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Contaminant transport in a thin layer: the influence of fuzzy orthotropic diffusivity

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Abstract

The classical model of Fickian diffusion forms the basis for the study of the diffusive transport of chemical species in fluid-saturated porous media. In this paper, we examine the problems related to in-plane diffusive transport in a thin layer, the thickness of which is significantly smaller than the lateral dimensions, and where the diffusion coefficients are considered to be orthotropic. In particular, the emphasis of the paper is on the incorporation of fuzzy set theory in the description of the two principal diffusivity coefficients governing the contaminant transport in the porous medium. The results presented in the paper illustrate the influence of fuzzy descriptions of the orthotropic diffusivity characteristics of the medium on the time-dependent distribution of the concentration.

1. Introduction

The study of contaminant migration in porous media constitutes a problem of significant interest to geo-environmental engineering (see, e.g. Bear and Verruijt (1990), Bear and Bachmat (1992), Banks (1994), Rowe *et al* (1995), Sun (1996), David (1998)). In general, such transport takes place through a combination of advective and diffusive processes. In addition to these fundamental mechanical and physical processes, the migration of the chemical is also governed by chemical characteristics of the porous medium and that of the contaminant. The advective processes are largely governed by Darcy type flow processes and the diffusive part governed by Fickian processes, which depend on concentration gradients (Bear and Verruijt 1990). Although the advective movement constitutes the major mode of transport in contaminants in porous media, there are instances, involving low advective velocities, or stagnant regions, where the contamination migration is largely due to diffusion. This paper focuses on certain problems where diffusive contaminant transport takes place in a porous medium where the diffusivity properties are, in general, orthotropic. In addition to the inherent orthotropy, the diffusivity characteristics are also assumed to exhibit characteristic attributes of fuzzy sets.

Interval arithmetic measures are used to characterize the fuzzy diffusivity properties of the porous medium. Attention is focussed on the study of the influence of orthotropic diffusivity as well as their fuzzy descriptions on the in-plane contaminant migration in a thin porous layer, the thickness of which is small in comparison to the lateral dimensions. The type of problem that is being investigated has applicability in the diffusive transport that can take place in either a fluid-filled open fracture or a porous seam in a geological medium. In such a situation, the fracture effectively behaves as a porous medium. In particular, attention is first directed to the study of contaminant movement from a quarter-plane region of a diffusively orthotropic porous medium with a constant initial concentration. The analysis is also extended to the study of contaminant movement from a semi-infinite layer as well as from a rectangular region with a constant initial concentration. The study of these diffusion problems is facilitated by the general theorem pertaining to *product solutions* applicable to the analysis of various types of diffusive phenomena. The solutions for the deterministic problem as defined by the 'crisp orthotropic diffusivity parameters' can be obtained in convenient forms, consisting of combination of special functions and in series form. These analytical solutions are extended to include fuzzy descriptions of the diffusivities which reflect possible *uncertainties* or *variabilities* in the diffusion parameters, e.g. due to uncertain parameter identification or spatial inhomogenities. Specific numerical results, developed for the contaminant movement from a rectangular region, illustrate the manner in which both the degree of orthotropy and the fuzzy description of the diffusivities influence the pattern of in-plane contaminant migration in the porous medium.

2. Governing equations

The fundamental law describing diffusive transport of a chemical species in a fluid-saturated porous medium is that originally proposed by Fick (1855). The analogy between the diffusive transport problem and the transient heat conduction problem is widely recognized and documented in the classical texts by Carslaw and Jaeger (1959), Crank (1975) and others. For completeness, however, we shall present a brief derivation of the relevant equations, to highlight in particular, the product solutions approach for the development of solutions to two-dimensional and three-dimensional problems associated with transient problems. In this model, the diffusive chemical flux vector $\boldsymbol{F}$ is related to the gradient of the concentration $C(\boldsymbol{x}, t)$, where $\boldsymbol{x}$ is a position vector and t is time, according to

$$\boldsymbol{F} = -\boldsymbol{D}\nabla C, \tag{1}$$

where $\boldsymbol{D}$ is a matrix of diffusivity coefficients which takes the form

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}, \tag{2}$$

where 1, 2 and 3 are arbitrary directions. The negative sign in equation (1) indicates that diffusive transport takes place form regions of higher concentration to regions of lower concentration. From considerations of thermodynamics, it can be shown that the diffusivity matrix is symmetric, i.e.

$$\boldsymbol{D} = \boldsymbol{D}^{\mathrm{T}}, \tag{3}$$

where the superscript T refers to the transpose. Also from theorems in linear algebra, it can be shown that the eigenvalues of $\boldsymbol{D}$ are real and positive definite. Since the orientation of the axes 1, 2, 3 is arbitrary, we can choose the reference coordinate system for the problem such that the principal axes of diffusion coincide with the reference Cartesian coordinate system, x, y, z, such that

$$\boldsymbol{D} = \begin{bmatrix} D_{xx} & 0 & 0 \\ 0 & D_{yy} & 0 \\ 0 & 0 & D_{zz} \end{bmatrix}. \tag{4}$$

A medium, which can be characterized by the diffusivity matrix equation (4) is said to be diffusively orthotropic. Following developments presented in Selvadurai (2000), we consider a porous medium of domain V with boundary S and porosity n^* in which the diffusion takes place in an orthotropic fashion. The mass influx to a control volume V is given by

$$m_i = \iiint_V n^* \boldsymbol{F}\boldsymbol{n} \,\mathrm{d}S. \tag{5}$$

Considering equations (1) and (4) and the divergence theorem, we can write equation (5) in the form

$$m_i = \iiint_V n^* \nabla \cdot \tilde{\nabla} C \,\mathrm{d}V. \tag{6}$$

where $\tilde{\nabla}$ is now a modified gradient operator given by

$$\tilde{\nabla} = D_{xx}\boldsymbol{i}_x \frac{\partial}{\partial x} + D_{yy}\boldsymbol{i}_y \frac{\partial}{\partial y} + D_{zz}\boldsymbol{i}_z \frac{\partial}{\partial z}, \tag{7}$$

and $\boldsymbol{i}_x$, $\boldsymbol{i}_y$ and $\boldsymbol{i}_z$ are the unit vectors in the principal directions. The rate of accumulation of the chemical species in the porous medium is given by

$$m_a = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V n^* C \,\mathrm{d}V = \iiint_V n^* \frac{\partial C}{\partial t} \,\mathrm{d}V. \tag{8}$$

If the mass of the chemical species is conserved, we obtain from equations (6) and (8)

$$\iiint_V \nabla \cdot \tilde{\nabla} C \,\mathrm{d}V = \iiint_V \frac{\partial C}{\partial t} \,\mathrm{d}V \tag{9}$$

which, by virtue of the Dubois–Reymond lemma, gives the diffusion equation

$$\nabla \cdot \tilde{\nabla} C = D_{xx} \frac{\partial^2 C}{\partial x^2} + D_{yy} \frac{\partial^2 C}{\partial y^2} + D_{zz} \frac{\partial^2 C}{\partial z^2} = \frac{\partial C}{\partial t}. \tag{10}$$

In equation (10), the diffusion coefficients are assumed to be ‘crisp’ parameters in the context of the theory of fuzzy sets.

3. Initial boundary value problems

In this section, we consider, purely for completeness, problems related to in-plane diffusion in a porous domain, the thickness of which is substantially smaller than its lateral dimensions. Such a region can be visualized either as a saturated porous layer or a fluid-saturated open fracture. In both cases, the layer or the fracture is contained between either impermeable layers or layers through which no diffusion takes place. In this instance, homogeneous Neumann-type boundary conditions are applicable to the boundaries of the thin porous medium in contact with the impervious confining layers. For such situations, it can be shown that the *two-dimensional solution constitutes the exact solution*, even for the instance when the thickness of the porous layer is finite.

