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Mechanics of a rigid circular disc bonded to a cracked elastic half-space

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Abstract

This paper examines the axial tensile loading of a rigid circular disc which is bonded to the surface of a half-space weakened by a penny-shaped crack. The integral equations governing the problem are solved numerically, to establish the influence of the extent of cracking on the axial stiffness of the bonded disc and on the stress intensity factors at the crack tip.

Keywords: Contact problem; Cracked half-space; Elastic stiffness of indentor; Crack-indentor interaction; Indentor-crack interaction; Crack

1. Introduction

The axisymmetric problem of the surface indentation of an isotropic elastic half-space by a smooth rigid circular punch or indentor was first examined by Boussinesq (1885), who developed the results for the contact stresses and the force–displacement relationship in exact closed form, using results of potential theory. In what is now acknowledged as a classic paper, Harding and Sneddon (1945) applied the Hankel integral transform techniques to reduce the mixed boundary problem to a system of dual integral equations. These equations were further reduced to a single integral of the Abel-type and the results for the contact stresses and the force–displacement relationship were again evaluated in exact closed form (see also Sneddon (1946)). A number of investigators, including Borowicka (1936); Elliott (1948, 1949); Korenev (1957); Shield (1967); Kassir and Chuapresert (1974), and Selvadurai (1980, 1996), have extended this classical contact problem to include transverse isotropy of the elastic half-space, flexibility of the circular indentor, and elastic non-homogeneity of the half-space region. The problem dealing with the adhesive contact between the elastic half-space and the rigid indentor is a non-classical contact problem which has been investigated by Mossakovskii (1954) and Ufliand (1956) while the problem of partial frictional contact

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between the indentor and the half-space was examined by Spence (1968). Very comprehensive accounts of contact problems in the classical theory of elasticity are given by Galin (1961); Ufliand (1965) de Pater and Kalker (1975); Selvadurai (1979, 2000a); Gladwell (1980); and Johnson (1985).

In these classical studies of the elastostatic contact problem, it is invariably assumed that the half-space region is free of defects such as cracks or inclusions. The analysis of the type of problem dealing with distributed defects in the form of micro-cracks within a half-space region is of interest to the study of indentational testing of materials which have experienced micro-mechanical damage. Similarly, problems which examine the influence of distributed inhomogeneities such as inclusions have applications to the study of multiphase composites (Fig. 1). These classes of problems are best examined by appeal to procedures dealing with micro-mechanical aspects of solids (Bilby et al., 1985; Mura, 1987; Weng et al., 1990) which derive effective elastic parameters for the elastic solid that are obtained via both analytical and numerical procedures.

In this paper, however, attention is restricted to the axisymmetric problem of the indentation of an isotropic half-space region containing a single crack, whose dimensions are comparable to those of the rigid disc that is *bonded* to the surface of the half-space. In particular, we focus on the problem where the crack is maintained in an open position by the application of tensile loading to a rigid circular disc which is bonded

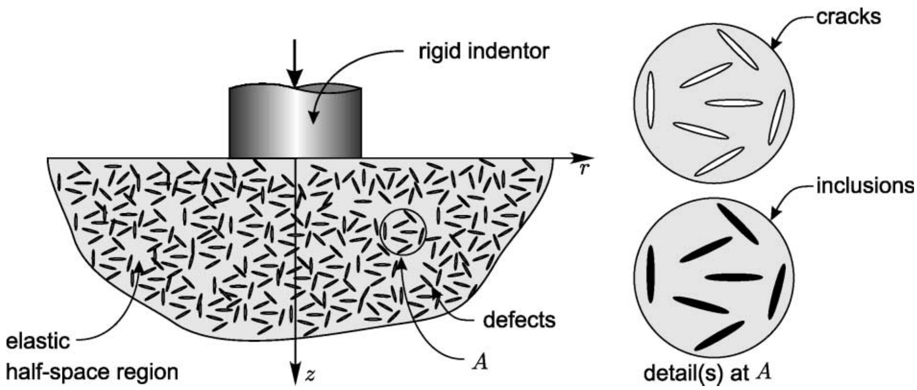

Fig. 1. Indentation of a half-space region containing distributed cracks.

