$m_a = \frac{d}{dt}\int\!\!\int\!\!\int_V n^* \widetilde{C} dV = \int\!\!\int\!\!\int_V \frac{\partial C}{\partial t} dV \tag{6}$$

The rate of production/loss of the contaminant due to either generation (+) or decay (−) processes within the porous medium is given by

$$m_p = \pm \int\!\!\int\!\!\int_V n^* \xi \widetilde{C}\, dV = \pm \int\!\!\int\!\!\int_V \xi C\, dV \tag{7}$$

where ξ is a generation/decay factor and measured per unit time. For conservation of mass of the contaminant we require $m_a = m_i + m_p$, which when combined with the Dubois-Reymond Lemma for the existence of a local equation, gives the partial differential equation governing advective transport of a chemical species as follows:

$$\frac{\partial C}{\partial t} + \nabla.(\mathbf{v}C) = \pm \xi C \tag{8}$$

The representation in terms of the volume-averaged contaminant concentration becomes more convenient and appropriate when considering integral representations applicable to the entire volume and the entire surface of a control volume.

[4] In this paper we restrict attention to flow fields characterized by Darcy flow. The spatially averaged flow velocities are governed by Darcy's law, which for a hydraulically isotropic medium is given by

$$\mathbf{v} = -k\nabla\phi \tag{9}$$

where k is the hydraulic conductivity and $\phi(\mathbf{x})$ is the reduced Bernoulli potential governing fluid flow through the porous medium, which includes only the datum potential and the pressure potential. For incompressible flow of the pore fluid and for nondeformability of the porous solid

$$\nabla.\mathbf{v} = 0 \tag{10}$$

and the partial differential equation governing the flow potential is Laplace's equation

$$\nabla^2\phi(\mathbf{x}) = 0 \tag{11}$$

The partial differential equations governing the advective transport problem are therefore (8) and (11) respectively for the time-dependent chemical concentration $C(\mathbf{x}, t)$ and the flow potential $\phi(\mathbf{x})$. These partial differential equations represent a weakly coupled system, which has a second-order elliptic equation for $\phi(\mathbf{x})$ and a first-order hyperbolic equation for $C(\mathbf{x}, t)$. The initial boundary value problem governing $C(\mathbf{x}, t)$ is subject to both an initial condition and, in the case of the advective transport problem discussed here, a Dirichlet type boundary condition on a given surface and the boundary value problem governing $\phi(\mathbf{x})$ is subject to Dirichlet type boundary conditions on the subregion S_D of S. The uniqueness of solution to the potential flow problem is well established [see, e.g., *Selvadurai*, 2000a], and the uniqueness of solution for the advective transport problem with Dirichlet boundary conditions can be proved in a concise manner (see Appendix A) and it is noted that uniqueness is assured for situations where only Dirichlet boundary conditions are prescribed for $C(\mathbf{x}, t)$. When Dirichlet and Neumann boundary conditions are prescribed for $C(\mathbf{x}, t)$ on subsets of S, uniqueness can be established for situations where the regions over which these boundary conditions are specified coincide with the regions over which Dirichlet and homogeneous Neumann boundary conditions are prescribed for $\phi(\mathbf{x})$ (A. P. S. Selvadurai, Some remarks on the uniqueness theorem for the classical advection-diffusion equation, manuscript in preparation, 2003).

3. Advective Transport From a Prolate Spheroidal Cavity

[5] We first consider the problem of a nondeformable porous medium of infinite extent, which is bounded internally by a cavity in the shape of a prolate spheroid where the length of the major axis is $2a_p$ and the length of the minor axis is $2b_p$ (Figure 1). The steady fluid flow into the porous medium is caused by a flow potential, which is constant at the boundary of the prolate cavity and reduces uniformly to zero at large distances from the cavity. When steady flow is maintained, the boundary of the cavity is subjected to a contaminant concentration $C_0F(t)$, where $F(t)$ is an arbitrary function of time. In view of the spheroidal nature of the cavity boundary, it is convenient to adopt a spheroidal coordinate formulation of the boundary value problem governing the hydraulic potential $\phi(\mathbf{x})$ and the initial boundary value problem governing the contaminant concentration $C(\mathbf{x}, t)$. We introduce a system of prolate spheroidal curvilinear coordinates (α, β, γ) defined by

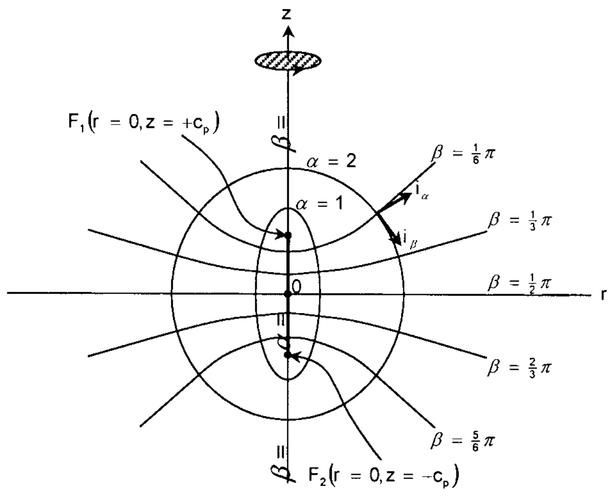
$$r = c_p \sinh\alpha \sin\beta \quad ; \quad z = c_p \cosh\alpha \cos\beta \tag{12}$$

such that the parametric surfaces α = const, say α_0, $\beta = \beta_0$, and $\gamma = \gamma_0$, form a triple orthogonal confocal family of prolate spheroids, hyperboloids of two sheets and meridional half planes respectively, and (r, θ, z) refers to the cylindrical polar coordinate system. Every point in space is represented by restricting the ranges of the prolate spheroidal coordinates (α, β, γ) as follows:

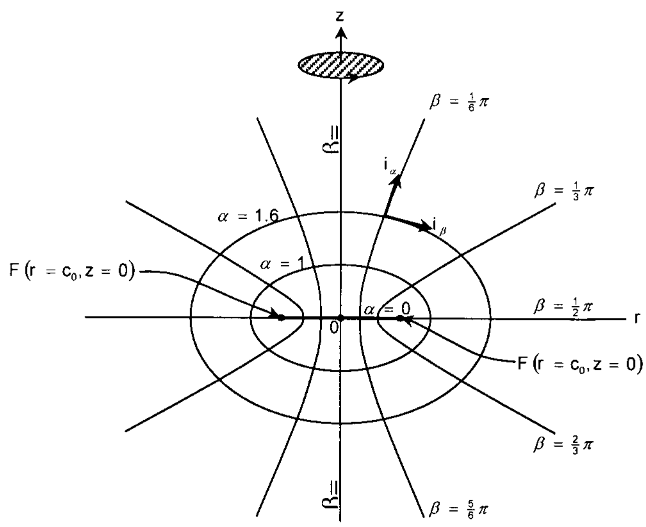
$$0 \le \alpha < \infty \quad ; \quad 0 \le \beta \le \pi \quad ; \quad 0 \le \gamma \le 2\pi \tag{13}$$

We consider the expression for a differential arc length (ds) given by

$$(ds)^2 = \left(\frac{d\alpha}{h_1^p}\right)^2 + \left(\frac{d\beta}{h_2^p}\right)^2 + \left(\frac{d\gamma}{h_3^p}\right)^2 \tag{14}$$



(a). The prolate spheroidal coordinate system



(b). The oblate spheroidal coordinate system

Figure 1. The spheroidal coordinate systems.

where the metric or local scale coefficients are given by [see, e.g., *Selvadurai*, 2000a]

$$h_1^p = h_2^p = \left[c_p^2\left(\sinh^2\alpha + \sin^2\beta\right)\right]^{-1/2} = h_p \tag{15}$$

$$h_3^p = \left(c_p \sinh\alpha \sin\beta\right)^{-1} \tag{16}$$

The focal distance c_p can be expressed in terms of the dimensions of the semimajor axis and the equatorial radius of the prolate spheroid conforming to the internal boundary of the porous medium, which is assumed to be defined by $\alpha = \alpha_0$, such that

$$a_p^2 = c_p^2 \cosh^2\alpha_0 \quad ; \quad b_p^2 = c_p^2 \sinh^2\alpha_0 \tag{17}$$

and

$$c_p = \sqrt{a_p^2 - b_p^2} \tag{18}$$

We first consider the axisymmetric flow problem where the axis $\beta = 0$ coincides with the z-axis, the axis of symmetry of the prolate spheroidal cavity. For axisymmetric problems, the hydraulic potential is independent of the azimuthal coordinate γ and in terms of the curvilinear coordinates (α,β), Laplace's equation (11) takes the form

$$\nabla^2\varphi(\alpha,\beta) = h_p^2\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \coth\alpha\frac{\partial}{\partial\alpha} + \cot\beta\frac{\partial}{\partial\beta}\right)\varphi(\alpha,\beta) = 0 \tag{19}$$

The partial differential equation is to be solved, subject to the boundary conditions

$$\varphi(\alpha,\beta) = \varphi_0 \quad \text{on} \quad \alpha = \alpha_0 \tag{20}$$

where $\alpha = \alpha_0$ corresponds to the boundary of the cavity and

$$\varphi(\alpha,\beta) \to 0 \quad \text{as} \quad \alpha \to \infty \tag{21}$$

For the solution of the boundary value problem we seek Lame' products associated with spheroidal coordinate systems [see, e.g., *Hobson*, 1931; *Morse and Feshbach*, 1953; *Selvadurai*, 1976; *Moon and Spencer*, 1988], the general expression for which can be obtained in the form

$$\varphi(\alpha,\beta) = \left[P_n^{(m)}(\cos\beta)\ or\ Q_n^{(m)}(\cos\beta)\right] \cdot \left[P_n^{(m)}(\cosh\alpha)\ or\ Q_n^{(m)}(\cosh\alpha)\right] \tag{22}$$

with m, n = 0, 1, 2, 3,..., and where $P_n^{(m)}$ and $Q_n^{(m)}$ are associated Legendre functions of the first and second kind [*Hobson*, 1931; *Morse and Feshbach*, 1953]. Considering the boundary condition (20) and the regularity condition (21) applicable to the flow problem, we need to select solutions of (19) for which $\varphi(\alpha, \beta) = \varphi(\alpha)$. The single solution that satisfies the regularity condition (21), can be obtained by selecting $m = n = 0$ and neglecting the remaining terms of the sequence (22); we have

$$\varphi(\alpha) = \frac{A_0}{2}\ln(\xi) \tag{23}$$

where A_0 is an arbitrary constant and

$$\xi = \left(\frac{\cosh\alpha + 1}{\cosh\alpha - 1}\right) \tag{24}$$

The arbitrary constant A_0 can be obtained by considering the boundary condition (20); this gives

$$\varphi(\alpha) = \frac{\varphi_0}{\ln\xi_0}\ln\xi \tag{25}$$

where

$$\xi_0 = \xi(\alpha_0) \tag{26}$$

The solution to the potential problem is now formally complete, in the sense that the velocity vector $\mathbf{v}(\alpha, \beta, \gamma)$ can be obtained in explicit closed form by using (25) in (9); in view of the axial symmetry we obtain

$$\mathbf{v}(\alpha,\beta) = \frac{2k\phi_0}{c_p\ln\xi_0\sinh\alpha\sqrt{\sinh^2\alpha + \sin^2\beta}}\mathbf{i}_\alpha \tag{27}$$

and note that all other components of the velocity vector are identically zero. Considering (27) and in view of the divergence free constraint (10) imposed by the incompressibility of the fluid, the partial differential equation (8) for advective transport of the contaminant now reduces to the form

$$\frac{\partial C}{\partial t} + \frac{2k\phi_0}{c_p^2\sinh\alpha\left(\sinh^2\alpha + \sin^2\beta\right)\ln\xi_0}\frac{\partial C}{\partial\alpha} = \pm\xi C \tag{28}$$