3.1. Diffusion in an orthotropic quarter-plane

As the first problem, we consider diffusion which takes place in the vicinity of a corner region of a quarter-plane region. The boundaries of the quarter-plane region correspond to $x = 0$ and $y = 0$, and the coordinate axes correspond to the principal directions of diffusivity. Within the context of this two-dimensional idealization, we can pose the initial boundary value problem in the following manner. A porous quarter-plane region with orthotropic diffusivity properties contains a chemical at a concentration C_0. At time $t = 0$, the chemical concentration at the boundaries of the quarter-plane region is reduced to zero. The first objective is to determine the solution to the initial boundary value problem where the diffusivity parameters are considered to be crisp parameters.

Since the problem is two-dimensional, the initial boundary value problem requires the solution of the governing partial differential equation

$$D_{xx}\frac{\partial^2 C}{\partial x^2} + D_{yy}\frac{\partial^2 C}{\partial y^2} = \frac{\partial C}{\partial t}, \qquad x \in (0, \infty), \quad y \in (0, \infty), \tag{11}$$

subject to the boundary conditions

$$C(x, 0, t) = 0, \qquad x \in (0, \infty), \tag{12}$$

$$C(0, y, t) = 0, \qquad y \in (0, \infty), \tag{13}$$

and the initial condition

$$C(x, y, 0) = C_0. \tag{14}$$

In addition to the boundary conditions in equations (12) and (13), we also require the solution to be bounded and finite as $(x, y) \to \infty$. This initial boundary value problem is well-posed in a Hadamard sense and the uniqueness of the solution is assured provided the diffusivity coefficients satisfy the constraints indicated previously (see, e.g. Selvadurai 2000).

The initial boundary value problem posed by equations (11) to (14) can be solved in a variety of ways, including the combined applications of Laplace and Fourier transforms to remove, respectively, the time and spatial variables and to reduce the resulting problem to one of transform inversion. An alternative technique involves the application of a 'product solutions approach', which states that the solution to the diffusion can be obtained as a product of solutions to two *one-dimensional initial boundary value problems* with appropriate initial conditions and boundary conditions. To illustrate the procedure, we transform the partial differential equation (11) by introducing the spatial variables

$$X = \frac{x}{D_{xx}}, \qquad Y = \frac{y}{D_{yy}}, \tag{15}$$

such that the partial differential equation now reduces to

$$\hat{\nabla}^2 C = \frac{\partial C}{\partial t}, \qquad X \in (0, \infty), \quad Y \in (0, \infty), \tag{16}$$

with

$$\hat{\nabla}^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} \tag{17}$$

and with the boundary conditions

$$C(X, 0, t) = 0, \qquad X \in (0, \infty), \quad t > 0, \tag{18}$$

$$C(0, Y, t) = 0, \qquad Y \in (0, \infty), \quad t > 0, \tag{19}$$

and the initial condition

$$C(X, Y, 0) = C_0. \tag{20}$$

The product solutions approach assumes that the solution to this revised initial boundary value problem can be expressed in the form

$$C(X, Y, t) = C_x(X, t)C_y(Y, t), \tag{21}$$

where the solutions $C_x(X, t)$ and $C_y(Y, t)$ are governed by the one-dimensional initial boundary value problems

$$\frac{\partial^2 C_x}{\partial X^2} = \frac{\partial C_x}{\partial t}, \qquad X \in (0, \infty), \tag{22}$$

$$C_x(0, t) = 0, \qquad t > 0, \tag{23}$$

$$C_x(X, 0) = C_0 \tag{24}$$

and

$$\frac{\partial^2 C_y}{\partial Y^2} = \frac{\partial C_y}{\partial t}, \qquad Y \in (0, \infty), \tag{25}$$

$$C_y(0, t) = 0, \qquad t > 0, \tag{26}$$

$$C_y(Y, 0) = 1. \tag{27}$$

The proof of the applicability of the product solutions approach for the solution of the diffusion equation is further discussed by Selvadurai (2000), and will not be repeated here. It is sufficient to note here the solutions of the reduced one-dimensional initial boundary value problems defined by equations (22)–(24) and equations (25)–(27). The solution to the first one-dimensional initial boundary value problem can be obtained in the form

$$C_x(X, t) = C_0 \operatorname{erf}\left(\frac{X}{2\sqrt{t}}\right), \tag{28}$$

where $\operatorname{erf}(X/2\sqrt{t})$ is the error function defined by

$$\operatorname{erf}\left(\frac{X}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{X/2\sqrt{t}} \mathrm{e}^{-\zeta^2}\, \mathrm{d}\zeta. \tag{29}$$

Similarly, the solution to the second one-dimensional initial boundary value problem is given by

$$C_y(Y, t) = \operatorname{erf}\left(\frac{Y}{2\sqrt{t}}\right). \tag{30}$$

Transforming equations (28) and (29) to the problem domain and making use of equation (20), we can now write the ‘crisp solution’ to the diffusive transport in the vicinity of the corner region of a quarter-plane with orthotropic diffusivity characteristics as follows:

$$C(x, y, t) = C_0 \operatorname{erf}\left(\frac{x}{2\sqrt{D_{xx}t}}\right) \operatorname{erf}\left(\frac{y}{2\sqrt{D_{yy}t}}\right). \tag{31}$$

3.2. Diffusion from an orthotropic semi-infinite layer

We next consider the problem of the in-plane contaminant migration from a semi-infinite layer of finite width l_y, which is embedded between impermeable regions. The semi-infinite layer is initially at a constant concentration C_0 throughout the semi-infinite layer. The boundaries of the layer are maintained at a zero value for $t > 0$. The initial boundary value problem governing the diffusion problem is posed in the following.

The partial differential equation governing the diffusion problem is given by

$$\hat{\nabla}^2 C = \frac{\partial C}{\partial t}, \qquad X \in (0, \infty), \quad Y \in \left(0, \frac{l_y}{\sqrt{D_{yy}}}\right), \tag{32}$$

where X,Y are the transformed spatial variables. The boundary conditions governing the initial boundary value problem are

$$C(X, 0, t) = 0, \qquad X \in (0, \infty), \quad t > 0, \tag{33}$$

$$C\left(X, \frac{l_y}{\sqrt{D_{yy}}}, t\right) = 0, \qquad X \in (0, \infty), \quad t > 0, \tag{34}$$

$$C(0, Y, t) = 0, \qquad Y \in \left(0, \frac{l_y}{\sqrt{D_{yy}}}\right), \quad t > 0. \tag{35}$$

The initial condition is

$$C(X, Y, 0) = C_0, \qquad X \in (0, \infty), \quad Y \in \left(0, \frac{l_y}{\sqrt{D_{yy}}}\right). \tag{36}$$

We adopt a 'product solutions approach' to the analysis of the initial boundary value problem given by equations (32)–(36). Using the representation in equation (21), the original initial boundary value problem is reduced to two appropriate one-dimensional problems. The details of the method of analysis will not be pursued here; suffice it to note that the solution for the one-dimensional problem involving the domain $X \in (0, \infty)$ can be obtained in terms of the error function and the solution to the problem involving the domain $Y \in (0, l_y/\sqrt{D_{yy}})$ can be obtained in a series form (Selvadurai 2000). The final solution for the time-dependent decay of concentration in the layer, expressed in terms of the spatial variables x and y takes the form

$$C(x, y, t) = C_0 \operatorname{erf}\left(\frac{x}{2\sqrt{D_{xx}t}}\right) \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - \cos(n\pi)] \exp\left(-\frac{n^2\pi^2 D_{yy} t}{l_y^2}\right) \sin\left(\frac{n\pi y}{l_y}\right). \tag{37}$$

