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On certain bounds for the in-plane translational stiffness of a disc inclusion at a bi-material elastic interface

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Abstract

This paper presents a set of bounds that can be used to estimate the in-plane translational stiffness of a rigid circular disc inclusion that is embedded at the interface between two dissimilar elastic half-space regions.
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Keywords: Elastic bounds; Disc inclusion problem; In-plane loading of inclusion; Bi-material elastic interface

1. Introduction

In the context of geomechanics, the embedded inclusion serves as model for an anchoring region, which can be created by the injection of a cementitious material (Selvadurai, 1994). The evaluation of the elastic stiffness of these embedded anchoring devices is of particular interest, since the deviations from the elastic stiffness provides a useful indicator of local failure at either the interface or in adjacent regions. The idealization of the anchoring region as a disc shaped inclusion is, of course, a simplification; nonetheless, the introduction of the cementitious material under pressure can lead to hydraulic fracturing of the material in a plane normal to the least principal value of the geostatic stresses and the migration of the cementitious fluid within the narrow fracture invariably leads to a disc shaped anchoring region. Also the migration pattern of the viscous cementitious fluid within the fracture is a complex problem in itself. Often, the flat anchoring region will have an irregular shape that is largely determined by local inhomogeneities at the plane of the fracture. In this paper we focus on a study that deals with the mechanics of the in-plane loading of a rigid disc that is embedded in bonded contact at the interface between two dissimilar elastic media. An analytical solution to this problem is not available in the literature in elasticity dealing with either inclusion problems or composite materials (Mura, 1987, 1988). Such a result can be obtained by solving the integral equations resulting from a consideration of the continuity conditions at the bi-material interface exterior to the inclusion and the displacement conditions within the inclusion region. The objective of this paper is to present an alternative approximate procedure that allows the development of a set of "bounds" for the

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in-plane elastic stiffness of the circular disc inclusion that is embedded at the bi-material elastic interface. In a previous study, Selvadurai (1984) developed a set of bounds for the *axial stiffness* of a circular rigid disc inclusion that is embedded in bonded contact at a bi-material elastic interface, by imposing two sets of constraints at the interface; one refers to an *inextensibility constraint* at the bi-material elastic interface and the second refers to a *frictionless interface in bilateral contact*. In the limit of material incompressibility of both half-space regions, the bounds converge to the exact result. The exact formulation of the same problem was also considered by Selvadurai (2000), who reduced the analysis to the solution of a pair of coupled Fredholm integral equations of the second-kind, which were solved numerically to evaluate the axial stiffness of the rigid disc inclusion. Gladwell (1999) re-examined this problem through an alternative formulation that systematically uses Fourier and Abel transform techniques (see also Keer, 1975), and was able to develop an exact closed form result for the axial stiffness of the disc inclusion. Both investigations (Selvadurai, 2000; Gladwell, 1999) indicate that the result for the axial stiffness derived via either the numerical solution of the governing integral equations or from the exact closed form result is always within the *bounds* developed previously (Selvadurai, 1984). Selvadurai (1985) further extended the investigations to develop bounds for the axial and rotational elastic stiffness of rigid elliptical disc inclusions that are embedded in bonded contact at a bi-material elastic interface.

2. The inclusion problem

We consider the problem of a rigid circular disc inclusion of radius a, which is embedded in bonded contact at the interface between two isotropic elastic half-space regions with elastic constants G_i and ν_i where the subscripts $i = 1$ and 2 refer to the half space regions occupying $r \in (0, \infty); z \in (0, \infty)$ and $r \in (0, \infty); z \in (0, -\infty)$, respectively. The circular rigid disc inclusion is subjected to an in-plane force T in the x-direction, which induces a rigid translation Δ in the plane of the inclusion, and, in view of the dissimilar nature of the elastic properties of the two regions, we shall also assume that the bonded inclusion also experiences a rigid rotation about the diametral axis y (Fig. 1). The boundary conditions governing the problem can be stated in relation to the inclusion region and the region exterior to the inclusion. In the *interior* inclusion region we require