The initial condition and the boundary condition applicable to the problem are as follows: we assume that the porous medium is initially free of the contaminant and that the entire boundary of the prolate spheroidal cavity, $\alpha = \alpha_0$ is subjected to a contaminant loading with an arbitrary time variation. These give

$$C(\alpha,\beta,0) = 0 \quad ; \quad C(\alpha,\beta,t) = C_0F(t) \tag{29}$$

where C_0 is a constant contaminant concentration and $F(t)$ is an arbitrary function of time. While there exists a number of analytical and computational methods for the solution of first-order partial differential equations of the type (28) [see, e.g., *Lapidus and Pinder*, 1982; *Ninomiya and Onishi*, 1991; *Hill*, 1992; *Banks*, 1994; *Melikyan*, 1998], in the context of the present problem it is convenient and sufficient to consider a method of solution that is based on the application of Laplace transforms. We define the Laplace transform of $C(\alpha, \beta, t)$ with respect to the time variable as [see, e.g., *Sneddon*, 1972; *Watson*, 1981]

$$\bar{c}(\alpha,\beta,s) = L\{C(\alpha,\beta,t)\} = \int_0^\infty \exp(-st)C(\alpha,\beta,t)dt \tag{30}$$

and since the initial contaminant concentration in the porous medium is zero (see e.g., (29)), we have

$$L\left\{\frac{\partial C}{\partial t}\right\} = s\bar{c}(\alpha,\beta,s) \tag{31}$$

Applying the Laplace transform to (28) we obtain a first-order ordinary differential equation for the transformed dependent variable $\bar{c}(\alpha, \beta, s)$; further restricting attention to a natural attenuation processes in the porous medium, we obtain the solution as follows:

$$\bar{c}(\alpha,\beta,s) = C_0f(s)\exp\left[-(s+\xi)\Omega_p(\alpha,\beta,\lambda)\right] \tag{32}$$

where

$$\Omega_p(\alpha,\beta,\lambda) = \frac{c_p^2\ln\xi_0}{6k\varphi_0} \cdot \left[\cosh^3\alpha - \cosh^3\alpha_0 + 3\cos^2\beta\{\cosh\alpha - \cosh\alpha_0\}\right] \tag{33}$$

and $f(s)$ denotes the Laplace transform of $F(t)$. The exact form of $F(t)$ is not important to the discussion that follows; it is only sufficient to require that the Laplace transform of $F(t)$ exists.

[6] We also note the following relationships, which express the trigonometric and hyperbolic functions in (33) in terms of the cylindrical polar coordinates (r, z) and the geometric aspect ratio of the prolate spheroid:

$$\cosh\alpha = \left[\frac{\sqrt{\rho_p^2 + \left(\eta_p + \sqrt{1-\lambda^2}\right)^2} + \sqrt{\rho_p^2 + \left(\eta_p - \sqrt{1-\lambda^2}\right)^2}}{2\sqrt{1-\lambda^2}}\right] \tag{34}$$

$$\cos\beta = \left[\frac{\sqrt{\rho_p^2 + \left(\eta_p + \sqrt{1-\lambda^2}\right)^2} - \sqrt{\rho_p^2 + \left(\eta_p - \sqrt{1-\lambda^2}\right)^2}}{2\sqrt{1-\lambda^2}}\right] \tag{35}$$

$$\cosh\alpha_0 = \frac{1}{\sqrt{1-\lambda^2}} \quad ; \quad \ln\xi_0 = \ln\left[\frac{1+\sqrt{1-\lambda^2}}{1-\sqrt{1-\lambda^2}}\right] \tag{36}$$

$$\lambda = \frac{b_p}{a_p} < 1 \quad ; \quad \rho_p = \frac{r}{a_p} = \quad ; \quad \eta_p = \frac{z}{a_p} \tag{37}$$

The analysis of the advective transport problem is now reduced to the inversion of the expression (32) for $\bar{c}(\alpha, \beta, s)$; the formal result can be written as follows:

$$\frac{C(\alpha,\beta,t)}{C_0} = L^{-1}\left\{f(s)\exp\left[-(s+\xi)\Omega_p(\alpha,\beta,\lambda)\right\}\right. \tag{38}$$

where L^{-1} refers to the inverse Laplace transform. The general result (38) for the advective transport in a porous medium of infinite extent due to contaminant loading applied at the boundary of the prolate spheroidal cavity can be evaluated by specifying plausible time variations defined by $F(t)$, along with the use of the convolution theorem for Laplace transforms defined by

$$L^{-1}\{f(s)g(s)\} = F * G = \int_0^t F(u)G(t-u)du. \tag{39}$$

Alternatively, the inversion can be carried out by appeal to *Tables of Integral Transforms* [see, e.g., *Erdelyi et al.*, 1954] or through the use of symbolic mathematical manipulation software such as MAPLE® or MATHEMATICA® [see, e.g., *Wolfram*, 1999] or through the use of numerical inversion techniques. As an illustrative example, consider the case where the boundary of the prolate spheroidal cavity is subjected to a contaminant concentration, which is in the form of a Heaviside step function, i.e.,

$$F(t) = \begin{cases} 0 & ; \quad -\infty > t \geq 0 \\ 1 & ; \quad 0 \leq t < \infty \end{cases} \tag{40}$$

In this case the result (38) gives

$$\frac{C(\alpha,\beta,t)}{C_0} = \exp\left[-\xi\Omega_p(\alpha,\beta,\lambda)\right]H\left[t-\Omega_p(\alpha,\beta,\lambda)\right] \tag{41}$$

where $H[t - \Omega_p(\alpha, \beta, \lambda)]$ is the Heaviside step function with the time shift $\Omega_p(\alpha, \beta, \lambda)$. The relevant solutions for situations involving no natural attenuation of the chemical concentration during its migration through the porous medium can be obtained by setting $\xi = 0$, in (41).

4. Advective Transport From an Oblate Spheroidal Cavity

[7] The developments presented in the preceding section can be extended, by considering a suitable coordinate transformation, to cover situations where the cavity region can be modeled as an oblate spheroidal cavity with the length of the major axis as $2b_o$ and the length of the minor axis as $2a_o$ (Figure 1b). For completeness, however, we shall present a brief development of the problem in relation to a system of oblate spheroidal coordinates defined by

$$r = c_o\cosh\alpha\sin\beta \quad ; \quad z = c_o\sinh\alpha\cos\beta \tag{42}$$

Again, each point in space is obtained once by limiting the ranges of the oblate spheroidal coordinates to (α, β, γ) to those given by (12).

[8] The metric coefficients are

$$h_1^o = h_2^o = \left[c_o^2\left(\cosh^2\alpha - \sin^2\beta\right)\right]^{-1/2} = h_o \tag{43}$$

$$h_3^o = (c_o\cosh\alpha\sin\beta)^{-1} \tag{44}$$

$$c_o^2 = b_o^2 - a_o^2 \tag{45}$$

and the equivalent form of Laplace's equation (19), expressed in oblate spheroidal coordinates takes the form

$$\nabla^2\varphi(\alpha,\beta) = h_o^2\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2} + \tanh\alpha\frac{\partial}{\partial\alpha} + \cot\beta\frac{\partial}{\partial\beta}\right)\varphi(\alpha,\beta) = 0 \tag{46}$$

The boundary and regularity conditions (20) and (21), respectively, applicable to the problem of fluid flow in a porous region bounded internally by a prolate cavity also apply to the oblate spheroidal region. The relevant solution of (46), [see, e.g., *Hobson*, 1931; *Morse and Feshbach*, 1953], which also satisfies these boundary conditions, takes the form

$$\varphi(\alpha,\beta) = \varphi_0\frac{\cot^{-1}(\sinh\alpha)}{\cot^{-1}(\sinh\alpha_0)} \tag{47}$$

The fluid velocity vector $\mathbf{v}(\alpha,\beta)$ in the porous medium is given by

$$\mathbf{v}(\alpha,\beta) = \frac{k\phi_0}{c_p\cosh\alpha\cot^{-1}(\sinh\alpha_0)\sqrt{\cosh^2\alpha - \sin^2\beta}}\mathbf{i}_\alpha \tag{48}$$

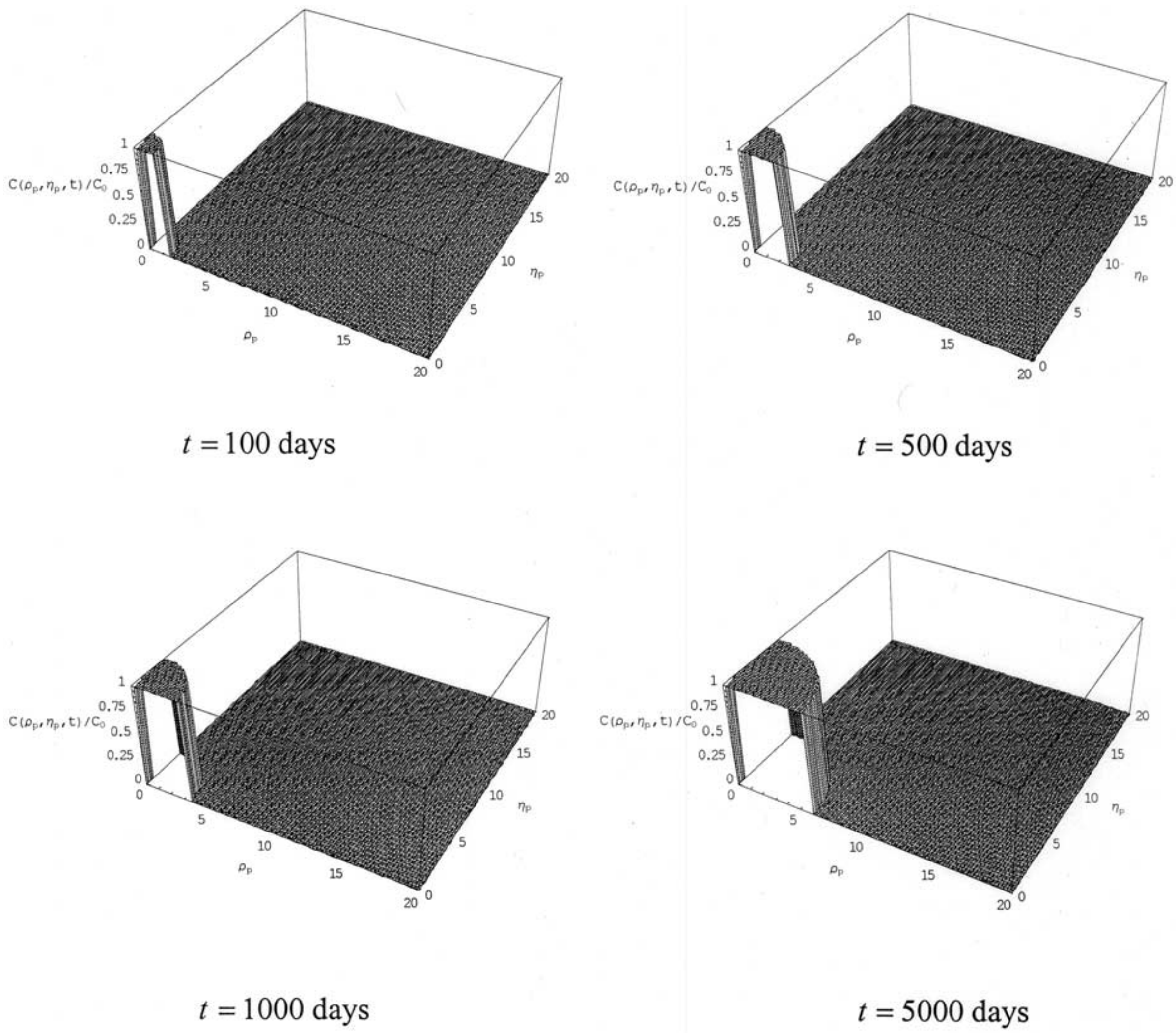


Figure 2. Advective transport of the contaminant from a prolate spheroidal cavity with $b_p/a_p = 1/8$ located in a porous medium with attenuation coefficient $\xi = 0$.