3.3. Diffusion from an orthotropic rectangular region

We next consider the related problem of the in-plane diffusion from a rectangular region with orthotropic diffusivity properties. The in-plane diffusion takes place from a rectangular region of dimensions $x \in (0, l_x)$ and $y \in (0, l_y)$. The rectangular region has a constant concentration C_0, and at time t, the boundaries of the rectangular region are reduced to zero. The resulting initial boundary value problem is governed by the partial differential equation

$$\hat{\nabla}^2 C = \frac{\partial C}{\partial t}, \qquad X \in \left(0, \frac{l_x}{\sqrt{D_{xx}}}\right), \qquad Y \in \left(0, \frac{l_y}{\sqrt{D_{yy}}}\right), \tag{38}$$

where X, Y are the transformed spatial variables. The boundary conditions governing the initial boundary value problem are

$$C(X, 0, t) = 0, \qquad X \in \left(0, \frac{l_x}{\sqrt{D_{xx}}}\right), \quad t > 0, \tag{39}$$

$$C\left(X, \frac{l_y}{\sqrt{D_{yy}}}, t\right) = 0, \qquad X \in \left(0, \frac{l_x}{\sqrt{D_{xx}}}\right), \quad t > 0, \tag{40}$$

$$C(0, Y, t) = 0, \qquad Y \in \left(0, \frac{l_y}{\sqrt{D_{yy}}}\right), \quad t > 0, \tag{41}$$

$$C\left(\frac{l_x}{\sqrt{D_{xx}}}, Y, t\right) = 0, \qquad Y \in \left(0, \frac{l_y}{\sqrt{D_{yy}}}\right), \quad t > 0. \tag{42}$$

The initial condition is

$$C(X, Y, 0) = C_0, \qquad X \in \left(0, \frac{l_x}{\sqrt{D_{xx}}}\right), \quad Y \in \left(0, \frac{l_y}{\sqrt{D_{yy}}}\right). \tag{43}$$

Again, the solution to the initial boundary value problem posed by equations (38)–(43) can be obtained by adopting the product solutions approach as outlined previously. Avoiding details, it can be shown that the time-dependent distribution of concentration in the rectangular region can be obtained in the form

$$C(x, y, t) = C_0 \sum_{m=1}^{\infty} \frac{2}{m\pi} [1 - \cos(m\pi)] \exp\left(-\frac{m^2\pi^2 D_{xx} t}{l_x^2}\right) \sin\left(\frac{m\pi x}{l_x}\right) \times \sum_{n=1}^{\infty} \frac{2}{n\pi} [1 - \cos(n\pi)] \exp\left(-\frac{n^2\pi^2 D_{yy} t}{l_y^2}\right) \sin\left(\frac{n\pi y}{l_y}\right). \tag{44}$$

This completes the development of analytical solutions which will form the basis for the extension of the diffusion problem to include non-crisp or fuzzy data with respect to the orthotropic diffusivity parameters.

4. Fuzzy arithmetical concept

4.1. Fuzzy sets and fuzzy numbers

The theory of fuzzy sets was introduced by Zadeh (1965) as an extension or generalization of classical set theory. According to the formulation of Cantor presented in the late nineteenth century, a classical or so-called crisp set A, $A \subseteq U$, can be defined as a collection of objects or elements $u \in U$, which are characterized by some well-defined property. Hence, if an element shows this property, it belongs to the set A, otherwise, it is excluded. Such a classical set can be described in different ways: in case of sets with a finite, countable number of elements, we can list the elements that belong to the set, e.g. $A = \{1, 3, 7\}$, or else, we can describe the set analytically, for instance by prescribing the conditions for membership; e.g. $A = \{u | u \leqslant 7\}$.

When we consider, as an example, the continuous and non-countable universal set U of possible outside temperatures u in degrees Celsius, we can use classical set theory and define the set A of 'freezing temperatures' by

$$A = \{u \in U | u < 0\}. \tag{45}$$

Thus, the property 'u is the freezing temperature' allows a non-ambiguous definition of the set A, i.e. it allows a clear distinction between the elements that belong to A, and those which do not.

Classical set theory, however, reaches its limits when the property that determines the membership of an element to a set, is formulated in such a way that a clear distinction between either membership or exclusion is no longer possible. As an example, let us consider the following subjective question: How does the set $\tilde{A}$ of 'low temperatures' look like? Even though the classification of the temperatures is, of course, very much dependent on the personal perception of 'low temperature', or 'cold', respectively, it is obvious that a clear split of the universal set into elements that definitely belong to the set, and those that are completely excluded, does no longer make sense. The notion of a fuzzy property 'u is a low temperature' for the set $\tilde{A}$ necessitates an extension of classical set theory towards a generalized set theory, where in addition to membership and exclusion there is also the possibility for the provision of gradations between the two groups.

Against this background, so-called fuzzy sets can be introduced as a generalization of conventional sets by allowing elements of a universal set not only to entirely belong or not to belong to a specific set, but also to belong to the set to a certain degree (Zadeh 1965). Thus, fuzzy sets can be expressed by the elements u of a universal set U with a certain degree of pre-assumed membership $\mu(u) \in [0, 1]$. The elements u belonging to conventional sets, instead, are characterized by degrees of membership that can only be equal to either zero or unity, i.e. by a membership function $\mu(u) \in \{0, 1\}$. A very promising application of the theory of fuzzy sets, which is rather different from the well-established use of fuzzy set theory in fuzzy control, is the numerical implementation of uncertain model parameters as fuzzy numbers (Hanss 1999). Basically, fuzzy numbers can be considered as a special class of fuzzy sets (Zadeh 1965) showing some specific properties (Kaufmann 1991). On this basis, closed intervals and crisp numbers of the form

$$[a, b] = \{u | a \leqslant u \leqslant b\} \quad \text{and} \quad c = \{u | u = c\}, \qquad u \in \mathbb{R}, \tag{46}$$

can be considered as conventional subsets of the universal set $\mathbb{R}$ which can also be expressed by

$$\mu_{[a,b]}(u) = \begin{cases} 1 & \text{for} \quad a \leqslant u \leqslant b \\ 0 & \text{for other values} \end{cases} \qquad \text{and} \qquad \mu_c(u) = \begin{cases} 1 & \text{for } u = c \\ 0 & \text{for other values} \end{cases} \tag{47}$$

when using the membership function $\mu(u) \in \{0, 1\}$, $u \in \mathbb{R}$ (figure 1). Fuzzy numbers, instead, are defined as convex fuzzy sets over the universal set $U = \mathbb{R}$ with their membership functions $\mu(u) \in [0, 1]$, where $\mu(u) = 1$ is true only for one single value $u = \bar{m} \in \mathbb{R}$. As an example, symmetric fuzzy numbers of a quasi-Gaussian shape can be defined by the membership function

$$\begin{aligned} &\mu(u) = \exp\left[-\frac{(u - \bar{m})^2}{2\sigma^2}\right] \qquad \text{for } |u - \bar{m}| \leqslant 3\sigma \\ &\mu(u) = 0 \qquad \text{for } u > \bar{m} + 3\sigma \quad \text{or} \quad u < \bar{m} - 3\sigma \end{aligned} \tag{48}$$

with $\bar{m}$ and σ denoting the mean value and the standard deviation of the Gaussian distribution (figure 1). In this paper, fuzzy numbers of a quasi-Gaussian type will be used to represent the uncertain model parameters, where the mean value $\bar{m}$ can then be considered as the most likely value of the parameter.