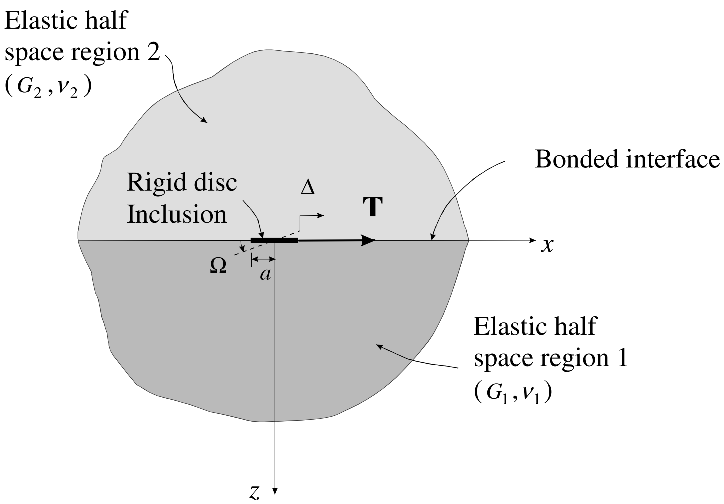

Fig. 1. Rigid disc inclusion embedded at a bonded bi-material elastic interface.

$$u_r^{(1)}(r,\theta,0) = \Delta\cos\theta = u_r^{(2)}(r,\theta,0); \quad u_\theta^{(1)}(r,\theta,0) = -\Delta\sin\theta = u_\theta^{(2)}(r,\theta,0); \quad r\in(0,a)$$
$$u_z^{(1)}(r,\theta,0) = \Omega r\cos\theta = u_z^{(2)}(r,\theta,0); \quad r\in(0,a) \tag{1}$$

and in the interface region *exterior* to the embedded disc inclusion we require the displacement continuity conditions

$$u_r^{(1)}(r,\theta,0) - u_r^{(2)}(r,\theta,0) = 0; \quad u_\theta^{(1)}(r,\theta,0) - u_\theta^{(2)}(r,\theta,0) = 0; \quad r\in(a,\infty)$$
$$u_z^{(1)}(r,\theta,0) - u_z^{(2)}(r,\theta,0) = 0; \quad r\in(a,\infty) \tag{2}$$

and the traction continuity conditions are

$$\sigma_{rz}^{(1)}(r,\theta,0) - \sigma_{rz}^{(2)}(r,\theta,0) = 0; \quad \sigma_{\theta z}^{(1)}(r,\theta,0) - \sigma_{\theta z}^{(2)}(r,\theta,0) = 0; \quad r\in(a,\infty)$$
$$\sigma_{zz}^{(1)}(r,\theta,0) - \sigma_{zz}^{(2)}(r,\theta,0) = 0; \quad r\in(a,\infty) \tag{3}$$

The formulation of the problem can be approached by adopting the solution of the equations of elasticity presented by Muki (1960), which are specific reductions of the Boussinesq–Somigliana–Galerkin approach (Gurtin, 1972), which states that the solution to the equations of elasticity can be represented in terms of a bi-harmonic function $\Phi^{(i)}(r,\theta,z)$ and harmonic function $\Psi^{(i)}(r,\theta,z)$: i.e.

$$\nabla^2\nabla^2\Phi^{(i)}(r,\theta,z) = 0; \quad \nabla^2\Psi^{(i)}(r,\theta,z) = 0; \quad (i=1,2) \tag{4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \tag{5}$$

is Laplace's operator in cylindrical polar coordinates. The displacements and stresses in the elastic medium can be obtained from the relationships

$$2G_i u_r^{(i)} = -\frac{\partial^2\Phi^{(i)}}{\partial r\partial z} + \frac{2}{r}\frac{\partial\Psi^{(i)}}{\partial\theta}; \quad 2G_i u_\theta^{(i)} = -\frac{1}{r}\frac{\partial^2\Phi^{(i)}}{\partial\theta\,\partial z} + 2\frac{\partial\Psi^{(i)}}{\partial r}$$
$$2G_i u_z^{(i)} = 2(1-\nu_i)\nabla^2\Phi^{(i)} - \frac{\partial^2\Phi^{(i)}}{\partial z^2} \tag{6}$$