The relevant partial differential equation governing the advective transport of the contaminant from the oblate spheroidal cavity is given by

$$\frac{\partial C}{\partial t} + \frac{k\phi_0}{c_o^2 \cosh\alpha \cot^{-1}(\sinh\alpha_0)\left(\cosh^2\alpha - \sin^2\beta\right)} \frac{\partial C}{\partial \alpha} = \pm\xi C \quad (49)$$

The boundary and initial conditions governing the problem are given by (28) and the solution to the problem is given by

$$\frac{C(\alpha,\beta,t)}{C_0} = L^{-1}\{f(s)\exp[-(s+\xi)\Omega_o(\alpha,\beta,\mu)\} \quad (50)$$

where

$$\Omega_o(\alpha,\beta,\mu) = \frac{b_o^2(1-\mu^2)\cot^{-1}(\sinh\alpha_0)}{3k\varphi_0}\Big[\sinh\alpha\cosh^2\alpha - \sinh\alpha_0\cosh^2\alpha_0 + \left(2 - 3\sin^2\beta\right)(\sinh\alpha - \sinh\alpha_0)\Big] \quad (51)$$

and

$$\cosh\alpha = \left[\frac{\sqrt{\eta_o^2 + \left(\rho_o + \sqrt{1-\mu^2}\right)^2} + \sqrt{\eta_o^2 + \left(\rho_o - \sqrt{1-\mu^2}\right)^2}}{2\sqrt{1-\mu^2}}\right] \quad (52)$$

$$\sin\beta = \left[\frac{\sqrt{\eta_o^2 + \left(\rho_o + \sqrt{1-\mu^2}\right)^2} - \sqrt{\eta_o^2 + \left(\rho_o - \sqrt{1-\mu^2}\right)^2}}{2\sqrt{1-\mu^2}}\right] \quad (53)$$

$$\cosh\alpha_0 = \frac{1}{\sqrt{1-\mu^2}} \quad ; \quad \sinh\alpha_0 = \frac{\mu}{\sqrt{1-\mu^2}} \quad (54)$$

$$\mu = \frac{a_o}{b_o} < 1 \quad ; \quad \rho_o = \frac{r}{b_o} \quad ; \quad \eta_o = \frac{z}{b_o} \quad (55)$$

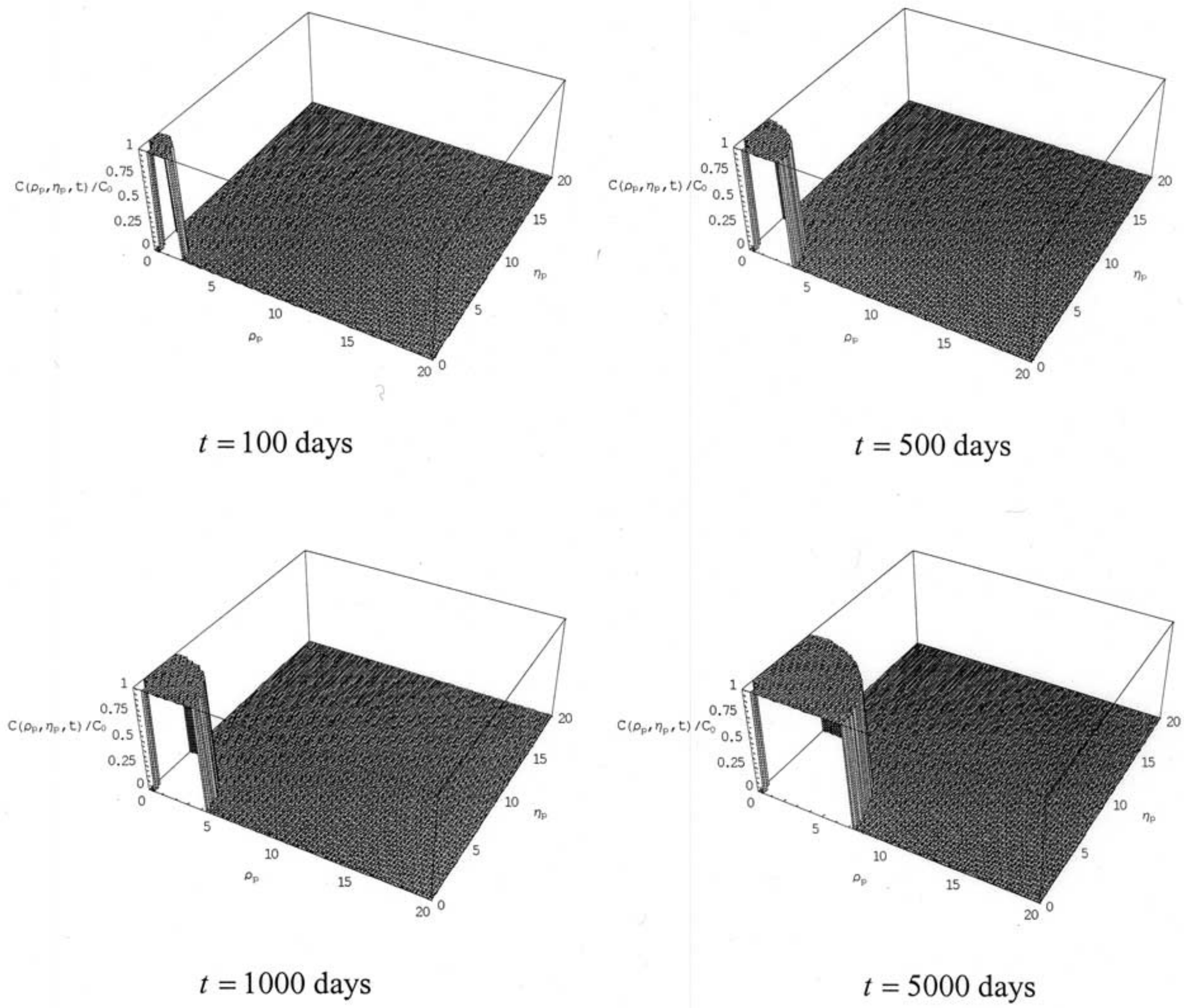


Figure 3. Advective transport of the contaminant from a prolate spheroidal cavity with b_p/a_p = 1/2 located in a porous medium with attenuation coefficient ξ = 0.

Therefore when the exact form of the function $F(t)$ is known, the result (50) can be evaluated through Laplace transform inversion. Again, as an illustrative example, consider the case where the boundary of the prolate spheroidal cavity is subjected to a contaminant concentration, which is in the form of a Heaviside step function as defined by (40).

[9] In this case the result (50) gives

$$\frac{C(\alpha,\beta,t)}{C_0} = \exp[-\xi\Omega_o(\alpha,\beta,\mu)]H[t-\Omega_o(\alpha,\beta,\mu)] \qquad (56)$$

Results for other forms of time-dependent variations in the contaminant dosing at the boundary of both the oblate and prolate spheroidal cavity regions can be obtained by using an appropriate Laplace transform inversion technique.

5. Numerical Results

[10] The expressions for the time-dependent variation of the advective flow-induced distribution of the contaminant concentration in the porous medium, due to the spheroidal sources are in forms amenable to evaluation either numerically or in exact closed form, depending upon the form of the function $F(t)$ that defines the time history of the boundary concentration. The numerical results presented in this section are therefore restricted to the special case where the boundary chemical concentration is represented by a Heaviside step function of time. The resulting closed form analytical solution represents a useful result that can be used to estimate the maximum contaminant concentration that can be encountered within the porous medium for specified values of the attenuation parameter ξ and the parameters $a_p^2/6k\varphi_0$ and $b_o^2/3k\varphi_0$. Prior to presentation of specific numerical results, it is instructive to examine the limiting cases that are applicable to spheroidal cavity geometries that correspond to a spherical shape and to discuss an alternative formulation of the advective transport from a disc shaped region.

[11] Consider the formal solution (38) and the specific result (41) for the time-dependent spatial distribution of the contaminant resulting from the embedded source with a prolate spheroidal shape. The influence of the cavity geometry is contained in the function $\Omega_p(\alpha, \beta, \lambda)$. We can therefore