4.2. Transformation method

To carry out arithmetical operations between fuzzy numbers, a 'standard fuzzy arithmetic' has been defined (Alefeld 1983, Kaufmann 1991), where each fuzzy number is decomposed into a

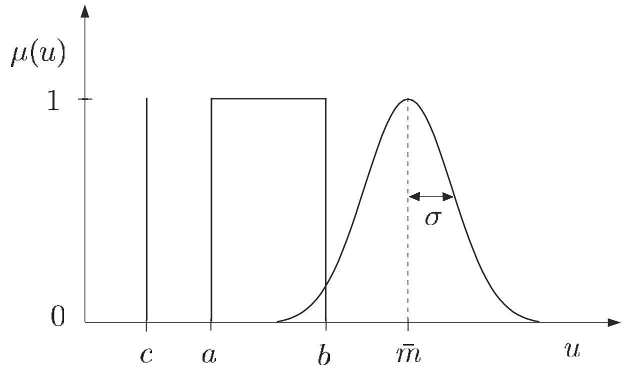

Figure 1. Closed interval $[a, b]$, crisp number c and symmetric fuzzy number of quasi-Gaussian shape (mean value $\bar{m}$, standard deviation σ) expressed by their membership functions.

set of intervals for the different levels of membership, so that conventional interval arithmetic can be applied separately for each level of membership. Upon closer examination, however, it is clear that the application of *standard fuzzy arithmetic* to the simulation of *real-world systems* with *uncertain model parameters* is undesirable. It has been shown that, as a serious drawback of this method, the results calculated by standard fuzzy arithmetic normally do not reflect the real result of the problem, but a result of much higher uncertainty (Hanss 2000, 2002). This drawback, however, can effectively be avoided by using the transformation method proposed by Hanss (2000, 2002) as a practical implementation of fuzzy arithmetic. In the following, the transformation method is presented only in its reduced form which is sufficient for the problem discussed. It can be used for both the simulation and the analysis of systems with uncertain parameters.

4.2.1. Simulation of systems with uncertain parameters. Given a problem with n independent parameters, which are assumed to be uncertain, the parameters can be represented by fuzzy numbers $\tilde{p}_i$ with the membership functions $\mu_{\tilde{p}_i}(x_i)$, $i = 1, 2, \ldots, n$, each decomposed into a set P_i of $(m + 1)$ intervals $X_i^{(j)}$, $j = 0, 1, \ldots, m$, of the form

$$P_i = \left\{X_i^{(0)}, X_i^{(1)}, \ldots, X_i^{(m)}\right\} \tag{49}$$

with

$$X_i^{(j)} = \left[a_i^{(j)}, b_i^{(j)}\right], \quad a_i^{(j)} \leqslant b_i^{(j)}, \qquad i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, m. \tag{50}$$

For the purpose of decomposition, the μ-axis is subdivided into m segments, equally spaced by $\Delta\mu = 1/m$ (figure 2). The $(m + 1)$ levels of membership μ_j are then given by

$$\mu_j = \frac{j}{m}, \qquad j = 0, 1, \ldots, m. \tag{51}$$

Now, instead of applying standard interval arithmetic directly to the intervals $X_i^{(j)}$, $i = 1, 2, \ldots, n$, for each level of membership μ_j, $j = 0, 1, \ldots, m$, the intervals can now be transformed into arrays $\hat{X}_i^{(j)}$ of the following form:

$$\hat{X}_i^{(j)} = \big(\overbrace{\alpha_i^{(j)}, \beta_i^{(j)}, \alpha_i^{(j)}, \beta_i^{(j)}, \ldots, \alpha_i^{(j)}, \beta_i^{(j)}}^{2^{i-1}\text{pairs}}\big) \tag{52}$$

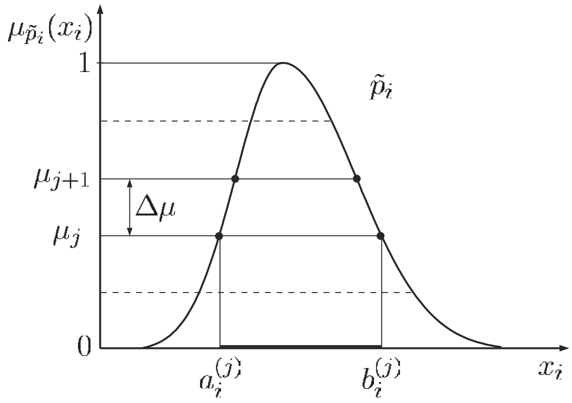

Figure 2. Implementation of the ith uncertain parameter as a fuzzy number $\tilde{p}_i$ decomposed into intervals.

with

$$\alpha_i^{(j)} = \big(\underbrace{a_i^{(j)}, \ldots, a_i^{(j)}}_{2^{n-i}\ \text{elements}}\big), \qquad \beta_i^{(j)} = \big(\underbrace{b_i^{(j)}, \ldots, b_i^{(j)}}_{2^{n-i}\ \text{elements}}\big). \tag{53}$$

Note that $a_i^{(j)}$ and $b_i^{(j)}$ are the lower and upper bounds of the interval at the membership level μ_j for the ith uncertain model parameter.

Assuming that the system which is to be simulated is given by an arithmetical expression F with the functional form

$$\tilde{q} = F(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n), \tag{54}$$

its estimation is then carried out by evaluating the expression separately at each of the positions of the arrays using the conventional arithmetic for crisp numbers. Thus, if the output $\tilde{q}$ of the system can be expressed in its decomposed and transformed forms by the arrays $\hat{Z}^{(j)}$, $j = 0, 1, \ldots, m$, the kth element ${}^k\hat{z}^{(j)}$ of the array $\hat{Z}^{(j)}$ is then given by

$${}^k\hat{z}^{(j)} = F\big({}^k\hat{x}_1^{(j)}, {}^k\hat{x}_2^{(j)}, \ldots, {}^k\hat{x}_n^{(j)}\big), \qquad k = 1, 2, \ldots, 2^n, \tag{55}$$

where ${}^k\hat{x}_i^{(j)}$ denotes the kth element of the array $\hat{X}_i^{(j)}$. Finally, the fuzzy-valued result $\tilde{q}$ of the problem can be achieved in its decomposed form

$$Z^{(j)} = [a^{(j)}, b^{(j)}], \qquad j = 0, 1, \ldots, m, \tag{56}$$

by retransforming the arrays $\hat{Z}^{(j)}$ according to the recursive formulae

$$\begin{aligned} a^{(j)} &= \min_k(a^{(j+1)}, {}^k\hat{z}^{(j)}), \\ b^{(j)} &= \max_k(b^{(j+1)}, {}^k\hat{z}^{(j)}), \end{aligned} \qquad j = 0, 1, \ldots, m-1, \tag{57}$$

and

$$a^{(m)} = \min_k({}^k\hat{z}^{(m)}) = \max_k({}^k\hat{z}^{(m)}) = b^{(m)}. \tag{58}$$

4.2.2. Analysis of systems with uncertain parameters. Until now, the fuzzy-valued result for a problem only shows the overall combined influence of all the uncertain parameters. However, it is possible to determine the proportions to which the n uncertain parameters of the system *separately contribute* to the overall uncertainty of the system output. Instead of reducing the array $\hat{Z}^{(j)}$ to the interval $Z^{(j)}$, as done in the retransformation procedure defined by equation (57), the supplementary information given by the values and the arrangement of the elements in $\hat{Z}^{(j)}$ can be used. For this purpose, the coefficients $\eta_i^{(j)}$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, (m-1)$, are to be determined according to

$$\eta_i^{(j)} = \frac{1}{2^{n-1}(b_i^{(j)} - a_i^{(j)})} \sum_{k=1}^{2^{n-i}} \sum_{l=1}^{2^{i-1}} ({}^{k_2}\hat{z}^{(j)} - {}^{k_1}\hat{z}^{(j)}) \tag{59}$$

with

$$k_1 = k + (l-1)\, 2^{n-i+1} \qquad \text{and} \qquad k_2 = k + (2l-1)\, 2^{n-i}. \tag{60}$$