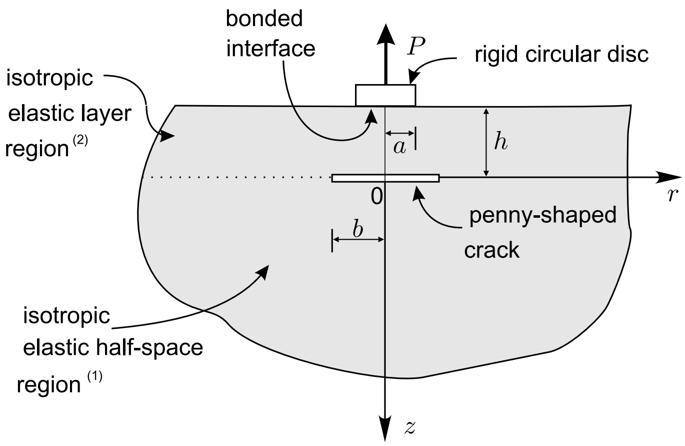

Fig. 2. Tensile loading of a rigid circular disc bonded to a cracked half-space.

to the surface of the half-space region (Fig. 2). Such a discrete penny-shaped crack can be initiated as a result of the coalescence of penny-shaped micro-cracks with their planes oriented normal to the tensile stress state in the half-space region. The orientation of the penny-shaped discrete crack with reference to the position of the disc can be arbitrary; in this study we restrict attention to the axisymmetric problem where a stable penny-shaped crack of radius b is located at a depth h below the surface of the half-space containing the bonded rigid disc. The axisymmetric problem is examined using a Hankel transform development of the governing equations. The integral equations resulting from the Hankel transform development of boundary and continuity conditions, respectively, at $z = -h$ and $z = 0$ (Fig. 2) are further reduced to four coupled Fredholm integral equations of the second-kind. These integral equations are solved numerically to determine results of interest to engineering applications. These include the evaluation of the influence of the penny-shaped crack on the axial stiffness of the bonded rigid disc and the evaluation of both Mode I and Mode II stress intensity factors at the crack tip.

2. Governing equations

We consider the idealized axisymmetric problem where an isotropic elastic half-space region containing a penny-shaped crack at a finite depth is loaded by a rigid circular disc which is bonded to the surface of the isotropic elastic half-space (Fig. 2). Axisymmetric problems in the classical theory of elasticity can be formulated in relation to either the strain potential function approach proposed by Love (1927) or its generalizations given in terms of the Papkovich–Neuber potentials (see e.g. Westergaard (1952)). We employ the latter approach for which, in the absence of body force fields, the displacement and stress components can be derived from the potential functions $\varphi(r,z)$ and $\chi(r,z)$ which satisfy

$$\nabla^2\varphi(r,z) = 0; \quad \nabla^2\chi(r,z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system. The displacement and stress components derived from $\varphi(r,z)$ and $\chi(r,z)$ referred to the cylindrical polar coordinate system are given by

$$u_r(r,z) = \frac{\partial\varphi}{\partial r} - z\frac{\partial\chi}{\partial r} \tag{3}$$

$$u_z(r,z) = (3-4v)\chi + \frac{\partial\varphi}{\partial z} - z\frac{\partial\chi}{\partial z} \tag{4}$$

and

$$\sigma_{rr}(r,z) = 2\mu\left[\frac{\partial^2\varphi}{\partial r^2} + 2v\frac{\partial\chi}{\partial z} - z\frac{\partial^2\chi}{\partial r^2}\right] \tag{5}$$

$$\sigma_{\theta\theta}(r,z) = 2\mu\left[\frac{1}{r}\frac{\partial\varphi}{\partial r} + 2v\frac{\partial\chi}{\partial z} - \frac{z}{r}\frac{\partial\chi}{\partial r}\right] \tag{6}$$

$$\sigma_{zz}(r,z) = 2\mu\left[2(1-v)\frac{\partial\chi}{\partial z} + \frac{\partial^2\varphi}{\partial z^2} - z\frac{\partial^2\chi}{\partial z^2}\right] \tag{7}$$

$$\sigma_{rz}(r,z) = 2\mu\left[(1-2v)\frac{\partial\chi}{\partial r} + \frac{\partial^2\varphi}{\partial r\partial z} - z\frac{\partial^2\chi}{\partial r\partial z}\right] \tag{8}$$

where v is Poisson's ratio and μ is the linear elastic shear modulus.

For the analysis of the mixed boundary value problem resulting from the surface indentation of the half-space containing a penny-shaped crack, it is convenient to identify a *half-space region* (superscript (1)) occupying the region $r \in (0,\infty)$; $z \in (0,\infty)$ and a *layer region* (superscript (2)) occupying the region $r \in (0,\infty)$; $z \in (-h,0)$.