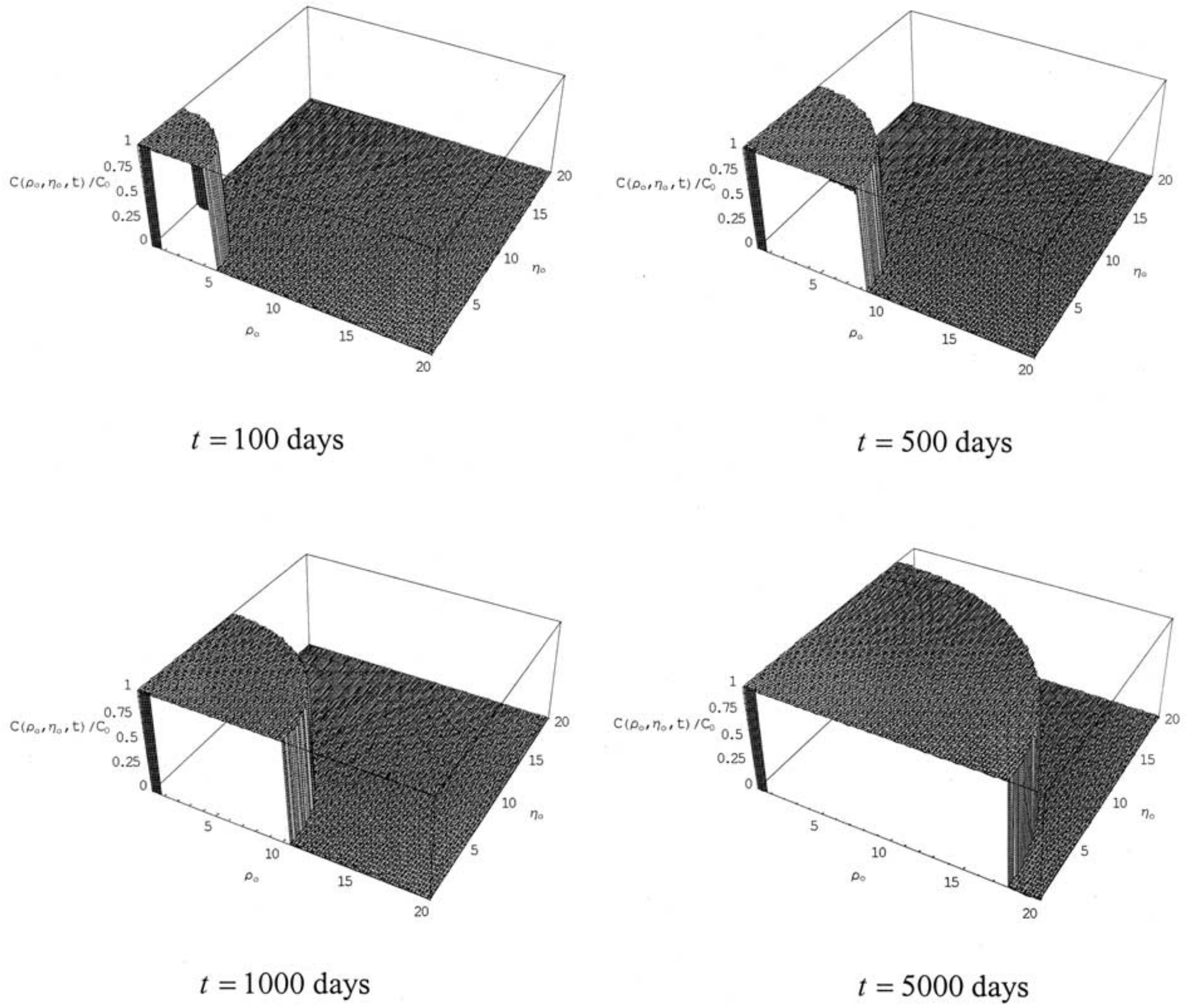


Figure 4. Advective transport of the contaminant from an oblate spheroidal cavity with $a_o/b_o = 1/8$ located in a porous medium with attenuation coefficient $\xi = 0$.

consider the limiting cases for this function in order to examine the dependency of the solution of the geometric aspect ratio of the prolate spheroid.

5.1. Spherical Cavity

[12] In the particular case when $\lambda \to 1$, the oblate spheroidal cavity reduces to a spherical cavity of radius $a_p = a$. Taking the limit of (33) as $\lambda \to 1$, we obtain

$$\Omega(\rho, \eta) = \frac{a^2}{3k\varphi_0}\left[\left(\rho^2 + \eta^2\right)^{3/2} - 1\right] \tag{57}$$

where

$$\rho = \frac{r}{a} \quad ; \quad \eta = \frac{z}{a} \tag{58}$$

and r and z are the cylindrical polar coordinates. This result can be easily derived by formulating the problem at the outset in terms of spherical coordinates [*Selvadurai*, 2002]. The corresponding result for the contaminant concentration in the porous medium can be obtained by using the result (57) in (38) and (41). A limiting result similar to (57) can also be derived by using the solutions (50) and (56) involving the oblate spheroidal cavity. Again, this limiting case is recovered by considering the behavior of the function $\Omega_o(\rho_o, \eta_o, \mu)$ as $\mu \to 1$ (spherical cavity). Considering the limit of (51) as $\mu \to 1$, we recover the result (57) that was obtained from the result (33) for the prolate spheroidal cavity.

5.2. Disc-Shaped Cavity

[13] The importance of the analytical results (50) and (56), which could be used to obtain results for the disc-shaped cavity can be better appreciated by examining the following alternative development of the problem concerning advective transport from a disc-shaped cavity in the form of a penny-shaped crack. We consider the problem of a saturated porous medium of infinite extent, which is bounded internally by a penny-shaped crack of radius a, located on the plane $z = 0$, referred to a cylindrical polar coordinate system (r, θ, z). The boundary of the penny-shaped crack is maintained at a constant potential φ_0, to initiate flow in the porous medium. At steady flow conditions, the boundary of the crack is subjected to a contaminant concentration $C_0F(t)$, and the

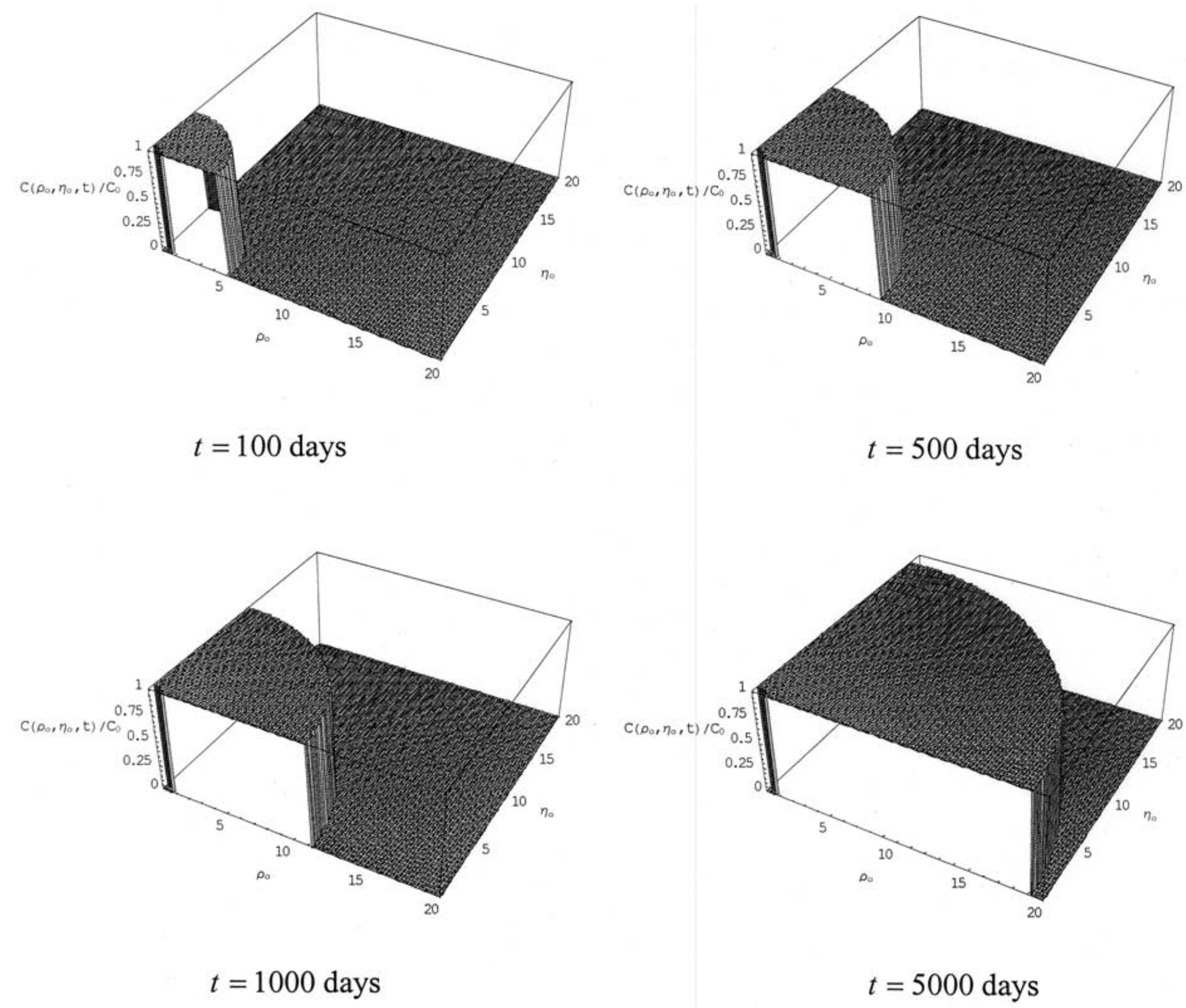


Figure 5. Advective transport of the contaminant from an oblate spheroidal cavity with a_o/b_o = 1/2 located in a porous medium with attenuation coefficient $\xi = 0$.

advective flow of the contaminant is governed by the partial differential equation (8) while the potential flow is governed by (11). Since the problem exhibits a state of axial symmetry about the z-axis and about the plane $z = 0$, we can formulate the advective transport problem in relation to the half-space region occupying $r \in (0, \infty)$ and $z \in (0, \infty)$. To determine the velocity field we consider the mixed boundary value problem governing the potential $\varphi(r, z)$, which should satisfy the mixed boundary conditions

$$\phi(r,0) = \varphi_0 \quad ; \quad r \in (0,a) \tag{59}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad r \in (a,\infty) \tag{60}$$

Considering a Hankel transform development of (11), the solution applicable to the half-space region occupying $r \in (0, \infty)$ and $z \in (0, \infty)$ takes the form

$$\phi(r,z) = \int_0^\infty \zeta A(\zeta)\exp(-\zeta z) J_0(\zeta r) d\zeta \tag{61}$$

where $A(\zeta)$ is an arbitrary function. The mixed boundary conditions now give rise to the system of dual integral equations, of the form

$$\int_0^\infty \zeta A(\zeta) J_0(\zeta r) d\zeta = \varphi_0 \quad ; \quad r \in (0,a) \tag{62}$$

$$\int_0^\infty \zeta^2 A(\zeta) J_0(\zeta r) d\zeta = 0 \quad ; \quad r \in (a,\infty) \tag{63}$$

The solution of the dual system is standard [see, e.g., *Sneddon*, 1972; *Selvadurai*, 2000b] and the distribution of the hydraulic potential in the half-space region is given by

$$\phi(r,z) = \frac{2\varphi_0}{\pi}\int_0^\infty \frac{\sin(\zeta a)}{\zeta}\exp(-\zeta z) J_0(\zeta r) d\zeta \tag{64}$$

The partial differential equation governing axisymmetric advective transport of the contaminant from the disc source is now given by

$$\frac{\partial C}{\partial t} + \frac{2k\varphi_0}{\pi}\left[\begin{array}{l} -\dfrac{\partial C}{\partial r}\displaystyle\int_0^\infty \exp(-\zeta z)\sin(\zeta a) J_1(\zeta r) d\zeta + \\ \dfrac{\partial C}{\partial z}\displaystyle\int_0^\infty \exp(-\zeta z)\sin(\zeta a) J_0(\zeta r) d\zeta \end{array}\right] = \pm \xi C \tag{65}$$