The values $a_i^{(j)}$ and $b_i^{(j)}$ denote the lower and upper bound of the interval $X_i^{(j)}$, and ${}^k\hat{z}^{(j)}$ is the kth element of the array $\hat{Z}^{(j)}$. The coefficients $\eta_i^{(j)}$ can be interpreted as gain factors that express the effect of the uncertainty of the ith parameter on the uncertainty of the output z of the problem at the membership level μ_j. More explicitly, within the range of uncertainty covered at the membership level μ_j, deviations $\Delta z^{(j)}$ from the central value of the output fuzzy number $\tilde{q}$ can be considered as being related to the corresponding deviations $\Delta x_i^{(j)}$ from the central value of the fuzzy parameters $\tilde{p}_i$, $i = 1, 2, \ldots, n$, by the approximation

$$\Delta z^{(j)} \approx \sum_{i=1}^{n} \eta_i^{(j)} \Delta x_i^{(j)}. \tag{61}$$

To achieve a non-dimensional form of the uncertain parameters $\tilde{p}_i$, the standardized mean gain factors κ_i can be determined as an overall measure of influence according to

$$\kappa_i = \frac{\sum_{j=1}^{m-1} \mu_j |\eta_i^{(j)}(a_i^{(j)} + b_i^{(j)})|}{2\sum_{j=1}^{m-1} \mu_j} = \frac{1}{m-1} \sum_{j=1}^{m-1} \mu_j \big|\eta_i^{(j)}(a_i^{(j)} + b_i^{(j)})\big|. \tag{62}$$

And finally, as a relative measure of influence, the normalized values ρ_i can be determined for $i = 1, 2, \ldots, n$ according to

$$\rho_i = \frac{\kappa_i}{\sum_{q=1}^{n} \kappa_q} = \frac{\sum_{j=1}^{m-1} \mu_j |\, \eta_i^{(j)}(a_i^{(j)} + b_i^{(j)})|}{\sum_{q=1}^{n} \sum_{j=1}^{m-1} \mu_j |\eta_q^{(j)}(a_q^{(j)} + b_q^{(j)})|} \tag{63}$$

satisfying the consistency condition

$$\sum_{i=1}^{n} \rho_i = 1. \tag{64}$$

The gain factors κ_i quantify the effect of the ith varying parameter $\tilde{p}_i$ on the overall variation of the problem output $\tilde{q}$, assuming every parameter to be varied relatively to the same extent. The origin of equations (59), (60) and (63) can be seen in the existence of special patterns in the elements of $\hat{Z}^{(j)}$ in case of complete independence of the output $\tilde{q}$ from a specific model parameter $\tilde{p}_i$. This pattern can numerically be characterized by the average difference between the values of specific columns which on its part can be generalized to the quantification of arbitrary degrees of dependence.

The question arises as to the basic differences or advantages offered by the fuzzy method as opposed to a conventional sensitivity analysis involving the material parameters and their appropriate ranges. It should be remarked that the fuzzy approach is a more complete analysis that involves a systematic choice of ranges in the values of the parameters. This is in contrast to a sensitivity analysis that will incorporate only the extreme values associated with the parameters. The mathematical basis for the inclusivity of the choice of parameters in the fuzzy methodology is further discussed by Hanss and Klimke (2003).

5. Numerical results

We now apply the fuzzy arithmetical concept described in the previous section to examine the problem of in-plane diffusion in an orthotropic porous square region of dimension $x \in (0, l_x)$ and $y \in (0, l_y)$ with $l_x = l_y = 0.1$ m, as presented in section 3.3. The region has constant initial concentration C_0, and for $t > 0$, the concentration at the boundaries of the rectangular plate is reduced to zero.

For the simulation of the system, the diffusivity coefficients D_{xx} and D_{yy} are regarded as uncertain model parameters, represented by fuzzy numbers with a distribution, which is of a symmetric quasi-Gaussian shape (figure 1). This can empirically be considered as a practical assumption. We assume that the mean values $\bar{m}_x$ and $\bar{m}_y$ of the fuzzy-valued diffusivities $\tilde{D}_{xx}$ and $\tilde{D}_{yy}$ are related through

$$\bar{m}_y = \lambda \bar{m}_x, \qquad \lambda \geqslant 1, \tag{65}$$

where $\lambda > 1$ expresses the orthotropic case with the dominating diffusivity D_{yy}, and $\lambda = 1$ leads to the *marginal case of isotropy*.

As an illustation, the porous medium is assumed to be orthotropic with $\lambda = 5$. The actual settings for the mean values $\bar{m}_x$ and $\bar{m}_y$ as well as for the standard deviations σ_x and σ_y of the fuzzy-valued diffusivities $\tilde{D}_{xx}$ and $\tilde{D}_{yy}$ are presented as

$$\begin{aligned} \bar{m}_x &= 1.0 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}, \qquad \sigma_x = 7\%\bar{m}_x, \\ \bar{m}_y &= 5.0 \times 10^{-6}\,\mathrm{m^2\,s^{-1}} = 5\bar{m}_x, \qquad \sigma_y = 2\%\bar{m}_x. \end{aligned} \tag{66}$$

Thus, the uncertain diffusivity coefficients $\tilde{D}_{xx}$ and $\tilde{D}_{yy}$ are set to cover a worst-case scenario with ranges of $\pm 21\%$ and $\pm 6\%$ from their mean values, $\bar{m}_x$ and $\bar{m}_y$, respectively.

By evaluating equation (44) using the reduced transformation method, the uncertain system can be simulated and the uncertain concentration $\tilde{C}(x, y, t)$ can be determined for any time t and at any location (x, y) within the square region. As an example, the uncertain (normalized) concentration $\tilde{C}/C_0$ at the location ($x^* = l_x/2 = 0.05$ m, $y^* = l_y/2 = 0.05$ m) and at time $t^* = 200$ s is shown in figure 3. The fuzzy-valued concentration is again of (nearly) a symmetric quasi-Gaussian shape, which indicates that the fuzziness-induced nonlinearities in

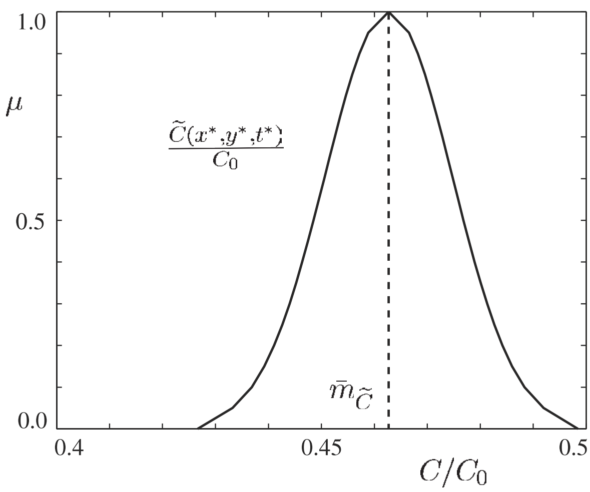

Figure 3. Uncertain normalized concentration $\tilde{C}(x^*, y^*, t^*)/C_0$ at the location $(x^*, y^*) = (l_x/2, l_y/2) = (0.05\,\mathrm{m}, 0.05\,\mathrm{m})$ and at the time $t^* = 200$ s.

the system equations have only a moderate effect on the simulated concentration within the considered ranges of uncertainty. The mean value of the normalized uncertain concentration amounts to $\bar{m}_{\tilde{C}} = 0.463$, and the worst-case range is given by the interval at $\mu = 0$, i.e. by [0.427, 0.499], which accounts for relative worst-case deviations of approximately $\pm 7.8\%$ from the mean value $\bar{m}_{\tilde{C}}$.

As a result of the analysis of the uncertain system, using equations (59), (60), (62) and (63), the relative degrees of influence $\rho_x^{\tilde{C}}(x, y, t)$ and $\rho_y^{\tilde{C}}(x, y, t)$ can be determined which for $x^* = l_x/2 = 0.05$ m, $y^* = l_y/2 = 0.05$ m and $t^* = 200$ s amount to

$$\rho_x^{\tilde{C}}(x^*, y^*, t^*) = 8.3\% \qquad \text{and} \qquad \rho_y^{\tilde{C}}(x^*, y^*, t^*) = 91.7\%. \tag{67}$$

Thus, as expected, the influence of the uncertainty associated with the diffusivity coefficient $\tilde{D}_{yy}$ on the concentration $\tilde{C}(x^*, y^*, t^*)$ is considerably larger than that resulting from the parameter $\tilde{D}_{xx}$.