and the stress components relevant to the formulation of the continuity conditions (3) are given by

$$\sigma_{zz}^{(i)} = \frac{\partial}{\partial z}\left[(2-\nu_i)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Phi^{(i)}; \quad \sigma_{\theta z}^{(i)} = \frac{1}{r}\frac{\partial}{\partial\theta}\left[(1-\nu_i)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Phi^{(i)} - \frac{\partial^2\Psi^{(i)}}{\partial r\,\partial z}$$
$$\sigma_{rz}^{(i)} = \frac{\partial}{\partial r}\left[(1-\nu_i)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]\Phi^{(i)} - \frac{1}{r}\frac{\partial^2\Psi^{(i)}}{\partial\theta\,\partial z} \tag{7}$$

The solutions for $\Phi^{(i)}(r,\theta,z)$ and $\Psi^{(i)}(r,\theta,z)$ should be such that, in addition to being single valued in the entire domain, they must satisfy the regularity conditions that ensure the vanishing of the displacements and stress fields as $(r^2+z^2)^{1/2}\to\infty$. The relevant solutions can be obtained through a Hankel transform development of the governing equations (see e.g. Sneddon, 1951): the solutions take the forms

$$\Phi^{(i)}(r,\theta,z) = \left\{\int_0^\infty \xi[A_i(\xi) + zB_i(\xi)]\exp\{(-1)^i\xi z\}J_1(\xi r)\,\mathrm{d}\xi\right\}\cos\theta; \quad (i=1,2) \tag{8}$$

$$\Psi^{(i)}(r,\theta,z) = \left\{\int_0^\infty \xi[C_i(\xi)]\exp\{(-1)^i\xi z\}J_1(\xi r)\,\mathrm{d}\xi\right\}\sin\theta; \quad (i=1,2) \tag{9}$$

where $A_i(\xi)$, $B_i(\xi)$ and $C_i(\xi)$ are six arbitrary functions and $J_1(\xi r)$ is the first-order Bessel function of the first kind. These solutions can be used in (6) and (7) to develop integral expressions for the displacement and stress components. Substituting the resulting expressions in the displacement boundary conditions (1) and the displacement and traction continuity conditions (2) and (3), we can arrive at a set of integral equations for the unknown functions. These integral equations can be further reduced by suitable substitutions and through the introduction of finite Fourier transforms, but the important deduction is that the solution of the resulting integral equations, even in their reduced forms, is a non-routine exercise (Selvadurai, 2000; Gladwell, 1999). The purpose of this paper is therefore to examine the development of a set of bounds, which is both convenient and accurate for the purposes of evaluation of the translational stiffness of the inclusion.

3. Bounds for the in-plane stiffness

The objective of the bounding technique is to introduce additional assumptions pertaining to the boundary and interface conditions at the bi-material interface, such that the resulting boundary value problem can be conveniently solved to generate the stiffness characteristics of the embedded inclusion, with the *reduced constraints*. We follow the approach employed by Selvadurai (1984) in the development of a set of bounds for the axial translation of the inclusion embedded in bonded contact at the bi-material interface and consider either a displacement–traction constraint or a traction–displacement constraint at the interface.

3.1. The displacement–traction constraint

The displacement–traction constraint is imposed by assuming that during the in-plane translation the rigid disc inclusion remains bonded to the two half-space regions, maintains frictionless contact in the region exterior to the inclusion and that the entire bi-material interface experiences zero displacements in the z-direction (Fig. 2). Even though there are mixed conditions related to the prescription of both dis-