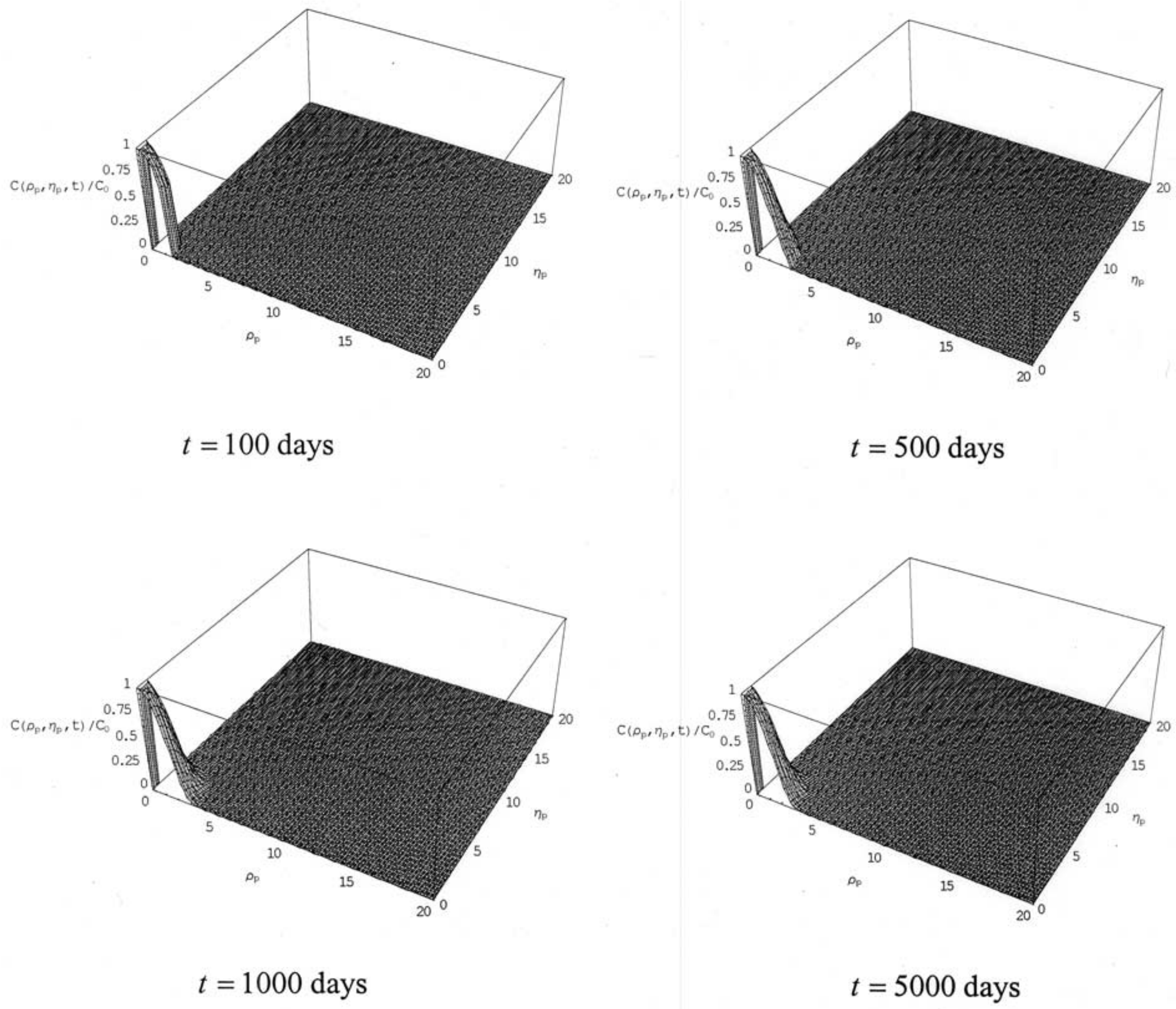


Figure 6. Advective transport of the contaminant from a prolate spheroidal cavity with $b_p/a_p = 1/8$ and located in a porous medium with attenuation coefficient $\xi = 0.005$/day.

This integro-partial differential equation is to be solved subject to the boundary condition

$$C(r,0,t) = C_0 F(t) \quad ; \quad r \in (0,a) \tag{66}$$

and the initial condition

$$C(r,z,0) = 0 \quad ; \quad r \in (0,\infty) \quad ; \quad z \in (0,\infty) \tag{67}$$

The solution of this initial boundary value problem is certainly nonroutine and requires the use of numerical procedures based on either finite difference schemes, or finite element or symbolic computational techniques [see, e.g., *Ganzha and Vorozhtsov*, 1996, 1998; *Zienkiewicz and Taylor*, 2000], where special care must be exercised to model the singular behavior of the velocity field at the boundary of the crack. The exact closed form result (50) can, however, be evaluated to obtain results applicable to the limiting case approximately resembling a penny shaped crack (i.e., as $\mu \rightarrow 0$).

[14] Since the results for the time-dependent distribution of the chemical within the porous medium have been evaluated in exact closed form, it is sufficient to present numerical results that demonstrate the basic facets of the analytical results as they relate to the geometry of the spheroidal region and attenuation effects in the porous medium. For the purposes of the calculations, we set $a_p^2/6\varphi_0 k \approx 0.833$ days. This advective transport coefficient can be identified, for example, for the case of a prolate spheroidal source of axial length $2a_p \approx 10$ m, embedded in a porous medium of hydraulic conductivity $k \approx 0.05$ metres/day and subjected to a flow potential $\varphi_0 \approx 100$ m. The attenuation coefficient is assigned the value $\xi = 0.005$/day. Similarly, for the purposes of presenting the results for the oblate spheroidal cavity problem in relation to the time-scales adopted for the prolate spheroidal cavity problem we set $b_o^2/3\varphi_0 k \approx 1.666$ days. This would correspond to, say, the advective transport from an oblate spheroidal cavity of equatorial diameter $2b_o \approx 10$ m located in a porous medium of hydraulic conductivity $k \approx 0.05$ m/day and subjected to a flow potential $\varphi_0 \approx 100$ m.

[15] Figures 2 and 3 illustrate the pattern of contaminant migration from prolate spheroidal cavity regions with $b_p/a_p = 1/8$ and 1/2 and in the absence of any attenuation. Similar results are presented in Figures 4 and 5 for the cases where the cavity has an oblate spheroidal shape with

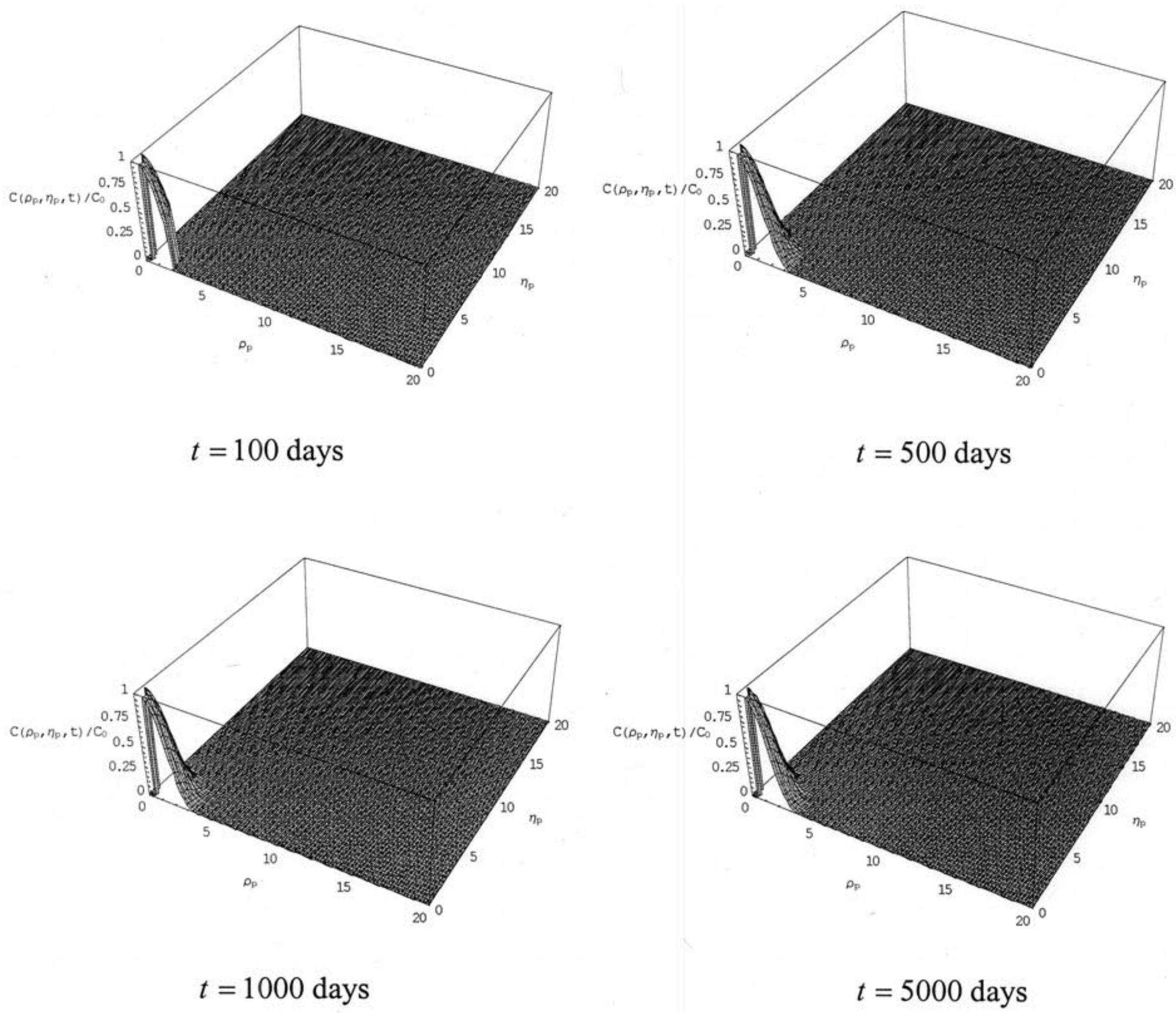


Figure 7. Advective transport of the contaminant from a prolate spheroidal cavity with b_p/a_p = 1/2 and located in a porous medium with attenuation coefficient ξ = 0.005/day.

geometric aspect ratios a_o/b_o = 1/8 and 1/2 respectively. In these graphical representations, full advantage is taken of the state of symmetry associated with both the prolate and oblate spheroidal geometries. The results shown in Figures 2–5 illustrate the influence of the surface area of the cavity region on the extent to which the contaminant is transported from the source. The region over which the contaminant plume extends into the porous medium increases as the surface area of the cavity region increases. For the cavity aspect ratios considered in the numerical evaluations, the lowest value of the cavity surface area is derived for the prolate spheroidal cavity with b_p/a_p = 1/8 and the largest value of the cavity surface area is derived for the oblate spheroidal cavity with a_o/b_o = 1/2. Additional results obtained indicate that for large values of the characteristic nondimensional time τ defined by

$$\tau = \left(\frac{6\varphi_0 kt}{a_s^2}\right) > 10^5$$

where $a_s = a_p$ or b_o, the profile of the contaminant migration front is relatively uninfluenced by the geometry of the axisymmetric source. The location of the contaminant migration front, however, continues to be influenced by the geometry of the axisymmetric source. As the geometry of the contaminant-emitting cavity approaches the limits applicable to a spherical shape, both the profile of the contamination front and its location calculated from both schemes approach the same result. Figures 6 and 7 illustrate the advective transport from a prolate spheroidal cavity with b_p/a_p = 1/8 and 1/2 in the presence of attenuation with constant magnitude ξ = 0.005/day. Analogous results for the case of an oblate spheroidal cavity where a_o/b_o = 1/8 and 1/2, are presented in Figures 8 and 9. These results clearly illustrate the significant influence of a time invariant natural attenuation factor in mitigating the migration of the contaminant from the axisymmetric source. From the results presented in the paper, it is evident that the results for the contaminant migration problem for both needle-shaped cavities and flat disc-shaped cavities located in porous media can be obtained quite conveniently, despite the fact that the potential problem for the limiting cases corresponding to $b_p/a_p \to 0$ and $a_o/b_o \to 0$ both involve singular velocity fields at, respectively, the extremities of the needle-shaped source and the boundary of the flat disc-shaped source. Finally, it is worthwhile to examine the influence of the