The boundary and continuity conditions governing the problem are the following:

$$u_z^{(2)}(r,-h) = -\Delta_0; \quad 0 \leqslant r \leqslant a \tag{9}$$

$$u_r^{(2)}(r,-h) = 0; \quad 0 \leqslant r \leqslant a \tag{10}$$

$$\sigma_{zz}^{(2)}(r,-h) = 0; \quad a < r < \infty \tag{11}$$

$$\sigma_{rz}^{(2)}(r,-h) = 0; \quad a < r < \infty \tag{12}$$

and

$$\sigma_{zz}^{(1)}(r,0) = 0; \quad 0 < r < b \tag{13}$$

$$\sigma_{zz}^{(2)}(r,0) = 0; \quad 0 < r < b \tag{14}$$

$$\sigma_{rz}^{(1)}(r,0) = 0; \quad 0 < r < b \tag{15}$$

$$\sigma_{rz}^{(2)}(r,0) = 0; \quad 0 < r < b \tag{16}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0); \quad b \leqslant r < \infty \tag{17}$$

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0); \quad b \leqslant r < \infty \tag{18}$$

$$\sigma_{zz}^{(1)}(r,0) = \sigma_{zz}^{(2)}(r,0); \quad b \leqslant r < \infty \tag{19}$$

$$\sigma_{rz}^{(1)}(r,0) = \sigma_{rz}^{(2)}(r,0); \quad b \leqslant r < \infty \tag{20}$$

The conditions (13)–(16) imply that the penny-shaped crack remains traction free during the "tensile" loading of the rigid circular punch which is bonded to the surface of the half-space region. In addition to these boundary and continuity conditions, it is assumed that the displacements and stresses should satisfy the appropriate regularity conditions as $r,z \to \infty$. The solutions of $\varphi(r,z)$ and $\chi(r,z)$ applicable to the regions (superscripts) (1) and (2) can be obtained from a Hankel transform development of the governing Eq. (1). Details of the approach are given by Sneddon (1951). The relevant expressions for $u_r^{(1)}$, $u_z^{(1)}$, $\sigma_{zz}^{(1)}$, and $\sigma_{rz}^{(1)}$ applicable to the half-space region are given by the following:

$$u_r^{(1)}(r,z) = \int_0^\infty [A(\xi) + \xi z B(\xi)]\mathrm{e}^{-\xi z} J_1(\xi r)\,\mathrm{d}\xi \tag{21}$$

$$u_z^{(1)}(r,z) = \int_0^\infty [A(\xi) + (3-4v)B(\xi) + \xi z B(\xi)]\mathrm{e}^{-\xi z} J_0(\xi r)\,\mathrm{d}\xi \tag{22}$$

$$\sigma_{zz}^{(1)}(r,z) = -2\mu \int_0^\infty \xi[A(\xi) + 2(1-\nu)B(\xi) + \xi z B(\xi)]\mathrm{e}^{-\xi z} J_0(\xi r)\,\mathrm{d}\xi \tag{23}$$

$$\sigma_{rz}^{(1)}(r,z) = -2\mu \int_0^\infty \xi[A(\xi) + (1-2\nu)B(\xi) + \xi z B(\xi)]\mathrm{e}^{-\xi z} J_1(\xi r)\,\mathrm{d}\xi \tag{24}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions. The appropriate forms of the solutions for $u_r^{(2)}$, $u_z^{(2)}$, $\sigma_{zz}^{(2)}$, and $\sigma_{rz}^{(2)}$ are given by

$$u_r^{(2)}(r,z) = -\int_0^\infty \{\sinh\{\xi(z+h)\}[C(\xi) - \xi z E(\xi)] + \cosh\{\xi(z+h)\}[D(\xi) - \xi z F(\xi)]\}\frac{J_1(\xi r)}{\sinh(\xi h)}\,\mathrm{d}\xi \tag{25}$$

$$\begin{aligned} u_z^{(2)}(r,z) = \int_0^\infty &[C(\xi)\cosh\{\xi(z+h)\} + (3-4\nu)[E(\xi)\sinh\{\xi(z+h)\} + F(\xi)\cosh\{\xi(z+h)\}] \\ &- \xi z[E(\xi)\cosh\{\xi(z+h)\} + F(\xi)\sinh\{\xi(z+h)\}] + D(\xi)\sinh\{\xi(z+h)\}]\frac{J_0(\xi r)}{\sinh(\xi h)}\,\mathrm{d}\xi \end{aligned} \tag{26}$$

$$\begin{aligned} \sigma_{zz}^{(2)}(r,z) = 2\mu \int_0^\infty &[\xi C(\xi)\sinh\{\xi(z+h)\} + 2(1-\nu)[E(\xi)\cosh\{\xi(z+h)\} + F(\xi)\sinh\{\xi(z+h)\}] \\ &- \xi^2 z[E(\xi)\sinh\{\xi(z+h)\} + F(\xi)\cosh\{\xi(z+h)\}] + \xi D(\xi)\cosh\{\xi(z+h)\}]\frac{J_0(\xi r)}{\sinh(\xi h)}\,\mathrm{d}\xi \end{aligned} \tag{27}$$

