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On an invariance principle for unilateral contact at a bimaterial elastic interface

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Abstract

This paper examines the axisymmetric elastostatic problem related to the unilateral receding contact at a pre-compressed smooth bimaterial elastic interface. The separation at the interface is caused by the action of axisymmetric stress fields of unequal magnitude, which are applied at any location of the separate halfspace regions. The analysis of the problem focuses on the determination of the zone of separation as a function of the pre-compression, the magnitudes and locations of the axisymmetric stress fields inducing the separation, and the elasticity characteristics of the halfspace regions. It is found that the radius of the separation zone can be evaluated in explicit form. In the particular instance when the loadings applied at the surface of the halfspace regions are equal in magnitude and distribution, the analysis reveals that the radius of the separation zone is *independent* of the elasticity properties of the halfspace regions and can be evaluated in *exact closed form*.

Keywords: Invariance principle; Unilateral contact; Bimaterial contact; Separation at pre-compressed interface; Contact problem; Contact of elastic halfspaces; Separation by Boussinesq forces

1. Introduction

The class of elastostatic problems that deal with the contact between smoothly compressed elastic bodies of differing elastic properties has applications to a number of areas in the engineering sciences. These include the study of separation at geomaterial interfaces with particular reference to pre-fractured, resource bearing formations, the study of the mechanics of contact

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between biomaterials and implants, and in the study of layered rubber-like elastic solids used as structural supports. Although the assumption of smooth contact at such interfaces represents a highly idealized condition, the smooth contact serves as a useful first approximation, as does the use of classical elasticity, in establishing the influence of the initial separation on the contact behaviour of such interfaces. The phenomenon of separation at such smoothly pre-compressed elastic bodies can be induced by defects and impurities, and by environmentally or thermally induced dilatation of such defects. For example, in order to extract resources from pre-fractured resource bearing geological strata, it is necessary to maintain the zone of separation in an open condition; this is achieved by introducing a granular proppant material. Also, an assessment of the zone of separation and the influence of the elasticity characteristics on the zone of separation is also of some interest to establishing the efficiency of load transfer at the interface and on any subsequent frictional response of the interface itself.

The subject of unilateral contact problems has well founded, elegant mathematical traditions, commencing largely with the works of Signorini [45], Prager [40] and Fichera [13], with further formal treatments by Fichera [14], Duvaut and Lions [12], Villaggio [50] and Ciarlet [6] and culminating with the studies by Kinderlehrer and Stampaccia [30], Haslinger and Janovsky [23], Panagiotopoulos [39], Moreau et al. [37], Fremond [15], Kalker [28], Klabring [31], and others to include friction, slip, adhesion and separation in contact zones where the boundaries of the separate contact zones themselves are unknowns functions. The more pragmatic approaches to the study of unilateral contact problems, however, have antecedents, commencing with the seminal works of Hertz [24,25], Boussinesq [5] and Love [33]. These studies, formulated largely within the context of the classical theory of elasticity, have resulted in an extensive body of literature dealing with the mathematical treatment of contact problems for elastic bodies of both finite and semi-infinite geometry. Extensive reviews of this class of problems are given, among others, by Mindlin and Deresiewicz [36], Galin [16], Ufliand [49], Lur'e [34], Dundurs and Stippes [9], de Pater and Kalker [8], Goodman [20], Selvadurai [41], Gladwell [17], Dundurs and Comninou [10], Johnson [26], Mura [38], Kikuchi and Oden [29], Curnier [7], Aliabadi and Brebbia [1] and Barnett et al. [4]. These latter studies evaluate the responses in unilateral contact problems with specific geometrical configurations of engineering interest; they invariably refer to unilateral contact between either elastically dissimilar indenting regions with regular geometries or halfspace regions and rigid objects with specified regular shapes.

This paper examines the idealized elastostatic problem of the separation at a pre-compressed bimaterial elastic interface caused by the action of distributed axisymmetric loads of unequal magnitude that can act at any location either within the separate halfspace regions or at the contacting smooth interface itself. Problems of related interest involving the perturbing of the unilateral smooth contact at an elastic interface by a rigid disc inclusion were examined by Gladwell and Hara [18], Selvadurai [42,43] and Gladwell [19]. Of related interest is a study by Johnson et al. [27], who extend the work of Dundurs et al. [11] dealing with the plane contact between wavy surfaces, to include three-dimensional wavy contact. In the studies by Selvadurai [42,43] and Gladwell [19] the separation at the pre-compressed bi-material elastic interface is induced by *displacements*, which are prescribed at the interface. In this paper we consider the traction boundary value problem, where the loads are applied at *any* location within the halfspace regions induce the separation but in a manner that will preserve the overall axial symmetry of the problem. The solution to the problem is obtained by using the results for classical dual integral equations obtained for a set of complementary sub-problems. The result for the radius of the zone of separation induced at the pre-compressed interface can be obtained in explicit form in terms of Hankel transforms of the loadings that induce the separation. An interesting invariance property, with certain similarities to Hertz's result, emerges from the general result for the case involving identical loads applied at identical distances from the interface itself; in this case it is found that the radius of the zone of separation is *independent* of the elasticity characteristic of *either* halfspace region. The results for the unilateral contact induced by loads acting at the *interior* of the half-space regions also provide restrictions on the invariance property.