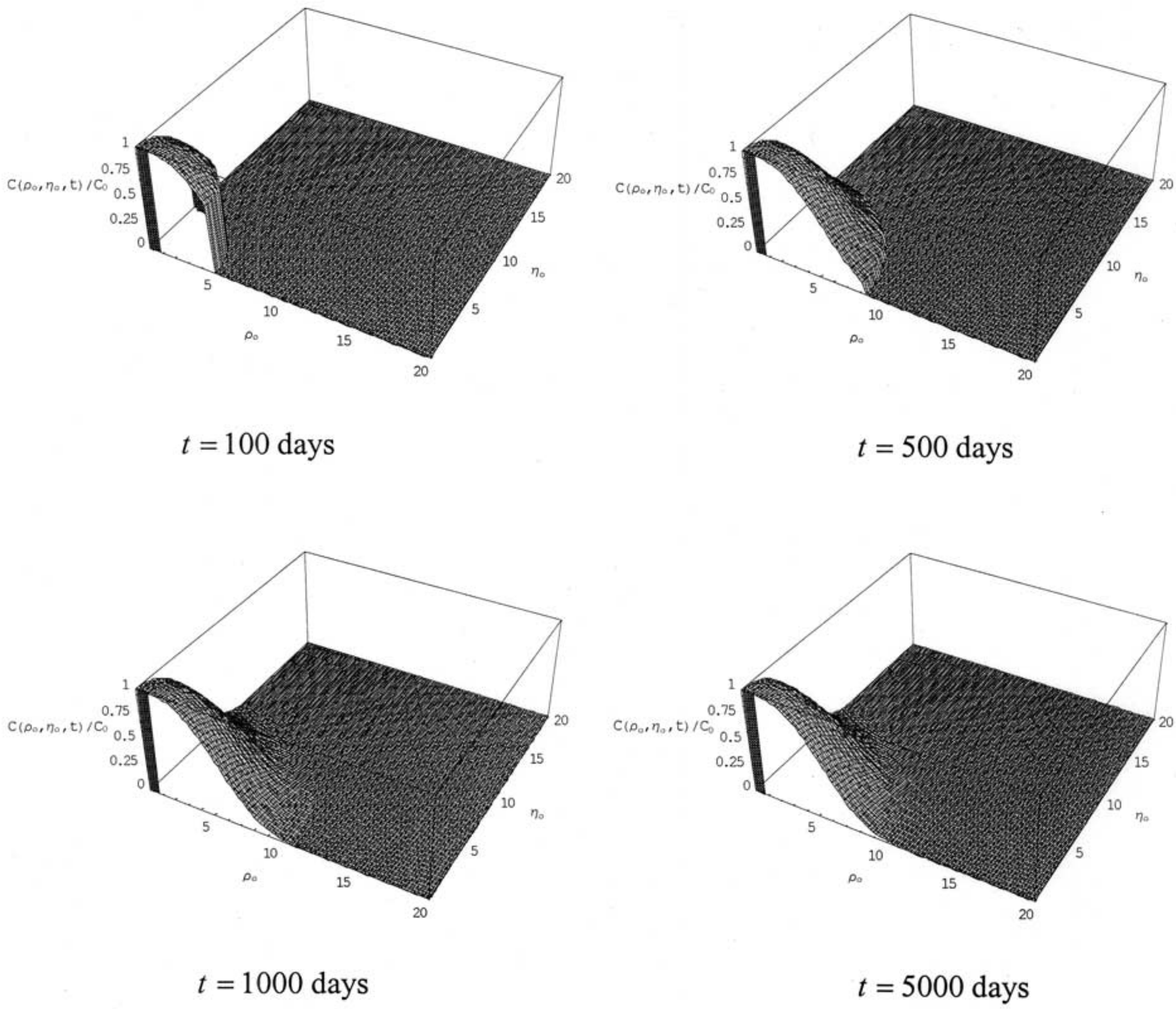


Figure 8. Advective transport of the contaminant from an oblate spheroidal cavity with $a_o/b_o = 1/8$ and located in a porous medium with attenuation coefficient $\xi = 0.005$/day.

attenuation factor ξ on the long-term distribution of the contaminant from the axisymmetric source. For purposes of illustration, we consider the contaminant distribution pattern from the prolate and oblate spheroidal cavities at time $t = 10{,}000$ hours, and the other parameters required for the numerical evaluations are identical to those prescribed previously. Figures 10–13 illustrate the typical influences of a range of values of the attenuation parameter, $\xi \in (5 \times 10^{-2}, 5 \times 10^{-5})$ and cavity geometries on the contaminant profile. It is clear that the presence of a time-invariant natural attenuation of a contaminant has a considerable influence on the extent to which the contaminant plume progresses spatially from the source.

6. Concluding Remarks

[16] The present analytical study deals with the modeling of the linear problem of advective contaminant transport from a spheroidal source that is located in a porous medium of infinite extent. The solution of this problem is facilitated by the fact that the advective flow velocity in the porous medium bounded internally by a spheroidal cavity can be evaluated in exact closed form. The resulting study represents a convenient three-dimensional solution to an advective transport problem from which sources with elongated, spherical and disc-shaped cavity regions can be modeled simply as limiting cases. These limiting cases involving elongated and disc-shaped cavity regions are nontrivial advective transport problems, in the sense that their formulation in the relevant reduced coordinate systems gives rise to integro-partial differential equations that can be solved only by appeal to numerical techniques. The formulation of these problems in relation to the spheroidal coordinates, however, can be used to generate convenient approximate analytical results, which can be expressed in terms of suitable nondimensional small parameters involving the spheroidal cavity geometries. The more general results for the advective transport problem involving both prolate and oblate spheroidal cavities and including linearly dependent attenuation in the porous medium can be evaluated in explicit form and these results are amenable to convenient numerical evaluation for special cases involving boundary contaminant concentrations that are maintained constant with time. The numerical results presented in the paper demonstrate the effectiveness of the mathematical solution as a benchmark for calibrating

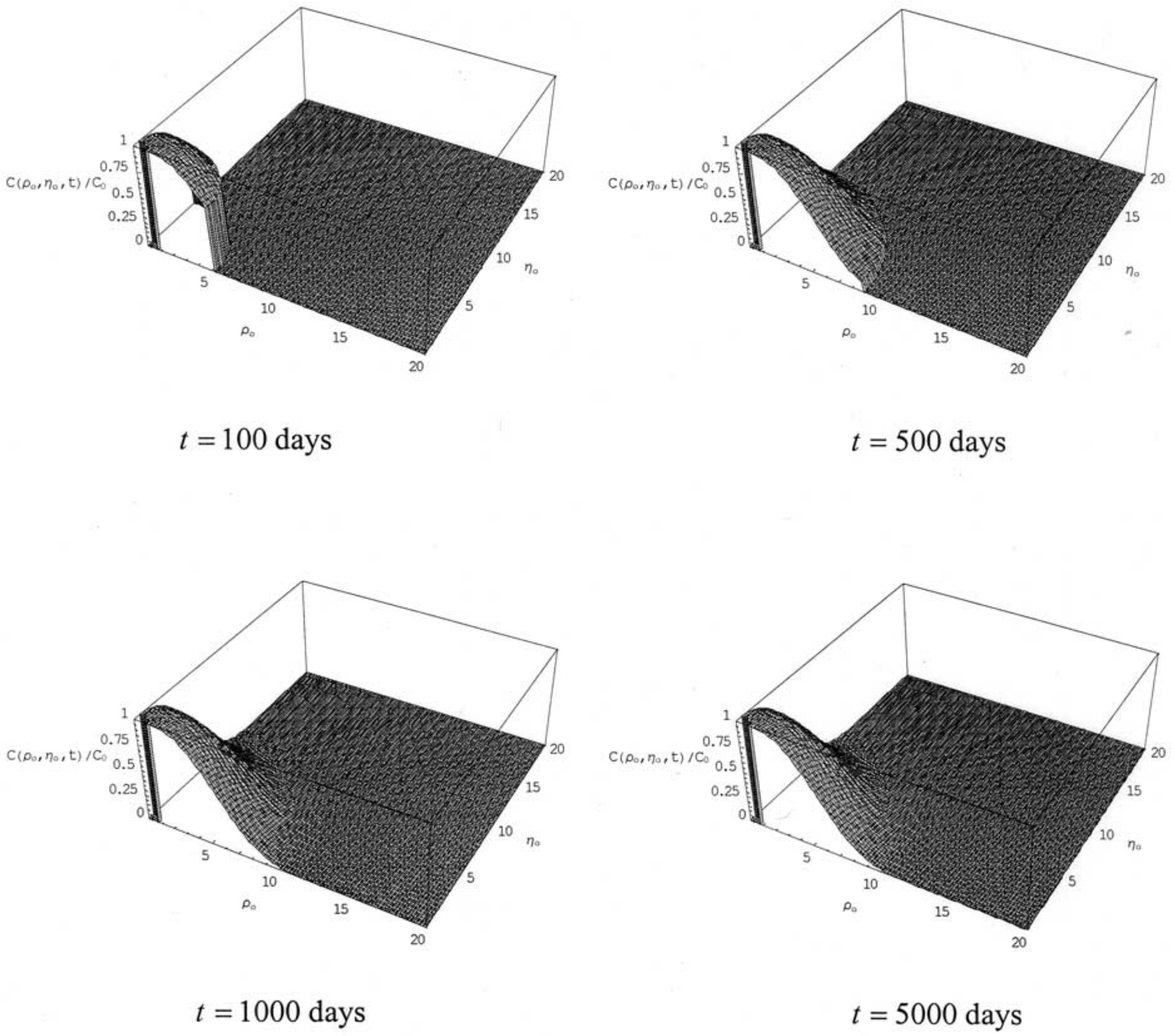


Figure 9. Advective transport of the contaminant from an oblate spheroidal cavity with $a_o/b_o = 1/2$ and located in a porous medium with attenuation coefficient $\xi = 0.005$/day.

computational results for three-dimensional advective transport problems from axisymmetric three-dimensional sources. The numerical results also indicate that with increasing time, the region of chemical migration also increases with the result that the influence of the boundary shape of the axisymmetric source, as distinguished by the prolate and oblate cavity configurations, has a diminishing influence on the shape of the contaminant front (i.e., both spheroidal surfaces of the leading edge of the plume tend to become spherical). This can indirectly be inferred from the observation that the far-field behavior of the velocity field associated with both prolate and oblate spheroidal shapes corresponds to that of the velocity field resulting from potential flow from a spherical cavity (see e.g., (57)). The position of the contaminant plume is, however, influenced by the shape of the cavity region. In this sense, the surface area through which the contaminant release takes place governs the extreme position of the leading edge at any particular time. The numerical results also demonstrate the strong influence of a persistent ''natural attenuation process'' in mitigating the advective transport of the contaminant from the spheroidal source. It must be borne in mind that the time-invariant attenuation assumed in the calculations is an idealization and that in an actual setting the attenuation process can be both time-dependent and nonlinear. The analytical solutions and the methodologies presented in the paper can be easily extended to include both time-dependent variations of the chemical concentration at the boundary of the spheroidal cavity region and plausible time-dependent decay of the attenuation coefficient itself. It is also important to note that the solutions presented here specifically address cavity regions with specified geometries. This is a particular advantage that avoids the thorny issue of treating the region of contaminant recharge as a distribution of ''point sources'', which have no geometry associated with them, except for the spatial extent that is prescribed a priori. The region from which the contaminant migrates now has specific dimensions and the advective flow velocities are governed by a potential prescribed on the surface of this region. The consideration of the specific attributes of the surface region over which the contaminant is introduced into the porous medium becomes important particularly in situations where the contaminant concentration in the vicinity of the cavity region needs to be accurately estimated.