$$\begin{aligned} \sigma_{rz}^{(2)}(r,z) = -2\mu \int_0^\infty &[C(\xi)\cosh\{\xi(z+h)\} + (1-2\nu)[E(\xi)\sinh\{\xi(z+h)\} + F(\xi)\cosh\{\xi(z+h)\}] \\ &- \xi z[E(\xi)\cosh\{\xi(z+h)\} + F(\xi)\sinh\{\xi(z+h)\}] + D(\xi)\sinh\{\xi(z+h)\}]\frac{\xi J_1(\xi r)}{\sinh(\xi h)}\,\mathrm{d}\xi \end{aligned} \tag{28}$$

where $C(\xi)$, $D(\xi)$, $E(\xi)$, and $F(\xi)$ are arbitrary functions derived from the potentials

$$\varphi^{(2)}(r,z) = \int_0^\infty [C(\xi)\sinh\{\xi(z+h)\} + D(\xi)\cosh\{\xi(z+h)\}]\frac{J_0(\xi r)\,\mathrm{d}\xi}{\xi\sinh(\xi h)} \tag{29}$$

$$\chi^{(2)}(r,z) = \int_0^\infty [E(\xi)\sinh\{\xi(z+h)\} + F(\xi)\cosh\{\xi(z+h)\}]\frac{J_0(\xi r)\,\mathrm{d}\xi}{\sinh(\xi h)} \tag{30}$$

3. The contact problem

Considering the traction boundary conditions on the surfaces of the penny-shaped crack (Eqs. (13)–(16)) and the continuity of tractions at the exterior region (Eqs. (19) and (20)) we note that

$$\begin{aligned} &\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0) = 0; \quad 0 < r < \infty \\ &\sigma_{rz}^{(1)}(r,0) - \sigma_{rz}^{(2)}(r,0) = 0; \quad 0 < r < \infty \end{aligned} \tag{31}$$

These conditions can be used in conjunction with the relevant expressions for $\sigma_{zz}^{(1)}$, $\sigma_{rz}^{(1)}$, $\sigma_{zz}^{(2)}$, and $\sigma_{rz}^{(2)}$ (Eqs. (23), (24), (27) and (28)), to reduce the number of unknown functions, $A(\xi)$, $B(\xi)$, ... etc. i.e.,

$$A(\xi) = 2(1-\nu)(1-2\nu)\{1+\coth(\xi h)\}\{E(\xi)+F(\xi)\} + \{(1-2\nu)+2(1-\nu)\coth(\xi h)\}C(\xi) + \{2(1-\nu)+(1-2\nu)\coth(\xi h)\}D(\xi) \tag{32}$$

and

$$B(\xi) = -[(1-2\nu)+2(1-\nu)\coth(\xi h)]E(\xi) - [2(1-\nu)+(1-2\nu)\coth(\xi h)]F(\xi) - [1+\coth(\xi h)][C(\xi)+D(\xi)] \tag{33}$$

Using these expressions and the integral representations for $u_r^{(1)}$, $u_r^{(2)}$, $u_z^{(1)}, \ldots$ etc., the boundary conditions (9)–(12) give the following set of integral equations:

$$\int_0^\infty [(3-4\nu)F(\xi) + C(\xi) + \xi h E(\xi)] \frac{J_0(\xi r)\,\mathrm{d}\xi}{\sinh(\xi h)} = -\Delta_0; \quad 0 \leqslant r \leqslant a \tag{34}$$

$$\int_0^\infty [D(\xi) + \xi h F(\xi)] \frac{J_1(\xi r)\,\mathrm{d}\xi}{\sinh(\xi h)} = 0; \quad 0 \leqslant r \leqslant a \tag{35}$$

$$\int_0^\infty \xi[2(1-\nu)E(\xi) + D(\xi) + \xi h F(\xi)] \frac{J_0(\xi r)\,\mathrm{d}\xi}{\sinh(\xi h)} = 0; \quad a < r < \infty \tag{36}$$

$$\int_0^\infty \xi[(1-2\nu)F(\xi) + C(\xi) + \xi h E(\xi)] \frac{J_1(\xi r)\,\mathrm{d}\xi}{\sinh(\xi h)} = 0; \quad a < r < \infty \tag{37}$$

and the remaining continuity conditions give the following integral equations:

$$\int_0^\infty \xi[2(1-\nu)\{F(\xi) + E(\xi)\coth(\xi h)\} + C(\xi) + D(\xi)\coth(\xi h)] J_0(\xi r)\,\mathrm{d}\xi = 0; \quad 0 < r < b \tag{38}$$

$$\int_0^\infty \xi[(1-2\nu)\{E(\xi) + F(\xi)\coth(\xi h)\} + D(\xi) + C(\xi)\coth(\xi h)] J_1(\xi r)\,\mathrm{d}\xi = 0; \quad 0 < r < b \tag{39}$$

$$\int_0^\infty [\{1+\coth(\xi h)\}(1-2\nu)[F(\xi)+E(\xi)] + C(\xi) + D(\xi)] J_1(\xi r)\,\mathrm{d}\xi = 0; \quad b < r < \infty \tag{40}$$

$$\int_0^\infty -2(1-\nu)\{1+\coth(\xi h)\}[2(1-\nu)\{F(\xi)+E(\xi)\} + C(\xi) + D(\xi)] J_0(\xi r)\,\mathrm{d}\xi = 0; \quad b < r < \infty \tag{41}$$