2. Governing equations

We consider the class of axisymmetric problems in classical elasticity referred to two bimaterial elastic halfspace regions where the common plane is smooth and located at $z = 0$. The interface is subjected to an axial pre-compression stress σ_0 in the z-direction. This normal pre-compression is, of course, necessitated by the assumption of smooth contact at the bimaterial interface. The halfspace regions designated by the superscript $^{(1)}$ and $^{(2)}$ occupy the regions $r \in (0, \infty)$; $z \in (0, \infty)$ and $r \in (0, \infty)$; $z \in (0, -\infty)$ respectively. The solution of axisymmetric problems in the classical theory of elasticity can be approached through formulations based either on Love's strain function approach or the Neuber–Papkovich potentials approach (see e.g. [17,22,32,44,48]). In the absence of body forces, the displacement equations governing axial symmetry take the forms

$$\nabla^2 u_r + \frac{1}{(1-2\nu)}\frac{\partial e}{\partial r} - \frac{u_r}{r^2} = 0 \tag{1}$$

$$\nabla^2 u_z + \frac{1}{(1-2\nu)}\frac{\partial e}{\partial z} = 0 \tag{2}$$

where u_r and u_z are the components of the displacement vector referred to the cylindrical polar coordinate system, ν is Poisson's ratio, ∇^2 is Laplace's operator for axial symmetry, defined by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{3}$$

and e is the divergence of the displacement vector defined by

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \tag{4}$$

The solution of the displacement equations of equilibrium can be approached by employing a variety of techniques for the representation of the displacement and stress fields. These range from the Papkovich–Neuber functions to Love's strain potential function approach. The completeness of these representations is further discussed by Truesdell [48] and Gurtin [22]. In the class of unilateral contact problem considered in this paper, the surface $z = 0$, referred to either halfspace region, is free of shear tractions; in this instance it is possible to represent the displacement and stress components in terms of a single *potential function* $\Phi(r,z)$, which automatically renders the surface $z=0$ free of shear tractions [17,21]. The displacement and stress components take the forms

$$u_r(r,z) = (1-2\nu)\frac{\partial\Phi}{\partial r} + z\frac{\partial^2\Phi}{\partial r\partial z} \tag{5}$$

$$u_z(r,z) = -2(1-\nu)\frac{\partial\Phi}{\partial z} + z\frac{\partial^2\Phi}{\partial z^2} \tag{6}$$

and

$$\sigma_{rr}(r,z) = 2\mu\left[(1-2\nu)\frac{\partial^2\Phi}{\partial r^2} - 2\nu\frac{\partial^2\Phi}{\partial z^2} + z\frac{\partial^3\Phi}{\partial r^2\partial z}\right] \tag{7}$$

$$\sigma_{\theta\theta}(r,z) = 2\mu\left[\frac{1}{r}\frac{\partial\Phi}{\partial r} + 2\nu\frac{\partial^2\Phi}{\partial r^2} + \frac{z}{r}\frac{\partial^2\Phi}{\partial r\partial z}\right] \tag{8}$$

$$\sigma_{zz}(r,z) = 2\mu\left[-\frac{\partial^2\Phi}{\partial z^2} + z\frac{\partial^3\Phi}{\partial z^3}\right] \tag{9}$$

$$\sigma_{rz}(r,z) = 2\mu\left[z\frac{\partial^3\Phi}{\partial r\partial z^2}\right] \tag{10}$$

where μ is the linear elastic shear modulus.

In view of the axial symmetry of the unilateral contact problem under discussion, and the semi-infinite geometries the bimaterial domains involved, it is convenient to select solutions of the potential functions based on a Hankel transform development of the governing partial differential equation

$$\nabla^2\Phi^{(n)}(r,z) = 0 \tag{11}$$

The relevant solutions for $\Phi^{(n)}(r,z)$, applicable to the halfspace regions $^{(1)}$ and $^{(2)}$ take the forms

$$\Phi^{(1)}(r,z) = \frac{1}{2\mu_1}\int_0^\infty \frac{A_1(\xi)}{\xi} e^{-\xi z} J_0(\xi r)\, d\xi \tag{12}$$

and

$$\Phi^{(2)}(r,z) = \frac{1}{2\mu_2}\int_0^\infty \frac{A_2(\xi)}{\xi} e^{\xi z} J_0(\xi r)\, d\xi \tag{13}$$

respectively, where $A_1(\xi)$ and $A_2(\xi)$ are arbitrary functions.

3. Problem formulation

We consider the problem of two elastic halfspace regions that are subjected to an axial compression σ_0 at the frictionless interface $z = 0$. The unilateral contact is perturbed by axisymmetric loadings of magnitude $p_n(r, \zeta_n)$, which act at specific locations within the halfspace regions, and ζ_n refer to dimensions, which are specified in relation to the contacting plane $z = 0$. It is assumed that the magnitudes and directions of application of the loads $p_n(r, \zeta_n)$ are such that separation occurs at the smooth interface over an as-yet undetermined region $r \in (0, a)$ (Fig. 1). The assertion is that the radius of the separation zone between the halfspace regions will depend on the relative magnitudes of the pre-compression σ_0, the magnitudes of the loads $p_n(r, \zeta_n)$ and the elasticity properties of both halfspace regions. In this sense, the separation at the interface is a receding contact or a moving boundary problem. The mixed boundary value problem relating to the unilateral contact problem can be posed as follows:

$$\sigma_{zz}^{(n)}(r, 0) = 0, \quad 0 \leqslant r \leqslant a \tag{14}$$

$$\sigma_{zz}^{(1)}(r, 0) = \sigma_{zz}^{(2)}(r, 0), \quad a \leqslant r \leqslant \infty \tag{15}$$

$$\sigma_{rz}^{(1)}(r, 0) = \sigma_{rz}^{(2)}(r, 0) = 0, \quad 0 < r < \infty \tag{16}$$