Another interesting problem of the uncertain system results from the following question: at what time $t = T$, would the concentration $C(x^*, y^*, t)$ at a given location (x^*, y^*) have fallen on a certain proportion ε of the initial concentration C_0. Due to the uncertain character of the diffusivity coefficients, the answer to this question will be given by a fuzzy-valued time $\tilde{T}_\varepsilon$, where the worst-case interval, at the membership level $\mu = 0$, then quantifies the time at which the concentration $C(x^*, y^*, t)$ may have the value εC_0. Figure 4(*a*) shows the uncertain normalized concentration $\tilde{C}(x^*, y^*, t)/C_0$ at the location ($x^* = l_x/2 = 0.05$ m, $y^* = l_y/2 = 0.05$ m), plotted over time along with bounding lines for the membership levels $\mu = 0.0$, $\mu = 0.5$ and $\mu = 1.0$. The corresponding uncertain time $\tilde{T}_\varepsilon$ at which the concentration $\varepsilon C_0 = 0.5 C_0$ is reached, can then be determined, as shown in figure 4(*b*).

The resulting uncertain time range $\tilde{T}_\varepsilon$ shows a mean value of $\bar{m}_{\tilde{T}_\varepsilon} = 185.6$ s and a worst-case range given by the interval [173.1 s, 199.4 s] at $\mu = 0$. Thus, the relative worst-case deviations of the slightly asymmetic fuzzy number $\tilde{T}_\varepsilon$ amount to -6.7% and +7.4% from the mean value $\bar{m}_{\tilde{T}_\varepsilon}$.

Finally, the values $\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*)$ and $\rho_y^{\tilde{T}_\varepsilon}(x^*, y^*)$ that quantify the relative degree of influence of the uncertain diffusivity coefficients $\tilde{D}_{xx}$ and $\tilde{D}_{yy}$ on the uncertain time $\tilde{T}_\varepsilon$ can be determined on the basis of the relative degrees of influence $\rho_{x/y}^{\tilde{C}}(x^*, y^*, t)$ through

$$\rho_{x/y}^{\tilde{T}_\varepsilon}(x^*, y^*) = \frac{\int_{t=0}^{\infty} \rho_{x/y}^{\tilde{C}}(x^*, y^*, t)\, \mu_{\tilde{T}_\varepsilon}(t)\, \mathrm{d}t}{\int_{t=0}^{\infty} \mu_{\tilde{T}_\varepsilon}(t)\, \mathrm{d}t}. \tag{68}$$

Using a time-discrete approximation with L steps, equation (68) can be evaluated from the result

$$\rho_{x/y}^{\tilde{T}_\varepsilon}(x^*, y^*) \approx \frac{\sum_{l=1}^{L-1} \rho_{x/y}^{\tilde{C}}(x^*, y^*, t_l)\, \mu_{\tilde{T}_\varepsilon}(t_l)}{\sum_{l=1}^{L-1} \mu_{\tilde{T}_\varepsilon}(t_l)}, \tag{69}$$

where

$$t_l = a_{\tilde{T}_\varepsilon} + \frac{l}{L}(b_{\tilde{T}_\varepsilon} - a_{\tilde{T}_\varepsilon}) \tag{70}$$

and where the open interval $(a_{\tilde{T}_\varepsilon}, b_{\tilde{T}_\varepsilon})$ represents the support set $\mathrm{supp}(\tilde{T}_\varepsilon)$ of the fuzzy number $\tilde{T}_\varepsilon$, defined by

$$\mathrm{supp}(\tilde{T}_\varepsilon) = \left\{ t \,|\, \mu_{\tilde{T}_\varepsilon}(t) > 0 \right\}. \tag{71}$$

The resulting values for $\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*)$ and $\rho_y^{\tilde{T}_\varepsilon}(x^*, y^*)$ are then

$$\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*) = 7.4\% \qquad \text{and} \qquad \rho_y^{\tilde{T}_\varepsilon}(x^*, y^*) = 92.6\%. \tag{72}$$

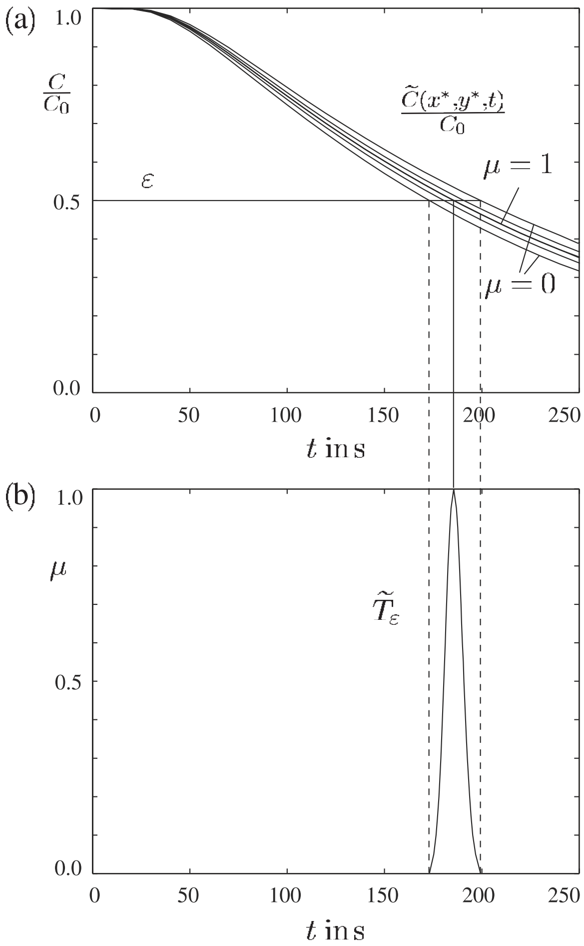

Figure 4. (*a*) Uncertain normalized concentration $\tilde{C}(x^*, y^*, t)/C_0$ at the location $(x^*, y^*) = (l_x/2, l_y/2) = (0.05\,\text{m}, 0.05\,\text{m})$ and (*b*) uncertain time $\tilde{T}_\varepsilon$, for $\lambda = 5.0$ (orthotropic).

The relative degrees of influence $\rho^{\tilde{C}}_{x/y}(x^*, y^*, t)$ as well as the gain factors $\kappa^{\tilde{C}}_{x/y}(x^*, y^*, t)$ are plotted for $0 \leqslant t \leqslant 250\,\text{s}$ in figure 5.

Avoiding the graphical construction of the fuzzy-valued time $\tilde{T}_\varepsilon$ as presented in figures 4 and 6, an estimation $\hat{\sigma}_{\tilde{T}_\varepsilon}$ for the standard deviation of a symmetrically-shaped approximation of the the fuzzy number $\tilde{T}_\varepsilon$ can be determined using the relative degrees of influence $\rho^{\tilde{T}_\varepsilon}_x(x^*, y^*)$ and $\rho^{\tilde{T}_\varepsilon}_y(x^*, y^*)$ by

$$\hat{\sigma}_{\tilde{T}_\varepsilon} = \rho^{\tilde{T}_\varepsilon}_x(x^*, y^*)\,\sigma_x + \rho^{\tilde{T}_\varepsilon}_y(x^*, y^*)\,\sigma_y = \sum_{i=x,y} \rho^{\tilde{T}_\varepsilon}_i(x^*, y^*)\,\sigma_i. \tag{73}$$

For this example, the standard deviation of the symmetrically-shaped approximation of $\tilde{T}_\varepsilon$ is estimated at $\hat{\sigma}_{\tilde{T}_\varepsilon} = 2.37\%\,\bar{m}_{\tilde{T}_\varepsilon}$, which is equivalent to a worst-case range of $\pm 7.11\%$ from the mean value $\bar{m}_{\tilde{T}_\varepsilon}$. In comparison with this, the average worst-case deviation for the real uncertain time $\tilde{T}_\varepsilon$ in figure 4 is $[(7.4\% + 6.7\%)/2]\,\bar{m}_{\tilde{T}_\varepsilon} = 7.05\%\,\bar{m}_{\tilde{T}_\varepsilon}$.