Considering (36), (37), (40) and (41) we introduce substitutions

$$[2(1-\nu)E(\xi) + D(\xi) + \xi h F(\xi)] = L_1(\xi)\sinh(\xi h) \tag{42}$$

$$[(1-2\nu)F(\xi) + C(\xi) + \xi h E(\xi)] = L_2(\xi)\sinh(\xi h) \tag{43}$$

$$[(1-2\nu)\{E(\xi)+F(\xi)\} + C(\xi) + D(\xi)] = L_3(\xi) \tag{44}$$

$$[2(1-\nu)\{E(\xi)+F(\xi)\} + C(\xi) + D(\xi)] = L_4(\xi) \tag{45}$$

and assume that $L_i(\xi)$ $(i = 1, 2, 3, 4)$ admit representations of the form

$$L_1(\xi) = \int_0^a \Omega_1(t)\cos(\xi t)\,\mathrm{d}t \tag{46}$$

$$L_2(\xi) = \int_0^a \Omega_2(t)\sin(\xi t)\,\mathrm{d}t \tag{47}$$

$$[1+\coth(\xi h)]L_3(\xi) = \frac{1}{\xi}\sqrt{\frac{2}{\pi}}\left[\frac{-\Omega_3(b)\sin(b\xi)}{\sqrt{b}} + \int_0^b \frac{\sin(\xi t)\,\mathrm{d}t}{t}\left[\sqrt{t}\Omega_3(t)\right]'\mathrm{d}t\right] \tag{48}$$

$$[1+\coth(\xi h)]L_4(\xi) = \int_0^b \Omega_4(t)\sin(\xi t)\,\mathrm{d}t \tag{49}$$

where $\Omega_i(t)$ $(i = 1, 2, 3, 4)$ are unknown functions and $[\ ]'$ denotes the derivative with respect to t. The representations (46)–(49) ensure that the integral Eqs. (36), (37), (40) and (41) are automatically satisfied and, by making use of the solutions to Abel-type integral equations (Sneddon, 1966), the remaining integral Eqs. (34), (35), (38) and (39) can be expressed in terms of $\Omega_i(t)$ as follows:

$$\begin{aligned}\Omega_1(t) = \frac{2\Delta_0}{\pi} &+ \int_0^a \Omega_1(u)K_{11}(u,t)\,\mathrm{d}u + \int_0^a \Omega_2(t)K_{12}(u,t)\,\mathrm{d}u + \int_0^b \Omega_3(t)K_{13}(u,t)\,\mathrm{d}u \\ &+ \int_0^b \Omega_4(t)K_{14}(u,t)\,\mathrm{d}u; \quad 0 < t < a\end{aligned} \tag{50}$$

$$\begin{aligned}\Omega_2(t) = &\int_0^a \Omega_1(u)K_{21}(u,t)\,\mathrm{d}u + \int_0^a \Omega_2(t)K_{22}(u,t)\,\mathrm{d}u + \int_0^b \Omega_3(t)K_{23}(u,t)\,\mathrm{d}u \\ &+ \int_0^b \Omega_4(t)K_{24}(u,t)\,\mathrm{d}u; \quad 0 < t < a\end{aligned} \tag{51}$$

$$\begin{aligned}\Omega_3(t) = &\int_0^a \Omega_1(u)K_{31}(u,t)\,\mathrm{d}u + \int_0^a \Omega_2(t)K_{32}(u,t)\,\mathrm{d}u + \int_0^b \Omega_3(t)K_{33}(u,t)\,\mathrm{d}u \\ &+ \int_0^b \Omega_4(t)K_{34}(u,t)\,\mathrm{d}u; \quad 0 < t < b\end{aligned} \tag{52}$$

$$\begin{aligned}\Omega_4(t) = &\int_0^a \Omega_1(u)K_{41}(u,t)\,\mathrm{d}u + \int_0^a \Omega_2(t)K_{42}(u,t)\,\mathrm{d}u + \int_0^b \Omega_3(t)K_{43}(u,t)\,\mathrm{d}u \\ &+ \int_0^b \Omega_4(t)K_{44}(u,t)\,\mathrm{d}u; \quad 0 < t < b\end{aligned} \tag{53}$$