$$u_z^{(1)}(r, 0) = u_z^{(2)}(r, 0), \quad a \leqslant r < \infty \tag{17}$$

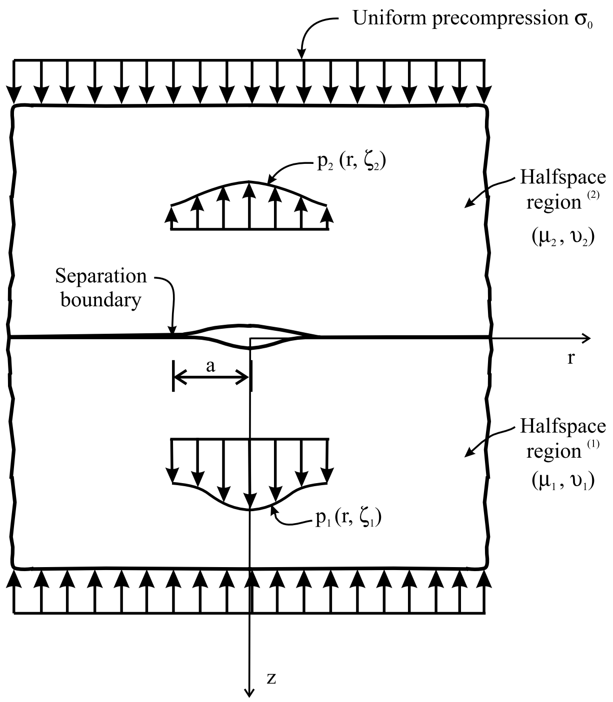

Fig. 1. Unilateral contact at a pre-compressed bimaterial elastic interface.

and tensile stresses are considered positive. In addition, the solution should satisfy the regularity conditions

$$\sigma_{zz}^{(1)}(r,z) = \sigma_{zz}^{(2)}(r,z) \to -\sigma_0, \quad \text{as } |z| \to \infty \tag{18}$$

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) \to 0, \quad \text{as } r \to a \tag{19}$$

and, referring to Fig. 1,

$$\sigma_{zz}^{(1)}(r,\zeta_1) = p_1(r,\zeta_1) \tag{20}$$

$$\sigma_{zz}^{(2)}(r,\zeta_2) = p_2(r,\zeta_2) \tag{21}$$

The condition (19) is *necessary and sufficient* to uniquely determine the radius a of the zone of separation.

The solution to the above mixed boundary value problem can be approached in a variety of ways; it is, however, convenient to formulate this unilateral contact problem as a combination of the solution to two complementary sub-problems as follows: The first sub-problem is associated with the mixed boundary value problem where the identified separation zone is subjected to an arbitrary tensile stress $q(r)$:

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0) = q(r), \quad 0 \leqslant r \leqslant a \tag{22}$$

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0) = 0, \quad 0 < r < \infty \tag{23}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0), \quad a \leqslant r < \infty \tag{24}$$

with regularity condition that $u_i^{(n)}$, $\sigma_{ij}^{(n)} \to 0$, $(n = 1,2)$ as $r, |z| \to \infty$. Considering the uniaxial compression of the halfspace regions, in order to render the region $r \in (0,a)$ traction free, we require

$$q(r) = \sigma_0 \tag{25}$$

The resulting stress state superposed on a state of *uniaxial compression* of σ_0, gives rise to zero normal tractions in the region $0 < r < a$. The displacement and stress components associated with the uniaxial stress state can simply be added to the solution derived from the first sub-problem without altering the conditions (23) and (24). The second sub-problem is associated with the mixed boundary value problem

$$\sigma_{zz}^{(n)}(r,0) = -\sigma^*(r), \quad 0 \leqslant r \leqslant a, \quad (n = 1,2) \tag{26}$$

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0) = 0, \quad 0 < r < \infty \tag{27}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0), \quad a \leqslant r < \infty \tag{28}$$

where $\sigma^*(r)$ is an axial (tensile) stress, acting over the entire interface $r \in (0, \infty)$ of the two halfspace regions that are in frictionless *bilateral contact* at the plane $z = 0$, and subjected to the loads $p_n(r, \zeta_n)$.

The combination of the solutions to the two sub-problems, along with the state of uniaxial compression and the states of stress corresponding to loads $p_n(r, \zeta_n)$, will satisfy the boundary conditions required for the unilateral contact problem. The division of the unilateral contact problem into two sub-problems also allows for the convenient discussion of various extensions to the problem.

3.1. Sub-problem 1

Considering the solutions for $\Phi^{(1)}(r,z)$ and $\Phi^{(2)}(r,z)$ given, respectively, by (12) and (13) and the expressions (6) and (9), we obtain integral expressions for $u_z^{(n)}$ and $\sigma_{zz}^{(n)}$. The zero shear boundary conditions applicable to the plane $z = 0$, defined by (23), are satisfied by the choice of the representation and the regularity conditions are also satisfied by the Hankel integral representations (12) and (13). The resulting expressions for the displacements $u_z^{(n)}(r, 0)$ and the stresses $\sigma_{zz}^{(n)}(r, 0)$ can be evaluated in the forms

$$u_z^{(n)}(r,0) = (-1)^{n+1} \frac{(1-\nu_n)}{\mu_n} \int_0^\infty A_n(\xi) J_0(\xi r)\,\mathrm{d}\xi \tag{29}$$

$$\sigma_{zz}^{(n)}(r,0) = -\int_0^\infty \xi A_n(\xi) J_0(\xi r)\,\mathrm{d}\xi \tag{30}$$

Furthermore, by virtue of the equilibrium of contact stresses in the region $r \in (a, \infty)$, as implied by (15), and due to the equality of applied tractions in the region $r \in (0, a)$, as implied by (22) we require

$$A_1(\xi) \equiv A_2(\xi) = A(\xi) \tag{31}$$

We can now reduce the boundary conditions (22) and (24) for the first sub-problem to a pair of dual integral equations of the general form

$$\int_0^\infty \xi A(\xi) J_0(\xi r)\,\mathrm{d}\xi = -q(r), \quad 0 < r < a \tag{32}$$

$$\left[\frac{(1-\nu_1)}{\mu_1} + \frac{(1-\nu_2)}{\mu_2}\right] \int_0^\infty A(\xi) J_0(\xi r)\,\mathrm{d}\xi = 0, \quad a \leqslant r \leqslant \infty \tag{33}$$