As a second example, the porous medium is now assumed to be quasi-isotropic, i.e. $\lambda = 1$. The settings for the mean value $\bar{m}_x$ as well as for the standard deviations σ_x and

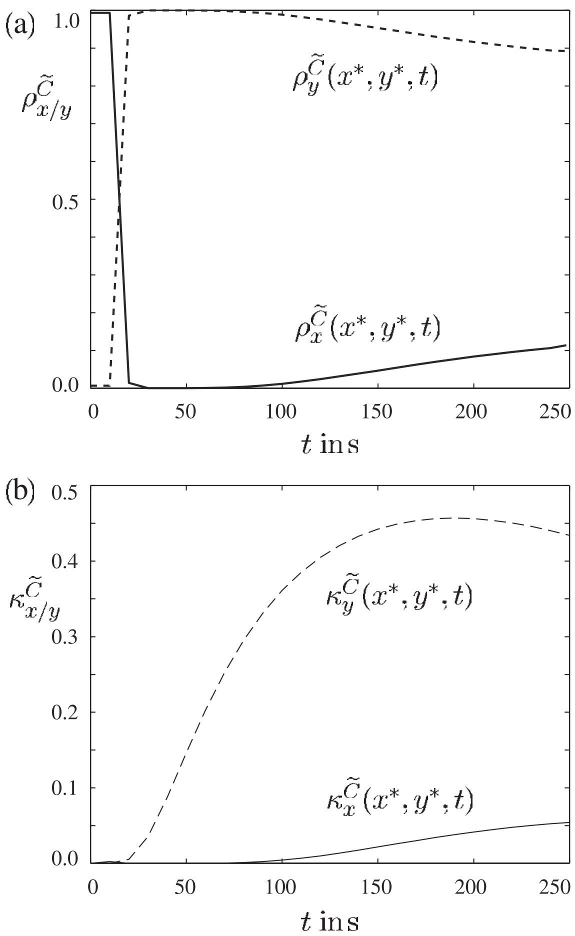

Figure 5. (*a*) Relative degrees of influence $\rho^{\tilde{C}}_{x/y}(x^*, y^*, t)$ and (*b*) gain factors $\kappa^{\tilde{C}}_{x/y}(x^*, y^*, t)$, for $\lambda = 5.0$ (orthotropic).

σ_y of the fuzzy-valued diffusivities $\tilde{D}_{xx}$ and $\tilde{D}_{yy}$ are kept identical to the orthotropic case in equation (66), so that the actual settings are given by

$$\begin{aligned} \bar{m}_x &= 1.0 \times 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}, \qquad && \sigma_x = 7\%\,\bar{m}_x, \\ \bar{m}_y &= 1.0 \times 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1} = \bar{m}_x, \qquad && \sigma_y = 2\%\,\bar{m}_x. \end{aligned} \tag{74}$$

Thus, the uncertain diffusivity coefficients $\tilde{D}_{xx}$ and $\tilde{D}_{yy}$ are again set to cover a worst-case range of ±21% and ±6% from their mean values, $\bar{m}_x$ and $\bar{m}_y$.

Posing again the problem, we seek the uncertain time $t = \tilde{T}_\varepsilon$ at which the fuzzy-valued concentration $\tilde{C}(x^*, y^*, t)$ at the given location ($x^* = l_x/2 = 0.05$ m, $y^* = l_y/2 = 0.05$ m) would reduce to the level $\varepsilon C_0 = 0.5C_0$. The relevant results for the isotropic case are given in figure 6. The resulting uncertain time $\tilde{T}_\varepsilon$ shows a mean value of $\bar{m}_{\tilde{T}_\varepsilon} = 592.8$ s and a worst-case range given by the interval [522.1 s , 685.0 s] at $\mu = 0$. Thus, the relative worst-case deviations of the fuzzy number $\tilde{T}_\varepsilon$ amount to −11.9% and +15.6% from the mean value. This noticeable augmentation of uncertainty in $\tilde{T}_\varepsilon$ from the orthotropic to the isotropic case becomes clear when we compare figures 4 and 6 (note that, although the absolute ranges of the time axes are different, they are plotted to the same scale).

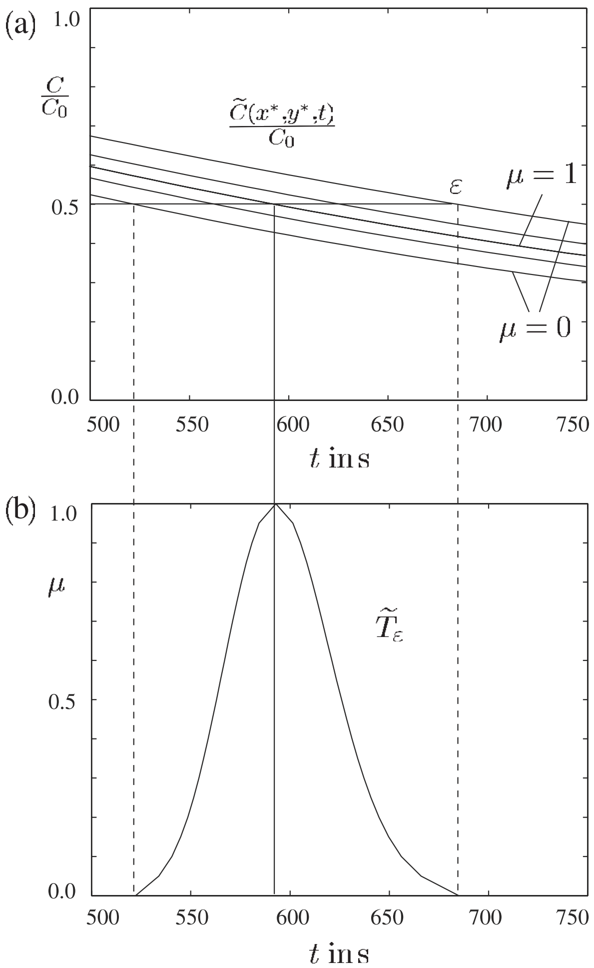

Figure 6. (*a*) Uncertain normalized concentration $\tilde{C}(x^*, y^*, t)/C_0$ at the location $(x^*, y^*) = (l_x/2, l_y/2) = (0.05\,\text{m}, 0.05\,\text{m})$ and (*b*) uncertain time $\tilde{T}_\varepsilon$, for $\lambda = 1.0$ (isotropic).

As a result of the evaluation of equations (69) and (70) for the isotropic case, the relative degrees of influence $\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*)$ and $\rho_y^{\tilde{T}_\varepsilon}(x^*, y^*)$ of the uncertain diffusivity coefficients $\tilde{D}_{xx}$ and $\tilde{D}_{yy}$ on the uncertain time $\tilde{T}_\varepsilon$ are now as expected

$$\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*) = 50.0\% \qquad \text{and} \qquad \rho_y^{\tilde{T}_\varepsilon}(x^*, y^*) = 50.0\%. \tag{75}$$

Figure 7 shows the corresponding relative degrees of influence $\rho_{x/y}^{\tilde{C}}(x^*, y^*, t)$ as well as the gain factors $\kappa_{x/y}^{\tilde{C}}(x^*, y^*, t)$, plotted for $500 \leqslant t \leqslant 750$ s.