This reduction procedure has been extensively used in connection with both elasto-static and elasto-dynamic problems associated with crack, inclusion and contact problems (see e.g. Kassir and Sih (1975); Selvadurai (1993, 1994, 2000b,c, 2001); Selvadurai et al. (1991); Yue and Selvadurai (1995)). The kernel functions K_{ij} $(i, j = 1, 2, 3, 4)$ occurring in (50)–(53) are given by

$$K_{11}(u,t) = -\frac{2(1-2\nu)}{\pi}\int_0^\infty \cos(\xi u)\cos(\xi t)\,\mathrm{d}\xi = -(1-2\nu)\delta(u-t) \tag{54}$$

$$K_{12}(u,t) = -\frac{2(1-2\nu)}{\pi}\int_0^\infty \sin(\xi u)\cos(\xi t)\,\mathrm{d}\xi = -\frac{2(1-2\nu)}{\pi}\frac{u}{(u^2-t^2)} \tag{55}$$

$$K_{13}(u,t) = -\frac{4(1-\nu)}{\pi}\int_0^\infty \sqrt{\xi}(\xi h)\mathrm{e}^{-\xi h}J_{3/2}(\xi u)\cos(\xi t)\,\mathrm{d}\xi$$
$$= -\frac{4(1-\nu)}{\pi}\left(\frac{2}{\pi u}\right)^{1/2} h\left\{\frac{(u+t)}{2u\left[h^2+(u+t)^2\right]} + \frac{(u-t)}{2u\left[h^2+(u-t)^2\right]} - \frac{h^2-(u+t)^2}{2\left[h^2+(u+t)^2\right]^2}\right.$$
$$\left. - \frac{h^2-(u-t)^2}{2\left[h^2-(u-t)^2\right]^2}\right\} \tag{56}$$

$$K_{14}(u,t) = \frac{4(1-\nu)}{\pi}\int_0^\infty (1+\xi h)\mathrm{e}^{-\xi h}\sin(\xi u)\cos(\xi t)\,\mathrm{d}\xi$$
$$= \frac{4(1-\nu)}{\pi}\left\{\frac{(u+t)}{2\left[h^2+(u+t)^2\right]} + \frac{(u-t)}{2\left[h^2+(u-t)^2\right]} + h^2\left[\frac{(u+t)}{\left[h^2+(u+t)^2\right]^2} + \frac{(u-t)}{\left[h^2+(u-t)^2\right]^2}\right]\right\} \tag{57}$$

$$K_{21}(u,t) = -\frac{2(1-2\nu)}{\pi}\int_0^\infty \cos(\xi u)\sin(\xi t)\,\mathrm{d}\xi = -\frac{2(1-2\nu)}{\pi}\frac{t}{(t^2-u^2)} \tag{58}$$

$$K_{22}(u,t) = -\frac{2(1-2\nu)}{\pi}\int_0^\infty \sin(\xi u)\sin(\xi t)\,\mathrm{d}\xi = -(1-2\nu)\delta(u-t) \tag{59}$$

$$K_{23}(u,t) = \frac{4(1-\nu)}{\pi}\int_0^\infty \sqrt{\xi}(1-\xi h)\mathrm{e}^{-\xi h}J_{3/2}(\xi u)\sin(\xi t)\,\mathrm{d}\xi$$
$$= \frac{4(1-\nu)}{\pi}\left(\frac{2}{\pi u}\right)^{1/2}\left\{\frac{1}{4u}\ln\left|\frac{h^2+(t+u)^2}{h^2+(t-u)^2}\right| - \frac{(t+u)}{2\left[h^2+(t+u)^2\right]} - \frac{(t-u)}{2\left[h^2+(t-u)^2\right]}\right.$$
$$\left. + \frac{h^2}{2u\left[h^2+(t+u)^2\right]} - \frac{h^2}{2u\left[h^2+(t-u)^2\right]} + \frac{h^2(t+u)}{\left[h^2+(t+u)^2\right]^2} + \frac{h^2(t-u)}{\left[h^2+(t-u)^2\right]^2}\right\} \tag{60}$$

$$K_{24}(u,t) = \frac{4(1-\nu)}{\pi}\int_0^\infty \xi h\mathrm{e}^{-\xi h}\sin(\xi u)\sin(\xi t)\,\mathrm{d}\xi = \frac{2(1-\nu)h}{\pi}\left\{\frac{\left[h^2-(t-u)^2\right]}{\left[h^2+(t-u)^2\right]^2} - \frac{\left[h^2-(t+u)^2\right]}{\left[h^2+(t+u)^2\right]^2}\right\} \tag{61}$$