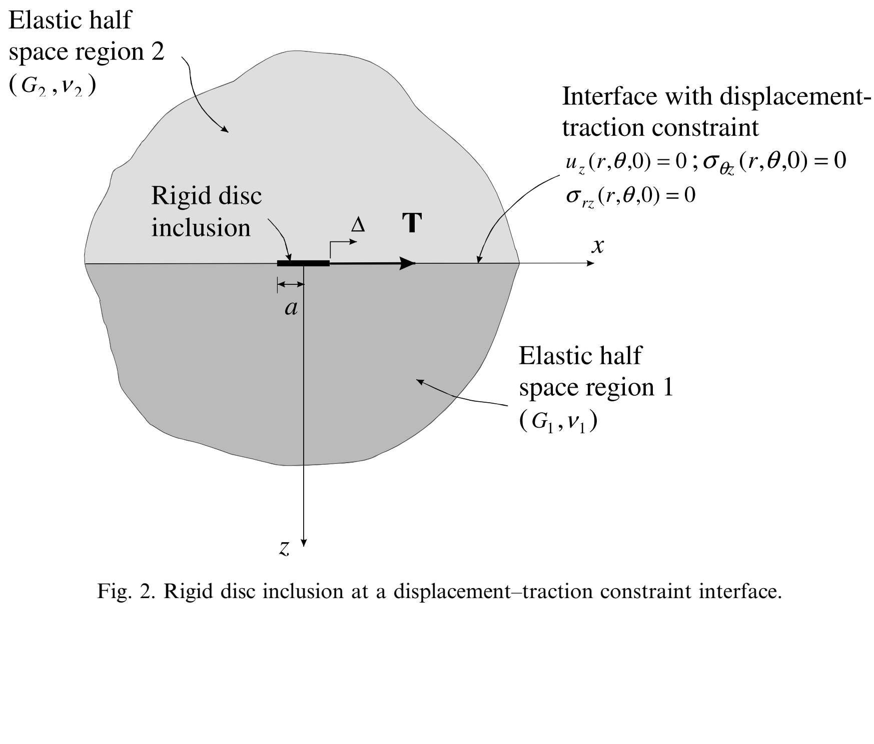

Fig. 2. Rigid disc inclusion at a displacement–traction constraint interface.

placement and traction conditions in the exterior region, the dominant constraint is the absence of axial displacements in the plane of the inclusion. The relevant displacement boundary conditions for the interface with the imposed constraint are

$$u_r^{(1)}(r,\theta,0) = \Delta\cos\theta = u_r^{(2)}(r,\theta,0); \quad u_\theta^{(1)}(r,\theta,0) = -\Delta\sin\theta = u_\theta^{(2)}(r,\theta,0); \\ r \in (0,a) \quad u_z^{(1)}(r,\theta,0) = u_z^{(2)}(r,\theta,0) = 0; \quad r \in (0,\infty) \tag{10}$$

and the corresponding traction boundary conditions are

$$\sigma_{rz}^{(1)}(r,\theta,0) = \sigma_{rz}^{(2)}(r,\theta,0) = 0; \quad \sigma_{\theta z}^{(1)}(r,\theta,0) = \sigma_{\theta z}^{(2)}(r,\theta,0) = 0; \quad r \in (a,\infty) \tag{11}$$

No assumption can be made with regard to the boundary conditions pertaining to the axial stresses $\sigma_{zz}^{(i)}$ in the region exterior to the disc inclusion. Even for the set of displacement and traction boundary conditions defined by (10) and (11), these will be non-zero. The mismatch in $\sigma_{zz}^{(i)}(r,0)$ in the exterior region is therefore an added constraint on the problem. The absence of compatibility of tractions is in the very nature of the approximation and has the advantage that the result for the in-plane stiffness can be evaluated in a convenient form. With these constraints, the problem is essentially reduced to the evaluation of the in-plane stiffness of a disc inclusion embedded in a homogeneous infinite space (Selvadurai, 1994; Keer, 1965; Kassir and Sih, 1968; Selvadurai and Singh, 1984), each part of the bi-material region corresponding to a part of an appropriate infinite space region. Considering *one-half of an infinite space region* with material properties corresponding to G_i and v_i, and assuming that a in-plane force T_i induces a in-plane translation Δ_i we have

$$T_i = \frac{32(1-v_i)G_i a\Delta_i}{(7-8v_i)} \tag{12}$$

Taking into account the total force on the inclusion and the requirement for the compatibility of the in-plane translation we have