The solution of the system of dual integral equations (32) and (33), through the introduction of a finite Fourier sine transform representation of $A(\xi)$ and the reduction of the resulting problem to an Abel integral equation for an unknown function, has been discussed in many texts on elasticity, fracture mechanics and integral equations (see e.g. [17,44,46,47]). The exact solution of the Abel integral equation can be used to calculate the displacements and stresses in the two elastic halfspace regions. We focus attention on a specific result of interest to the analysis of the unilateral contact problem that deals with the evaluation of the manner in which the stresses $\sigma_{zz}^{(n)}$ approach the boundary $r = a$. This requires the evaluation of the stress intensity factor K_I^a, associated with $\sigma_{zz}^{(n)}$ and defined by

$$K_I^a = \lim_{r \to a+} \sqrt{(r-a)}\sigma_{zz}^{(n)}(r,0) \tag{34}$$

Avoiding details, it can be shown that the stress intensity factor can be evaluated from the general result

$$\left[K_I^a\right]_{\sigma_0} = -\frac{2}{\pi\sqrt{a}} \int_0^a \frac{rq(r)\,\mathrm{d}r}{\sqrt{a^2 - r^2}} = -\frac{2\sigma_0\sqrt{a}}{\pi} \tag{35}$$

which is identical (*in its absolute value*) to the stress intensity factor for a penny-shaped crack of radius a in a *homogeneous isotropic elastic medium* and subjected to a uniaxial state of stress σ_0. The negative value of the stress intensity factor is interpreted in a manner consistent with the compressive nature of the applied stress σ_0.

3.2. Sub-problem 2

The second sub-problem can be examined in a variety of ways; here, we develop the solution by considering the combination of solutions for elastic halfspace problems, where the traction boundary conditions (21) are identically satisfied.

We first consider the problem of a halfspace region $^{(1)}$, where the surface is subjected to an arbitrary axisymmetric *tensile* normal traction $\sigma^*(r)$, the axisymmetric load $p_1(r, \zeta_1)$ is directed along the *positive* z-axis, and the shear stresses are zero on the plane boundary. The solution to this problem can be obtained by adopting a Hankel transform development of the Love strain function, as indicated previously. Considering (7) and satisfying the boundary conditions applicable to the problem it can be shown that

$$\tilde{u}_z^{(1)}(\xi) = \frac{(1-\nu_1)}{\xi\mu_1}\left[-\tilde{\sigma}^*(\xi) + \widetilde{F}_1(\xi, \zeta_1)\right] \tag{36}$$

where $\widetilde{F}_1(\xi, \zeta_n)$ is directly related to the zeroth-order Hankel transform of the axial displacement of the plane $z = 0$ due to the applied loading $p_1(r, \zeta_1)$.

Similarly, by considering the boundary value problem where the halfspace region $^{(2)}$ is subjected to a *tensile* normal traction $\sigma^*(r)$, an internal load $p_2(r, \zeta_2)$ directed along the negative z-axis, and zero shear stresses on the plane boundary, we obtain

$$\tilde{u}_z^{(2)}(\xi) = \frac{(1-\nu_2)}{\xi\mu_2}\left[\tilde{\sigma}^*(\xi) - \widetilde{F}_2(\xi, \zeta_2)\right] \tag{37}$$

where, following (36), $\widetilde{F}_2(\xi, \zeta_2)$, is related to the zeroth-order Hankel transform of the axial surface displacement of the halfspace region $^{(2)}$ due to the action of $p_2(r, \zeta_2)$.

In order for the two halfspaces to maintain a *fictitious* contact during the application of these forces and tractions we require

$$\tilde{u}_z^{(1)}(\xi) = \tilde{u}_z^{(2)}(\xi), \quad r \in (0, \infty) \tag{38}$$

which together with (36) and (37), gives

$$\tilde{\sigma}^*(\xi, \zeta_n) = \left[\frac{\widetilde{F}_1(\xi, \zeta_1)(1 - \nu_1)\mu_2 + \widetilde{F}_2(\xi, \zeta_2)(1 - \nu_2)\mu_1}{(1 - \nu_1)\mu_2 + (1 - \nu_2)\mu_1} \right] \tag{39}$$

The result (39) represents the Hankel transform of the *interface tensile stresses*. The corresponding tractions are obtained by inversion: i.e.

$$\sigma^*(r, \zeta_n) = \int_0^\infty \xi \tilde{\sigma}^*(\xi, \zeta_n) J_0(\xi r)\, d\xi \tag{40}$$

We consider the second mixed boundary value problem for the "*bilaterally linked bi-material region*" which experiences delamination over a region $r \in (0, a)$. The system of dual integral equations associated with the mixed boundary value problem for an unknown function $B^*(\xi)$ takes the form

$$\int_0^\infty \xi^4 B^*(\xi) J_0(\xi r)\, d\xi = \sigma^*(r, \zeta_n), \quad 0 < r < a \tag{41}$$

$$\left[\frac{(1 - \nu_1)}{\mu_1} + \frac{(1 - \nu_2)}{\mu_2} \right] \int_0^\infty B(\xi) J_0(\xi r)\, d\xi = 0, \quad a \leqslant r < \infty \tag{42}$$

Again, these dual integral equations can be solved in a standard fashion [46] to obtain integral relationships for the displacement and stress fields. The result of primary interest to the study of the unilateral contact problem relates to the stress intensity factor at the boundary of the circular delaminated region, which can be evaluated in integral form: i.e.