$$K_{31}(u,t) = -t\int_0^\infty \sqrt{\xi}(\xi h)\mathrm{e}^{-\xi h}J_{3/2}(\xi u)\cos(\xi u)\,\mathrm{d}\xi$$
$$= -\left(\frac{2t}{\pi}\right)^{1/2} h\left\{\frac{(t+u)}{2t\left[h^2+(t+u)^2\right]} + \frac{(t-u)}{2t\left[h^2+(t-u)^2\right]} - \frac{\left[h^2-(t+u)^2\right]}{2\left[h^2+(t+u)^2\right]^2} - \frac{\left[h^2-(t-u)^2\right]}{2\left[h^2+(t-u)^2\right]^2}\right\} \tag{62}$$

$$K_{32}(u,t) = t\int_0^\infty \sqrt{\xi}(1-\xi h)\mathrm{e}^{-\xi h}J_{3/2}(\xi u)\sin(\xi u)\,\mathrm{d}\xi$$

$$= \left(\frac{2t}{\pi}\right)^{1/2}\left\{\frac{1}{4t}\ln\left|\frac{h^2+(t+u)^2}{h^2+(t-u)^2}\right| - \frac{(t+u)}{2\left[h^2+(t+u)^2\right]} + \frac{(t-u)}{2\left[h^2+(t-u)^2\right]}\right.$$

$$\left. + \frac{h^2}{2t\left[h^2+(t+u)^2\right]} - \frac{h^2}{2t\left[h^2+(t-u)^2\right]} + \frac{h^2(u+t)}{\left[h^2+(t+u)^2\right]^2} - \frac{h^2(t-u)}{\left[h^2+(t-u)^2\right]^2}\right\} \tag{63}$$

$$K_{33}(u,t) = t\int_0^\infty \xi\left\{\frac{3}{2} - \left[\frac{1}{2} - \xi h + (\xi h)^2\right]\mathrm{e}^{-2\xi h}\right\}J_{3/2}(\xi t)J_{3/2}(\xi u)\,\mathrm{d}\xi$$

$$= \frac{3}{2}\left(\frac{t}{u}\right)^{1/2}\delta(u-t) - \frac{2}{\pi}\left(\frac{t}{u}\right)^{1/2}\left\{\frac{h}{2\left[4h^2+(t+u)^2\right]} + \frac{h}{2\left[4h^2+(t-u)^2\right]} + \frac{h(t+u)}{2u\left[4h^2+(t+u)^2\right]}\right.$$

$$- \frac{h(t-u)}{2u\left[4h^2+(t-u)^2\right]} + \frac{h(t+u)}{2t\left[4h^2+(t+u)^2\right]} + \frac{h(t-u)}{2t\left[4h^2+(t-u)^2\right]} - \frac{h\left[4h^2-(t+u)^2\right]}{2\left[4h^2+(t+u)^2\right]^2}$$

$$- \frac{h\left[4h^2-(t-u)^2\right]}{2\left[4h^2+(t-u)^2\right]^2} - \frac{h^3}{ut\left[4h^2+(t+u)^2\right]} + \frac{h^3}{ut\left[4h^2+(t-u)^2\right]} - \frac{2h^3(u+t)}{u\left[4h^2+(u+t)^2\right]^2}$$

$$+ \frac{2h^3(t-u)}{u\left[4h^2+(t-u)^2\right]^2} - \frac{2h^3(t+u)}{t\left[4h^2+(t+u)^2\right]^2} - \frac{2h^3(t-u)}{t\left[4h^2+(t-u)^2\right]^2} + \frac{2h^3\left[4h^2-3(t+u)^2\right]}{\left[4h^2+(t+u)^2\right]^3}$$

$$\left. + \frac{2h^3\left[4h^2-3(t-u)^2\right]}{\left[4h^2+(t-u)^2\right]^3}\right\} \tag{64}$$

$$K_{34}(u,t) = t\int_0^\infty \sqrt{\xi}(\xi h)^2\mathrm{e}^{-2\xi h}J_{3/2}(\xi t)\sin(\xi u)\,\mathrm{d}\xi$$

$$= \left(\frac{2t}{\pi}\right)^{1/2}h^2\left\{-\frac{\left[4h^2-(t+u)^2\right]}{2t\left[4h^2+(t+u)^2\right]^2} + \frac{\left[4h^2-(t-u)^2\right]}{2t\left[4h^2+(t-u)^2\right]^2} - \frac{(t+u)\left[12h^2-(t+u)^2\right]}{\left[4h^2+(t+u)^2\right]^3}\right.$$

$$\left. + \frac{(t-u)\left[12h^2-(t-u)^2\right]}{\left[4h^2+(t-u)^2\right]^3}\right\} \tag{65}$$