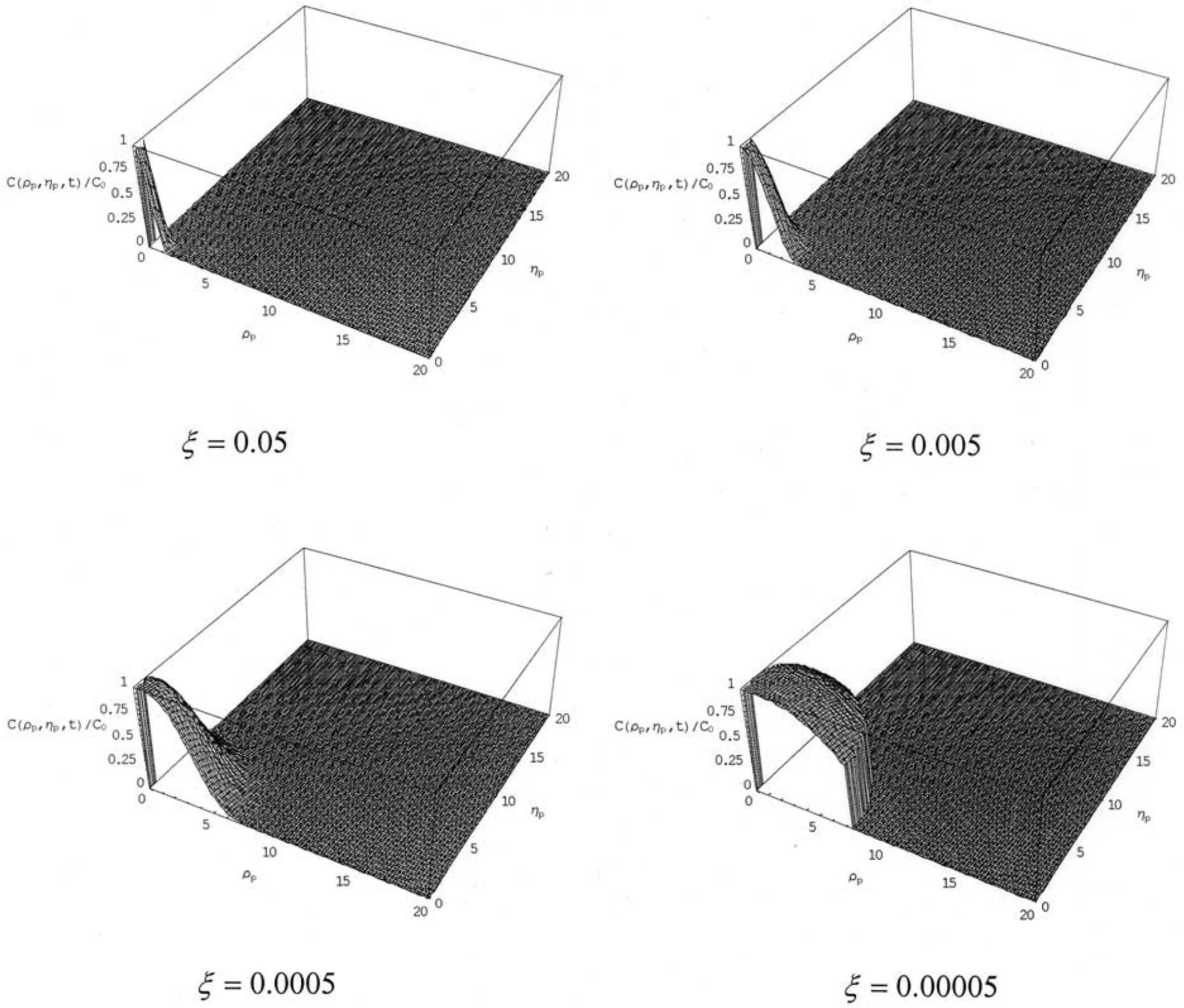


Figure 10. The influence of the attenuation coefficient on the pattern of contamination migration from a prolate spheroidal cavity with b_p/a_p = 1/8 and at time t = 10,000 hours.

[17] Despite the attractiveness and the convenient presentation of exact closed form results for the linear advective transport problem, it must be emphasized that the processes involving the migration of contaminants and other chemicals in naturally occurring porous geomaterials are generally much more complicated, involving complex nonlinear processes that result from fluid flow in the porous medium and the diffusion and attenuation of the chemical. Factors such as contaminant-induced alterations in the hydraulic conductivity of the porous medium and the spatial decay of the attenuation process are expected to exert a strong influence on the advective transport of the contaminant. Such influences are best addressed via computational schemes of the resulting nonlinear advective transport problem. The results presented in the paper not only serve as benchmark mathematical solutions for the validation of the linear component of such nonlinear computational schemes but also as convenient techniques for establishing the limits of behavior of the advective transport process by appeal to estimates that utilize the linear approximation. Finally, the paper also presents a very straightforward proof of the uniqueness of solution for the advective transport problem where Dirichlet boundary conditions are prescribed on the concentration. This topic is rarely discussed in either texts or research articles on the subject and should be of interest to students and educators alike.

Appendix A: A Uniqueness Theorem

[18] We consider the advective transport equation defined by (8) applicable to a domain of finite extent V with surface S. We proceed to show that the solution to initial boundary value problem governed by the partial differential equation

$$\frac{\partial C}{\partial t} + \nabla.(\mathbf{v}C) = \pm\xi C \quad ; \quad \mathbf{x} \in V \tag{A1}$$

subject to the boundary and initial condition

$$C(\mathbf{x}, t) = F(\mathbf{x}, t) \quad ; \quad \mathbf{x} \in S^* \tag{A2}$$

$$C(\mathbf{x}, 0) = G(\mathbf{x}) \quad ; \quad \mathbf{x} \in V \tag{A3}$$

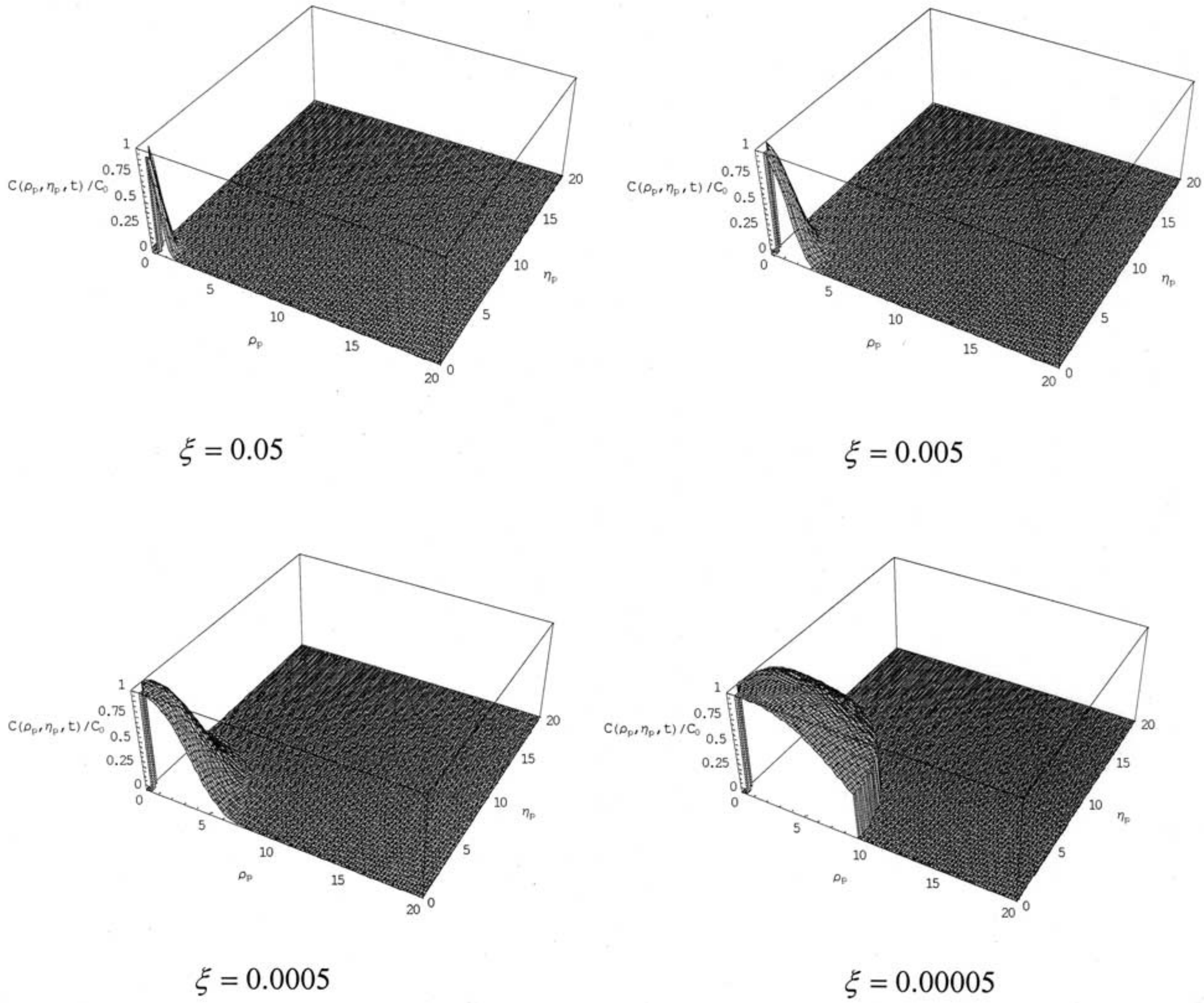


Figure 11. The influence of the attenuation coefficient on the pattern of contamination migration from a prolate spheroidal cavity with $b_p/a_p = 1/2$ and at time t = 10,000 hours.

where S^* is a subset of S, $F(\mathbf{x}, t)$ and $G(\mathbf{x})$ are, respectively, arbitrary functions, $\mathbf{v}(\mathbf{x})$ is a unique velocity field. We assume that the initial boundary value problem defined by (A1) to (A3), admits two solutions $C^{(1)}(\mathbf{x}, t)$ and $C^{(2)}(\mathbf{x}, t)$. Then the solution $C^*(\mathbf{x}, t)$ defined by

$$C^*(\mathbf{x}, t) = C^{(1)}(\mathbf{x}, t) - C^{(2)}(\mathbf{x}, t) \tag{A4}$$

satisfies

$$\frac{\partial C^*}{\partial t} + \nabla.(\mathbf{v}C^*) = \pm\xi C^* \quad ; \quad \mathbf{x} \in V \tag{A5}$$

and subject to the boundary and initial conditions

$$C^*(\mathbf{x}, t) = 0 \quad ; \quad \mathbf{x} \in S^* \tag{A6}$$

$$C^*(\mathbf{x}, 0) = 0 \quad ; \quad \mathbf{x} \in V \tag{A7}$$

We note that by virtue of the uniqueness theorem for the potential problem, the velocity field is the same for both states. We now consider the weak form of (A5) obtained by multiplying the equation by $C^*(\mathbf{x}, t)$ and integrating the result over the region V. We obtain

$$\iiint_V C^*\left[\frac{\partial C^*}{\partial t} + \nabla.(C^*\mathbf{v}) \mp \xi C^*\right] dV = 0 \tag{A8}$$