$$T = T_1 + T_2; \quad \Delta = \Delta_1 = \Delta_2. \tag{13}$$

We obtain the following result as a bound for the in-plane translation for the disc inclusion:

$$\frac{T}{32a\Delta(G_1+G_2)} = \left[\frac{(7-8v_1)(1-v_2)+\Gamma(7-8v_2)(1-v_1)}{(1+\Gamma)(7-8v_1)(7-8v_2)}\right] \tag{14}$$

where $\Gamma = G_1/G_2$ is the mismatch in the linear elastic shear modulus between the two half-space regions. The result (14) will be referred to as the "upper bound" in keeping with the general observation that the elastic stiffness of a "constrained elastic medium" is generally higher than that of the unconstrained equivalent. We note that when $v_1 = v_2$, the in-plane stiffness of the embedded disc inclusion is dependent on the parameter $(1+\Gamma)$, which is basically the average of the shear moduli for the two media.

3.2. The traction–displacement constraint

We next consider the embedded disc inclusion problem where tractions are prescribed at the bi-material interface. Ideally, the traction-free constraint should be applied over the entire interface. In this case, however, the specification of a traction-free interface between the inclusion and the half-space regions will, by virtue of the action of an in-plane load, leads to an improperly posed inclusion problem. The alternative is to assume that the inclusion is embedded in bonded contact at the interface of two dissimilar half-space regions and that the surfaces exterior to the inclusion are traction-free. During the application of the in-plane force T, the inclusion will experience an in-plane translation Δ and a rotation Ω as shown in Fig. 3. The displacement boundary conditions associated with this problem are as follows: for $(i = 1, 2)$

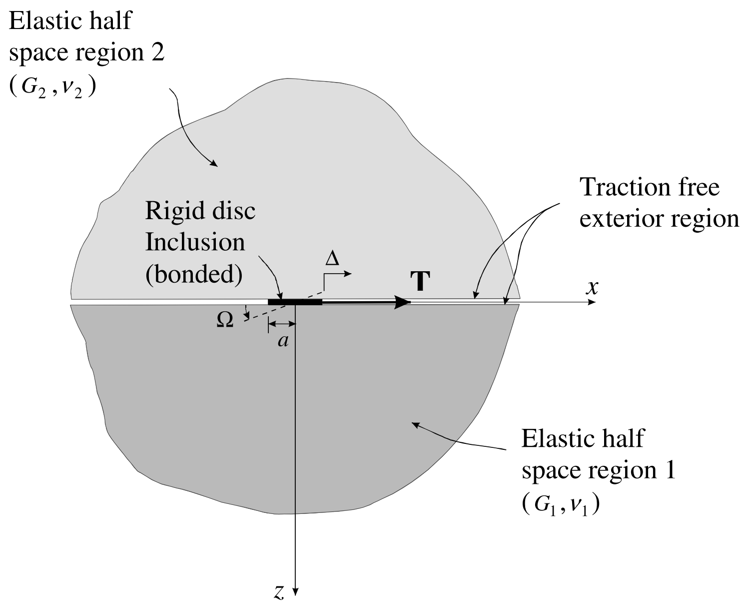

Fig. 3. Disc inclusion at a traction–displacement constraint interface.

$$u_r^{(i)}(r,\theta,0) = \Delta_i \cos\theta; \quad u_\theta^{(i)}(r,\theta,0) = -\Delta_i \sin\theta; \quad u_z^{(i)}(r,\theta,0) = \Omega_i r\cos\theta; \quad r \in (0,a) \tag{15}$$

and the traction boundary conditions are

$$\sigma_{rz}^{(i)}(r,\theta,0) = 0; \quad \sigma_{\theta z}^{(i)}(r,\theta,0) = 0; \quad \sigma_{zz}^{(i)}(r,\theta,0) = 0; \quad (i = 1,2); \quad r \in (a,\infty) \tag{16}$$