$$\left[K_{\mathrm{I}}^a\right]_{p_n(r)} = \frac{2}{\pi\sqrt{a}} \int_0^a \frac{r\sigma^*(r, \zeta_n)\, dr}{(a^2 - r^2)^{1/2}} \tag{43}$$

The corresponding stress intensity factor can be evaluated in the form

$$\left[K_{\mathrm{I}}^a\right]_{p_n} = \frac{2}{\pi\sqrt{a}} \int_0^a \frac{1}{(a^2 - r^2)^{1/2}} \left[\int_0^\infty \xi \left[C_1 \widetilde{F}_1(\xi, \zeta_1) + C_2 \widetilde{F}_2(\xi, \zeta_2) \right] J_0(\xi r)\, d\xi \right] r\, dr \tag{44}$$

where

$$C_1 = \frac{(1 - \nu_1)\mu_2}{[(1 - \nu_1)\mu_2 + (1 - \nu_2)\mu_1]}, \quad C_2 = \frac{(1 - \nu_2)\mu_1}{[(1 - \nu_1)\mu_2 + (1 - \nu_2)\mu_1]} \tag{45}$$

with $C_1 + C_2 = 1$.

4. The unilateral contact problem

The zone of separation induced at the pre-compressed bimaterial elastic interface region due to the action of the axisymmetric loads $p_n(r,\zeta_n)$, can be obtained by applying the constraint that, at the point of separation, the normal contact stress should uniformly approach zero. This requirement gives the constraint *originally proposed* by Barenblatt [2,3] for decohesion zones extending from cracks in *homogeneous brittle elastic solids* of infinite extent: i.e.

$$\left[K_{\mathrm{I}}^{a}\right]_{p_n}+\left[K_{\mathrm{I}}^{a}\right]_{\sigma_0}=0 \tag{46}$$

Substituting the expressions for the stress intensity factors defined by (35) and (44) in (46) we obtain the following constraint for the determination of the zone of separation at the pre-compressed bimaterial elastic interface:

$$\int_0^a\left[-q(r)+\int_0^\infty \xi\Big\{C_1\widetilde{F}_1(\xi,\zeta_1)+C_2\widetilde{F}_2(\xi,\zeta_2)\Big\}J_0(\xi r)\,\mathrm{d}\xi\right]\frac{r\,\mathrm{d}r}{\sqrt{a^2-r^2}}=0 \tag{47}$$

At this stage, it appears unlikely that one could proceed further to obtain an explicit expression for the radius of the zone of separation without specifying the exact nature of the loads $p_n(r,\zeta_n)$ that induce the separation. There are, however, certain conclusions that can be made regarding the invariance property, by restricting attention to the specific case where the location of the loads that induce separation are restricted to the unilateral interface itself; i.e. $p_n(r,\zeta_n)\to p_n(r)$. Considering the individual halfspace regions in which the *surfaces* are simultaneously subjected to $p_n(r)$ and $\sigma^*(r)$, we have, from (39)

$$\tilde{\sigma}^*(\xi)=\left[\frac{\tilde{p}_1(\xi)(1-\nu_1)\mu_2+\tilde{p}_2(\xi)(1-\nu_2)\mu_1}{(1-\nu_1)\mu_2+(1-\nu_2)\mu_1}\right] \tag{48}$$

Consider the case where separation at the interface is induced by uniform circular loads of stress intensities p_1 and p_2 of radii b_1 and b_2 that act at the unilateral interface, referred to the halfspace regions $^{(1)}$ and $^{(2)}$, respectively. In this case, the expressions for the Hankel transforms of the applied loads p_n are independent of the elastic constants of the separate halfspace regions. The constraint (47) for determining the radius of separation now reduces to

$$\sigma_0=\frac{1}{a}\left[\int_0^{b_i}\frac{r\,\mathrm{d}r}{\sqrt{a^2-r^2}}\int_0^\infty\sum_{i=1,}^{2}C_ip_ib_iJ_1(\xi b_i)J_0(\xi r)\,\mathrm{d}\xi\right] \tag{49}$$

where the upper limit of the finite integral in (49) has to be interpreted in relation to the limits applicable to the infinite integral. Since

$$\int_0^\infty J_1(\xi b_i)J_0(\xi r)\,\mathrm{d}\xi=\begin{cases}1/b_i, & r\in(0,b_i)\\ 0, & r\in(b_i,\infty)\end{cases} \tag{50}$$

we obtain from (49)

$$\sigma_0 = \sum_{i=1,}^{2} C_i p_i \left\{ 1 - \sqrt{1 - \left(\frac{b_i^2}{a^2} \right)} \right\} \tag{51}$$

Once p_i, b_i, C_i and σ_0 are specified, the radius of the separation zone is determined from (51). It is noted that, when the loadings applied at the interface are unequal either in magnitude or extent, the radius of the separation zone is *dependent* on the elasticity parameters of both halfspace regions. When the uniform loads are equal in magnitude and act over equal areas (Fig. 2); i.e. $p_i = p_0$ and $b_i = b$, the expression (51) reduces to

$$\frac{\sigma_0}{p_0} = \left[1 - \sqrt{1 - \left(\frac{b}{a} \right)^2} \right] \tag{52}$$

which is identical to the result obtained by Barenblatt [2,3], admittedly, for obtaining the dimension of a decohesion zone around a closed crack located in a *homogeneous elastic solid.*

In the particular instance when the interface separation is induced by concentrated forces of magnitude P_n which act at the origin (Fig. 3), for the second sub-problem, we have

$$\sigma^*(r) = \frac{\delta(r)}{2\pi} \left[\frac{P_1(1 - \nu_1)\mu_2 + P_2(1 - \nu_2)\mu_1}{(1 - \nu_1)\mu_2 + (1 - \nu_2)\mu_1} \right] \tag{53}$$

where $\delta(r)$ is the Dirac delta function. The result (47) now gives the following closed form expression for the radius of the separation zone:

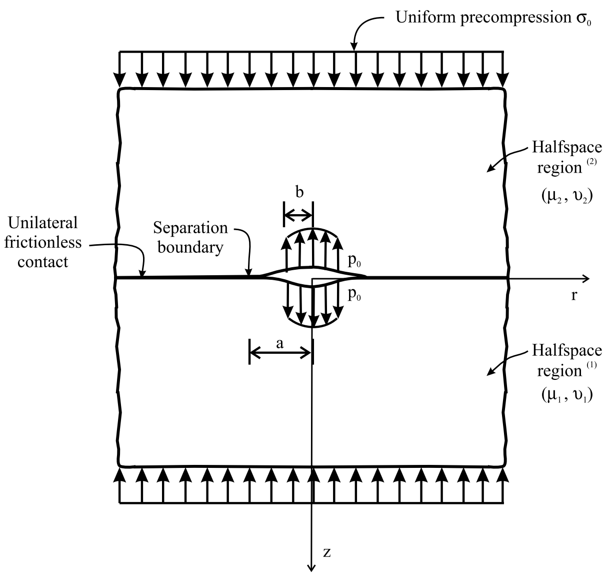

Fig. 2. Separation at the pre-compressed bimaterial elastic interface by tractions applied at the interface.

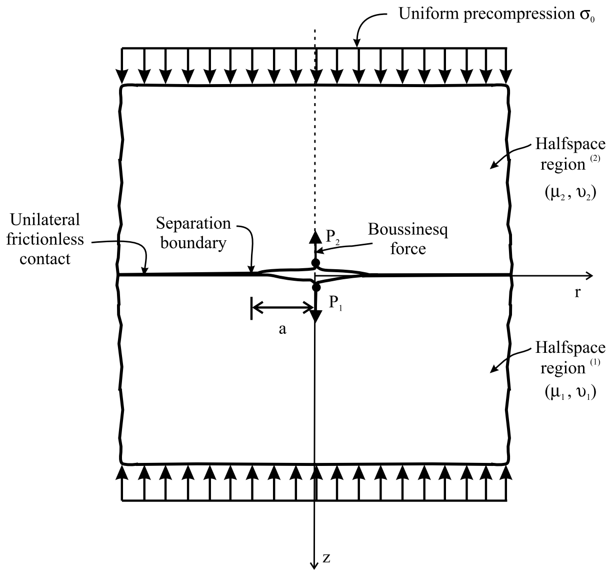

Fig. 3. Separation at the pre-compressed bimaterial elastic interface by unequal Boussinesq forces applied at the interface.

$$a = \left[\frac{1}{2\pi\sigma_0} \left(\frac{P_1(1-\nu_1)\mu_2 + P_2(1-\nu_2)\mu_1}{(1-\nu_1)\mu_2 + (1-\nu_2)\mu_1} \right) \right]^{1/2} \tag{54}$$

We note that this result can also be recovered from (51), for the problem of the separation induced by uniform circular loads of stress intensities p_i, by expanding the expression under the radical sign in (51) in powers of $(b_i/a_i)^2$ and taking the limit as $b_i \to 0$, with $\pi p_i b_i^2 \to P_i$.

It is of interest to examine result (54) more closely. The choice of the result for concentrated forces is dictated primarily by the fact that the expression for the radius of separation is obtained in a convenient exact closed form:

(i) For concentrated forces of equal magnitude P, the radius of the separation zone reduces to

$$a = \left[\frac{P}{2\pi\sigma_0} \right]^{1/2} \tag{55}$$

In other words, when Boussinesq-type concentrated loads of equal magnitude are applied at the *interface* of a uniformly pre-compressed *bimaterial elastic region*, the radius of the zone of separation is *independent* of the elasticity properties of either region.

(ii) When only *one* Boussinesq-type force acts at the *interface*, the radius of the zone of separation depends on the elasticity characteristics of *both* halfspace regions.

(iii) When the halfspace regions have the *same elasticity properties*, the radius of the zone of separation is *independent* of the elasticity characteristics of the halfspace region(s): i.e.

$$a = \left[\frac{P_1 + P_2}{4\pi\sigma_0}\right]^{1/2} \tag{56}$$

and in the instance when the Boussinesq forces are of equal magnitude, the result (56) reduces to (55).

(iv) When *one* elastic halfspace region is rigid, the radius of the separation zone is independent of the elasticity properties of the deformable halfspace region, and given by the result (55), with the Boussinesq force being interpreted appropriately.

We now focus attention to the case where a zone of separation is induced by loads, which are *located at the interior* of each halfspace region. Admittedly, it is rather cumbersome to develop generalized results in explicit form in view of the fact that the integrals encountered in the expressions such as (44) or (47) cannot be evaluated in a convenient closed form except in the simplest of cases. The implications of the action of loads, located at the interior of the halfspace regions can, nonetheless, be readily assessed through an evaluation of the explicit expression that is obtained for $\tilde{\sigma}^*(\xi)$. As an example, let us consider the problem of separation at the smoothly compressed bimaterial regions, induced by a set of Mindlin-type concentrated forces acting at finite distances from the interface (Fig. 4). Consider the action of a Mindlin force of magnitude R_1 acting at a distance c_1 from the traction free boundary of the halfspace region $^{(1)}$, and directed along the positive z-axis. The surface displacement of the halfspace region is given by Mindlin [35]

$$\left[u_z^{(1)}(r)\right]_{R_1} = \frac{R_1(1-\nu_1)}{2\pi\mu_1}\left[\frac{1}{(r^2+c_1^2)} + \frac{c_1^2}{2(1-\nu_1)(r^2+c_1^2)^{3/2}}\right] \tag{57}$$

Now consider the combined action of the Mindlin force R_1 and the interface tensile traction $\sigma^*(r)$, which acts over the entire surface of the halfspace region $r \in (0,\infty)$.