Finally, the relative degrees of influence $\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*)$ and $\rho_y^{\tilde{T}_\varepsilon}(x^*, y^*)$ of the uncertain diffusivity coefficients $\tilde{D}_{xx}$ and $\tilde{D}_{yy}$ on the uncertain time $\tilde{T}_\varepsilon$ shall be determined for various degrees of anisotropy $\lambda = \bar{m}_y/\bar{m}_x$. For the range from $\lambda = 1.0$ to 10.0, the relative degrees of influence $\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*)$ and $\rho_y^{\tilde{T}_\varepsilon}(x^*, y^*)$ are plotted in figure 8. It can be seen that with an increasing degree of anisotropy, the uncertain time $\tilde{T}_\varepsilon$ is influenced by the uncertainty of the dominant diffusivity coefficient. Empirically, the dependency of $\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*)$ and $\rho_y^{\tilde{T}_\varepsilon}(x^*, y^*)$

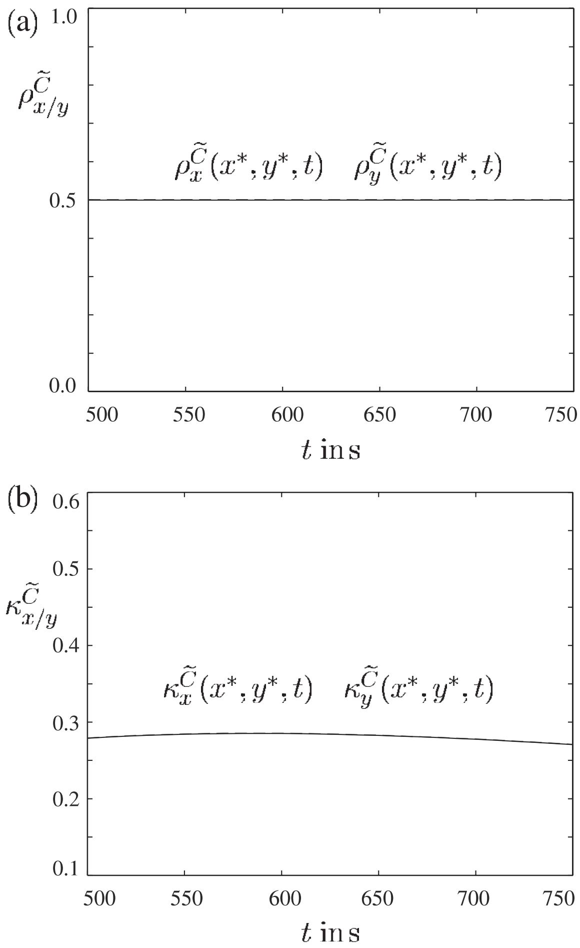

Figure 7. (*a*) Relative degrees of influence $\rho_{x/y}^{\tilde{C}}(x^*, y^*, t)$ and (*b*) gain factors $\kappa_{x/y}^{\tilde{C}}(x^*, y^*, t)$, for $\lambda = 1.0$ (isotropic).

on the degree of anisotropy λ can very well be approximated by the formulae

$$\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*) = \frac{1}{2} - \frac{1}{2}\left\{1 - \exp\left[-\frac{(\lambda - 1)}{2}\right]\right\} = -\frac{1}{2}\exp\left[-\frac{(\lambda - 1)}{2}\right], \tag{76}$$

$$\rho_y^{\tilde{T}_\varepsilon}(x^*, y^*) = \frac{1}{2} + \frac{1}{2}\left\{1 - \exp\left[-\frac{(\lambda - 1)}{2}\right]\right\} = \frac{1}{2}\left\{2 - \exp\left[-\frac{(\lambda - 1)}{2}\right]\right\} \tag{77}$$

satisfying the consistency condition

$$\rho_x^{\tilde{T}_\varepsilon}(x^*, y^*) + \rho_y^{\tilde{T}_\varepsilon}(x^*, y^*) = 1. \tag{78}$$

The incorporation of equations (76) and (77) into equation (73) then yields

$$\hat{\sigma}_{\tilde{T}_\varepsilon} = \sigma_x + \left\{1 - \frac{1}{2}\exp\left[-\frac{(\lambda - 1)}{2}\right]\right\}(\sigma_x + \sigma_y), \tag{79}$$

which is a very practical formula for the determination of the uncertainty to be expected in the time $\tilde{T}_\varepsilon$.

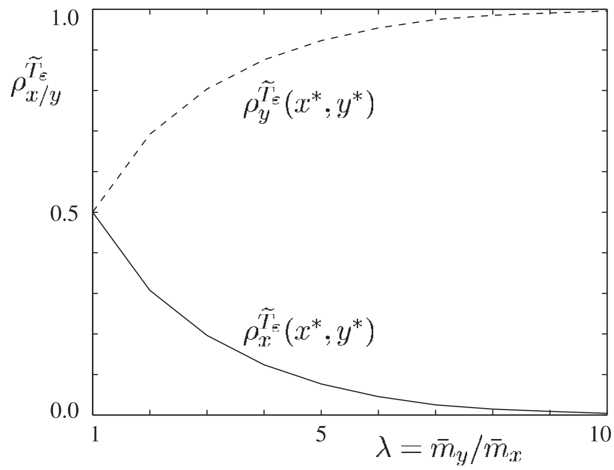

Figure 8. Relative degrees of influence $\rho_{x/y}^{\tilde{T}_\varepsilon}(x^*, y^*)$ for increasing degree of anisotropy $\lambda = \bar{m}_y/\bar{m}_x$.

6. Conclusions

The modelling of diffusive transport processes in fluid-saturated porous media is of some importance to the study of migration of chemicals and other hazardous substances in geological media. The mathematical studies in this area can be extended to include orthotropic characteristics of the Fickian diffusive behaviour, and the product solutions technique can then be used to generate a wide class of analytical solutions to problems of technological importance. The conventional studies in this area invariably assume that the parameters governing the orthotropic diffusive process are sensibly constant. In practice, the diffusion parameters are seldom 'crisp' values and a realistic assessment of the diffusive behaviour can only be gained by considering the inherent variabilities that can be assigned to the orthotropic diffusion parameters. The resulting analysis can be accomplished through a variety of procedures including Monte–Carlo simulations. This paper has presented a comprehensive treatment of the orthotropic diffusive processes in a two-dimensional rectangular region by posing the problem in the context of a 'fuzzy arithmetical concept'. It is shown that owing to the availability of the solution to the diffusion problem with 'crisp' diffusion parameters in explicit analytical form, it is possible to readily extend the analysis to include fuzzy estimates of the orthotropic diffusivity parameters. In particular, it is shown that for the use of the novel developments proposed in the paper, the fuzziness in the estimates for the diffusivities can be characterized by, say, symmetric quasi-Gaussian distributions and the attendant mean values and standard deviations. The uncertain diffusivity coefficients are also set to cover worst-case scenarios with respect to the ranges. The novel treatments of the fuzzy arithmetic-based formulation of the diffusion problem allows the development of a fuzzy-valued estimate of the time required for the concentration to reduce to a prescribed level at a given location. This estimate given by equation (73) is based on the weighted sum of the relative degrees of influence of the uncertain diffusivities defined by either equation (68) or equations (69) and (70), and the respective standard deviations. The theoretical developments are applied to the specific case of a square region with a finite initial concentration and the boundaries of which are maintained at zero value for a time greater than zero. The results presented are purely for purposes of illustration but, nonetheless, identify trends of interest. First, the 'crisp' or deterministic solution occurs as a special case of the formulation based on 'fuzzy' scheme. Second, the uncertain time at which the concentration at a point will reduce to a prescribed value can vary with the degree of anisotropy λ associated with the diffusivity coefficients. Finally, it should be remarked that the approach presented here can easily be extended to include three-dimensional problems involving the transient diffusion problem.

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