The inclusion problem is now reduced to identical mixed boundary problems associated with the in-plane translation of a disc bonded to a half-space region. This problem was examined by Ufliand (1956) and a detailed account of the developments is also given by Gladwell (1980). The results of primary interest to the development of the bounds are expressions that relate the applied in-plane force (T_i) to the in-plane translation (Δ_i) and the rotation (Ω_i). Further, we assume that the resultant moments M_i acting on the faces of the inclusion are identically zero; this gives

$$16G_i a\Delta_i = (2 + \beta_i + 3\varpi_i\alpha_i)T_i; \quad 16G_i a^2\Omega_i = 3\varpi_i T_i \tag{17}$$

where

$$\alpha_i = \frac{1}{2\pi}\ln(3 - 4\nu_i); \quad \beta_i = \frac{(1-2\nu_i)}{\pi\alpha_i}; \quad \varpi_i = \frac{\alpha_i\beta_i}{(1+\alpha_i^2)} \tag{18}$$

Combining these with the expressions for the force resultant and kinematic constraints

$$T = T_1 + T_2; \quad \Delta_1 = \Delta_2 = \Delta; \quad \Omega_1 = \Omega_2 = \Omega; \quad M_1 = M_2 = 0 \tag{19}$$

we obtain the following expression for the in-plane translational stiffness:

$$\frac{T}{32a\Delta(G_1+G_2)} = \left[\frac{(\varpi_1 + \Gamma\varpi_2)}{2\varpi_1\{2+\beta_2+3\varpi_2\alpha_2\} + 2\varpi_2\Gamma\{2+\beta_1+3\varpi_1\alpha_1\}}\right] \tag{20}$$

Again, we note that when the elastic half space regions have the same Poisson's ratios, the in-plane elastic stiffness of the embedded inclusion will be proportional to the average of the shear modulus of the two regions.

4. Concluding remarks

Combining the results presented in the previous section we obtain the following bounds for the *in-plane translational stiffness* of the rigid circular disc inclusion embedded at a bi-material elastic interface:

$$\left[\frac{(\varpi_1+\Gamma\varpi_2)}{2\varpi_1\{2+\beta_2+3\varpi_2\alpha_2\}+2\varpi_2\Gamma\{2+\beta_1+3\varpi_1\alpha_1\}}\right]\leqslant\frac{T}{32a\Delta(G_1+G_2)}\leqslant\left[\frac{(7-8\nu_1)(1-\nu_2)+\Gamma(7-8\nu_2)(1-\nu_1)}{(1+\Gamma)(7-8\nu_1)(7-8\nu_2)}\right] \tag{21}$$

As observed previously, when the values of Poisson's ratio for the two half-space regions are identical, the in-plane stiffness for the embedded disc inclusion is directly proportional to the average of the shear moduli of the two half-space regions. This is a useful result from the point of view of applications in geomechanics, where the shear modulus can exhibit wide variations but Poisson's ratio is sensibly constant. Also in the limit of incompressibility of the two half-space regions, $\nu_i \rightarrow 1/2$ with the result $\alpha_i \rightarrow 0$; $\beta_i \rightarrow 1$; $\varpi_i \rightarrow 0$ and the two bounds reduce to the exact result

$$\frac{T}{32a\Delta(G_1+G_2)}=\frac{1}{6} \tag{22}$$

Consider also the case when $\nu_i \equiv 0$; in this case the bounds (21) yield

$$\frac{\ln 3[4\pi^2+(\ln 3)^2]}{4[(1+\ln 3)(4\pi^2+(\ln 3)^2+3\ln 3)]}\cong\frac{1}{8}\leqslant\frac{T}{32a\Delta(G_1+G_2)}\leqslant\frac{1}{7} \tag{23}$$

Since the expression (21) is in an explicit form it can be evaluated to provide results of interest to applications in geomechanics. The set of bounds provides a useful approximate technique for the evaluation of the in-plane translational stiffness of disc-shaped inclusions such as anchoring regions embedded at the interface between dissimilar elastic geomaterials. Such bounds are of particular value in many engineering applications of a geomechanical nature, since the elastic properties of the geomaterials can be determined only in an approximate sense. The availability of a closed form result, albeit in the form of a set of bounds, allows the rapid estimation of the in-plane stiffness, for purposes of sensitivity analysis.

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