The Hankel transform of the resulting surface displacement of the halfspace region is given by

$$\tilde{u}_z^{(1)}(\xi) = \frac{(1-\nu_1)}{\xi\mu_1}\left[-\tilde{\sigma}^*(\xi) + \frac{R_1 e^{-\xi c_1}}{2\pi}\left\{1 + \frac{\xi c_1}{2(1-\nu_1)}\right\}\right] \tag{58}$$

We can develop a similar expression for the case where the halfspace region $^{(2)}$ is subjected to the combined action of a Mindlin force of magnitude R_2, which acts in the negative z-direction, while the surface is subjected to the tensile traction $\sigma^*(r)$, over the entire surface of the halfspace region $^{(2)}$, $r \in (0,\infty)$. For compatibility of the displacements at the interface for the sub-problem 2, given by (38), we require

$$\tilde{\sigma}^*(\xi) = \frac{1}{2\pi}\left[R_1 C_1 e^{-\xi c_1}\left\{1 + \frac{\xi c_1}{2(1-\nu_1)}\right\} + R_2 C_2 e^{-\xi c_2}\left\{1 + \frac{\xi c_2}{2(1-\nu_2)}\right\}\right] \tag{59}$$

which represents the tensile stress field for fictitious bilateral contact at the smoothly interacting halfspace regions due to the action of the Mindlin forces R_n. Unlike the expression (48) for the Hankel transform of the interface stresses due to the action of the Boussinesq forces at the plane boundary of the halfspace regions themselves, the expression (59) displays a dependency on the

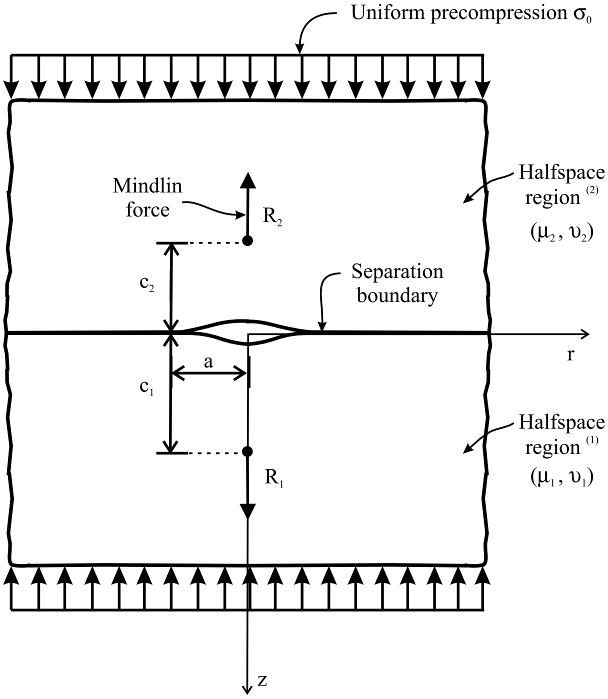

Fig. 4. Separation at the pre-compressed bimaterial elastic interface by unequal Mindlin forces applied at the interior of the halfspace regions.

elastic constants of both halfspace regions, even when the Mindlin forces are equal in magnitude and placed equidistant from the contact plane: i.e., when $R_1 = R_2 = R$ and $c_1 = c_2 = c$, (59) gives

$$\tilde{\sigma}^*(\xi) = \frac{R\mathrm{e}^{-\xi c}}{2\pi}\left[1 + \frac{\xi c}{2}\left\{\frac{\mu_1 + \mu_2}{(1 - v_1)\mu_2 + (1 - v_2)\mu_1}\right\}\right] \tag{60}$$

We observe that even when both halfspace regions are identical, the expression for $\tilde{\sigma}^*(\xi)$ depends on the elasticity properties of the halfspace region. Consequently, the radius of the separation zone will depend on the elasticity properties of both halfspaces in contact. Only when $R_n = R$ and $c_n \to 0$, will the dependency on the elastic constants vanish.

It is also natural to enquire whether the invariance property identified here could be attributed to the assumed uniform far field compression imposed on the bimaterial elastic halfspace regions. It is a relatively easy matter to prove that this is indeed the case. As an example, consider the case where the halfspace regions in frictionless contact are subjected to a non-uniform pre-compression by a pair of Mindlin-type forces of magnitude R which are situated at distances $z = \pm c$ from the smooth interface (Fig. 5). The distance c is assumed to be such that the Mindlin forces R acting by themselves do not initiate separation at the interface. In this case the contact stress distribution at the compressed frictionless interface can be obtained through an inversion of the result (60) followed by a change in sign to signify that the state of stress induced is compressive. i.e.

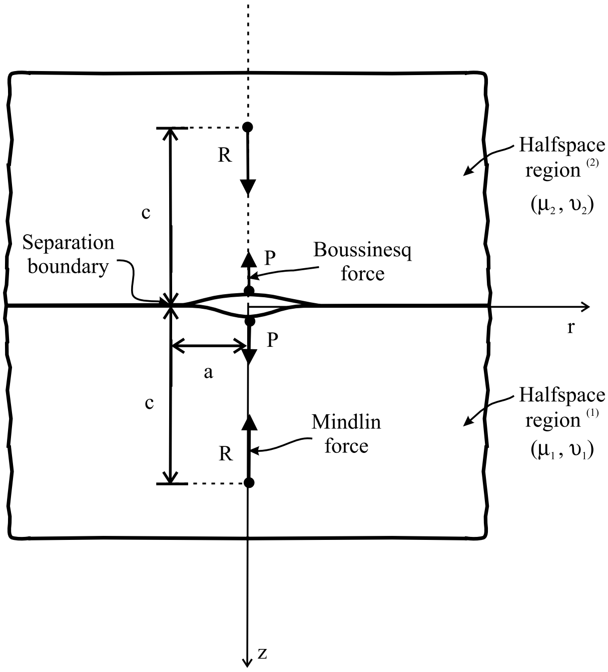

Fig. 5. Bimaterial elastic interface pre-compressed by equal Mindlin forces and separation induced by identical Boussinesq forces applied at the interface.