Noting that $\nabla.\mathbf{v} = 0$, we have

$$\begin{aligned}\iiint_V C^*\nabla.(C^*\mathbf{v})dV &= \iiint_V C^*\nabla C^*.\mathbf{v}dV \\ &= \iiint_V \frac{1}{2}\nabla.\left[(C^*)^2\mathbf{v}\right]dV\end{aligned} \tag{A9}$$

Using Green's theorem (A9) gives

$$\begin{aligned}\iiint_V \frac{1}{2}\nabla.\left[(C^*)^2\mathbf{v}\right]dV &= \frac{1}{2}\iint_S (C^*)^2\mathbf{v}.\mathbf{n}\,dS \\ &= \frac{1}{2}\iint_{S^*}(C^*)^2\mathbf{v}.\mathbf{n}\,dS + \frac{1}{2}\iint_{S-S^*}(C^*)^2\mathbf{v}.\mathbf{n}\,dS\end{aligned} \tag{A10}$$

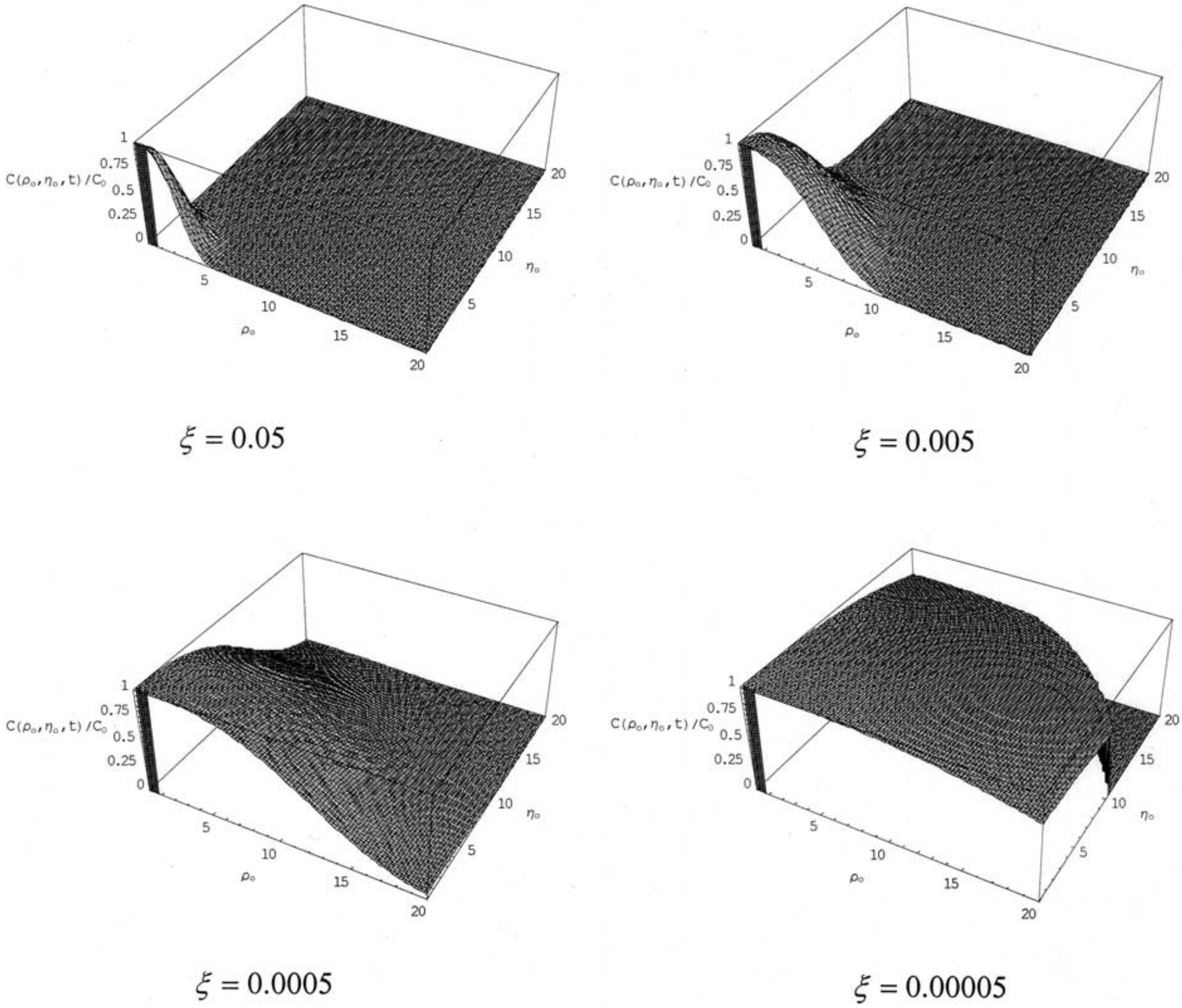


Figure 12. The influence of the attenuation coefficient on the pattern of contamination migration from an oblate spheroidal cavity with a_o/b_o = 1/8 and at time t = 10,000 hours.

The first term in the right hand side of the equation (A10) vanishes on the surface S^*. For the second term in the right hand side of (A10) to vanish, we require $\mathbf{v}(\mathbf{x})$ to vanish on the subset $(S - S^*)$. This is a condition that will be satisfied in three-dimensional domains of infinite extent, if we identify $(S - S^*)$ as the remotely located boundary at infinity, where for a well-posed potential problem the velocity fields will decay to zero. With this proviso, the result (A8) now reduces to

$$\frac{1}{2}\frac{d}{dt}\int\int\int_V (C^*)^2 dV = \pm\xi \int\int\int_V (C^*)^2 dV \qquad \text{(A11)}$$

Denoting

$$\mathrm{I} = \int\int\int_V (C^*)^2 dV \qquad \text{(A12)}$$

equation (A11) yields a first order ordinary differential equation for I. Integrating this equation we obtain

$$\int\int\int_V C^*(\mathbf{x}, t)dV = A^* \exp(\pm 2\xi t) \quad ; \quad \mathbf{x} \in V \quad ; \quad t \in (0, T) \qquad \text{(A13)}$$

where A^* is a constant of integration and T is any arbitrary time. We can use the initial condition (A7) to determine this arbitrary constant. i.e.

$$\int\int\int_V C^*(\mathbf{x}, 0)dV = A^* \equiv 0 \quad ; \quad \mathbf{x} \in V \qquad \text{(A14)}$$

which reduces (A13) to

$$\int\int\int_V C^*(\mathbf{x}, t)dV = 0 \quad ; \quad \mathbf{x} \in V \qquad \text{(A15)}$$

From the Dubois-Reymond Lemma, (A15) is equivalent to

$$C^*(\mathbf{x}, t) = 0 \qquad \text{(A16)}$$

which, from (A4) establishes the uniqueness of the solution to the initial boundary value problem with Dirichlet

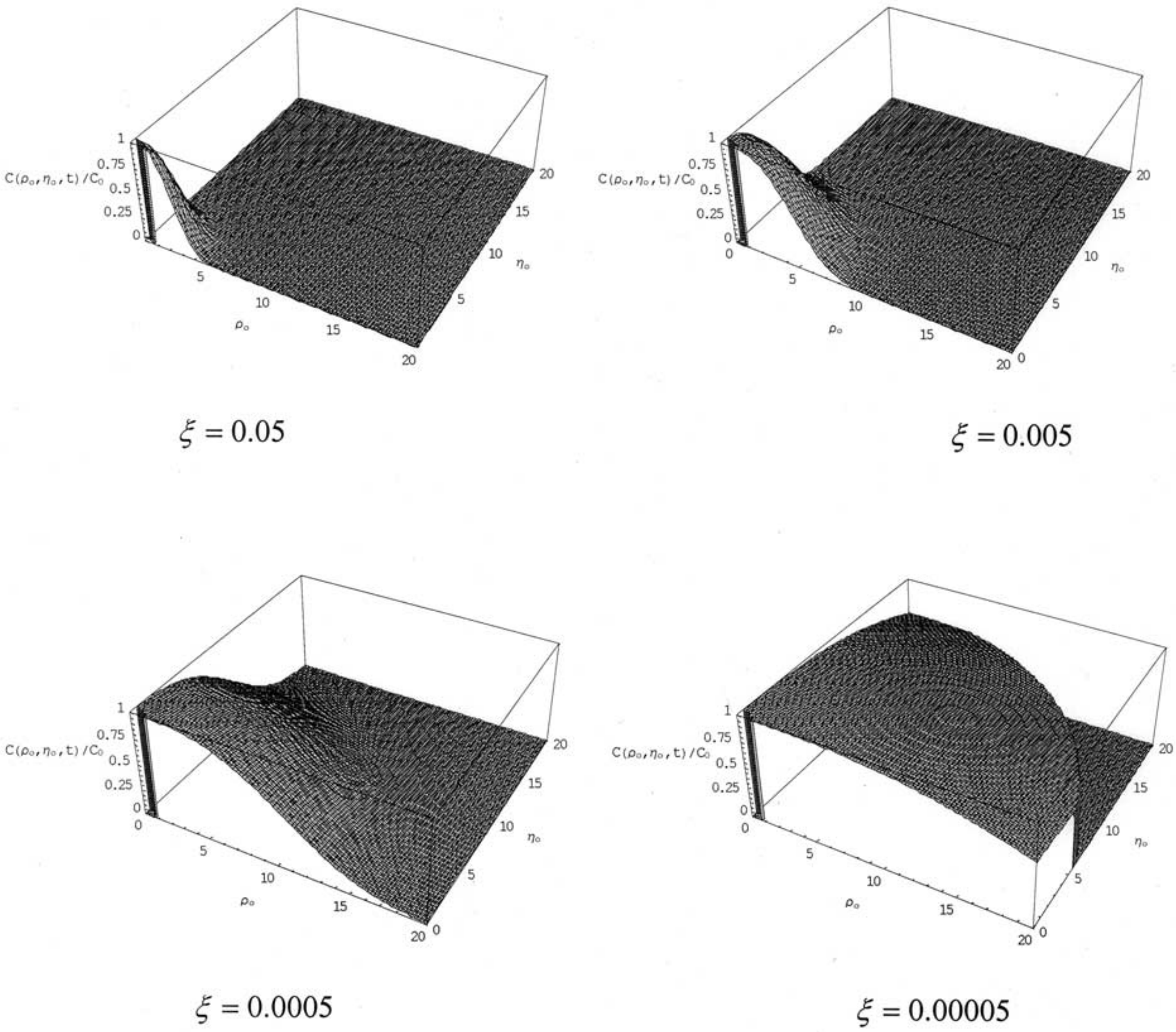


Figure 13. The influence of the attenuation coefficient on the pattern of contamination migration from an oblate spheroidal cavity with a_o/b_o = 1/2 and at time t = 10,000 hours.

boundary conditions, associated with the advective transport problem applicable to an infinite domain.

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