$$q(r) = -\frac{R}{2\pi}\left[\frac{c}{(r^2+c^2)^{3/2}} + \frac{c}{2}\left\{\frac{\mu_1+\mu_2}{(1-\nu_1)\mu_2+(1-\nu_2)\mu_1}\right\}\left\{\frac{2c^2-r^2}{(r^2+c^2)^{5/2}}\right\}\right] \tag{61}$$

The interface separation is induced by Boussinesq-type forces of equal magnitude P, which act directly on the interface. In this case, the stress intensity factor $[K_{\mathrm{I}}^a]_{\sigma_0}$ due to the initial pre-compression can be calculated by making use of the result (35). It is evident that this stress intensity factor is dependent on the elasticity parameters even in the instance when both halfspace regions are identical. For identical Boussinesq-type forces P acting on the faces of the two halfspace regions,

$$\sigma^*(r) = \frac{P\delta(r)}{2\pi} \tag{62}$$

and $[K_{\mathrm{I}}^a]_{p_n}$ is independent of the elasticity parameters. From (46) it is evident that, by virtue of (61), the radius of the zone of separation is dependent on the elasticity parameters of *both* halfspace regions.

5. Concluding remarks

This paper discusses the axisymmetric elastostatic problem of the unilateral smooth contact between dissimilar elastic halfspace regions, where the interface is subjected to uniform pre-compression, and separation is induced by axisymmetric loads applied at any location within the halfspace regions. A formal integral relationship can be developed for the evaluation of the radius of the zone of separation induced by these generalized loadings. In the particular instance when the loads inducing separation act directly at the pre-compressed interface, the radius of the zone of separation can be evaluated in explicit form. A *closed form* expression for the radius of the separation zone can be obtained for the specific case where the separation is caused by the action of *unequal Boussinesq-type forces*. It is shown that, in general, the radius of the separation zone is dependent on the elasticity characteristics of *both* halfspace regions. The result also reveals that, when the elastic halfspace regions are dissimilar and the Boussinesq-type loads are of equal magnitude, the radius of the zone of separation is *uninfluenced* by the elasticity characteristics of either elastic material. Also, when the elastic halfspace regions are identical, the radius of the zone of separation due to Boussinesq-type loadings is again uninfluenced by their elasticity properties. The examination of the problem of separation at the pre-compressed interface, which can be induced by unequal Mindlin-type forces located at the interior of the contacting halfspace regions, indicates that, in general, the radius of the zone of separation is dependent on the elasticity properties of both these contacting halfspace regions. Even in the case of an *identical doublet* of Mindlin-type forces acting at equidistant from the pre-compressed interface, the radius of the zone of separation will depend on the elasticity characteristics of both halfspace regions. As the Mindlin-type forces migrate towards the pre-compressed interface, the radius of the zone of separation is again uninfluenced by the elasticity characteristics of the halfspace regions. A continuation of the analysis to include a non-uniform axisymmetric pre-compression is performed by examining the case where the bimaterial interface is first compressed by a pair of Mindlin forces located remotely from the interface with separation being induced by Boussinesq-type forces that act at the plane boundaries. In this case it is shown that the radius of the separation zone now *depends* on the elastic constants of both halfspace regions. The results of this study can be expressed in the form of an invariance property, which can be stated as follows:

Lemma. *Consider two bimaterial elastic halfspace regions which are in smooth contact at the plane boundary and subjected to a far field uniform pre-compression normal to the contact plane. When separation is induced at the contact plane by axisymmetric stress fields of equal magnitude and distribution, which act directly at the contact plane, the radius of the separation zone is independent of the elasticity characteristics of both halfspace regions. When the loads inducing separation, and which act at the interface, are either unequal or located within the halfspace regions, the radius of the separation zone is dependent on the elasticity characteristics of both halfspace regions. The presence of a uniform pre-compression is an essential prerequisite to the validity of the invariance principle.*

Although unilateral contact problems are in general non-linear, we have shown that, owing to the assumed axial symmetry, the problem can be conveniently solved through the *superposition* of classical solutions derived for a penny shaped crack located at a bimaterial elastic interface, albeit with a relaxed interface constraint applied to the shear tractions acting at the interface. Several other limiting cases can be recovered from the exact closed form result. Furthermore, it is a relatively easy matter to show that the above Lemma also applies to dissimilar transversely isotropic elastic media where the planes of transverse isotropy coincide with the contacting plane. Further investigation is merited to establish whether the results of the invariance property, as covered by the above Lemma, extends to either non-Euclidean contact regions or problems dealing with non-axisymmetric states of deformation in the halfspace regions. In cases where *displacements* rather than *tractions* are imposed at the interface, existing studies clearly indicate that the radius of the separation zone does depend on the elasticity characteristics of both half-space regions. The invariance property establishes *a non-dependency on elasticity parameters* condition for a receding contact problem involving a bimaterial region. From the point of view of engineering applications involving *fluid pressure-induced separation* at a bi-material elastic interface, the advantages of the results are self-evident; the zone of separation at the bimaterial interface can be determined from only a knowledge of the pre-compression stress and the distribution of the normal pressure (usually induced by fluid pressures) either uniform or non-uniform, along the interface. Furthermore, certain results available in the literature for the decohesion problem for pressurized cracks with closed boundaries contained in a *homogeneous elastic solid* are equally applicable to the problem involving separation at a pre-compressed *bi-material elastic interface*.

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