$$K_{41}(u,t) = \frac{2}{\pi}\int_0^\infty (1+\xi h)\mathrm{e}^{-\xi h}\cos(\xi u)\sin(\xi t)\,\mathrm{d}\xi$$
$$= \frac{1}{\pi}\left\{\frac{(t+u)}{\left[h^2+(t+u)^2\right]} + \frac{(t-u)}{\left[h^2+(t-u)^2\right]} + \frac{2h^2(t+u)}{\left[h^2+(t+u)^2\right]^2} + \frac{2h^2(t-u)}{\left[h^2+(t-u)^2\right]^2}\right\} \tag{66}$$

$$K_{42}(u,t) = \frac{2}{\pi}\int_0^\infty \xi h\mathrm{e}^{-\xi h}\sin(\xi u)\sin(\xi t)\,\mathrm{d}\xi = \frac{h}{\pi}\left\{\frac{h^2-(t-u)^2}{\left[h^2+(t-u)^2\right]^2} - \frac{h^2-(t+u)^2}{2\left[h^2+(t+u)^2\right]^2}\right\} \tag{67}$$

$$K_{43}(u,t) = \frac{2}{\pi}\int_0^\infty \sqrt{\xi}(\xi h)^2\mathrm{e}^{-2\xi h}J_{3/2}(\xi u)\sin(\xi t)\,\mathrm{d}\xi$$
$$= \frac{2}{\pi}\left(\frac{2}{\pi u}\right)^{1/2} h\left\{\frac{4h^2-(t-u)^2}{2u\left[4h^2+(t-u)^2\right]^2} - \frac{4h^2-(t+u)^2}{2u\left[4h^2+(t+u)^2\right]^2} - \frac{(t+u)\left[12h^2-(t+u)^2\right]}{\left[4h^2+(t+u)^2\right]^3}\right.$$
$$\left. - \frac{(t-u)\left[12h^2-(t-u)^2\right]}{\left[4h^2+(t-u)^2\right]^3}\right\} \tag{68}$$

$$K_{44}(u,t) = \frac{2}{\pi}\int_0^\infty \left\{\frac{3}{2} - \left[\frac{1}{2}+\xi h+(\xi h)^2\right]\mathrm{e}^{-2\xi h}\right\}\sin(\xi u)\sin(\xi t)\,\mathrm{d}\xi$$
$$= \frac{3}{2}\delta(u-t) - \frac{2}{\pi}\left\{-\frac{h}{2\left[4h^2+(t+u)^2\right]} + \frac{h}{2\left[4h^2+(t-u)^2\right]} - \frac{h\left[4h^2-(t+u)^2\right]}{2\left[4h^2+(t+u)^2\right]^2}\right.$$
$$\left. + \frac{h\left[4h^2-(t-u)^2\right]}{2\left[4h^2+(t-u)^2\right]^2} - \frac{2h^3\left[4h^2-3(t+u)^2\right]}{\left[4h^2+(t+u)^2\right]^3} + \frac{2h^3\left[4h^2-3(t-u)^2\right]}{\left[4h^2+(t-u)^2\right]^3}\right\} \tag{69}$$

where $\delta(\ )$ is the Dirac delta function. It should be noted that the constraints that are imposed on the representations (46)–(49), are that the infinite integrals for the kernel functions (54)–(69) in the resulting Fredholm integral equations, should converge (Noble, 1963).

The analysis of the mechanics of a surface loading of a half-space containing a penny-shaped crack, by the bonded rigid disc, is thus reduced to the solution of the system of coupled Fredholm integral equations of the second-kind defined by (50)–(53). The solutions of these integral equations can be substituted in the relevant expressions to obtain results for displacements and stresses in the layer and half-space regions. Results of specific interest to engineering applications relate to the evaluation of the load–displacement relationship for the rigid circular disc bonded to a cracked half-space and the stress intensity factors at the tip of the penny-shaped crack due to surface loading of the half-space.

The load–displacement relationship for the bonded rigid circular disc is given by

$$P = 2\pi\int_0^a r\sigma_{zz}^{(2)}(r,-h)\,\mathrm{d}r \tag{70}$$

Avoiding details of calculations, it can be shown that (70) reduces to

$$P = 4\pi\mu \int_0^a \Omega_1(t)\,\mathrm{d}t \tag{71}$$

The stress intensity factors in the crack tip in Mode I and Mode II are defined, respectively, by

$$K_{\mathrm{I}}(b) = \lim_{r \to b^+} \sqrt{2(r-b)}\sigma_{zz}^{(2)}(r,0) \tag{72}$$

$$K_{\mathrm{II}}(b) = \lim_{r \to b^+} \sqrt{2(r-b)}\sigma_{rz}^{(2)}(r,0) \tag{73}$$

Again, these stress intensity factors can be evaluated directly in terms of the functions $\Omega_i(t)$ derived from the solution of the system of coupled Fredholm integral equations of the second-kind, (50)–(53). Simplifying these expressions we obtain

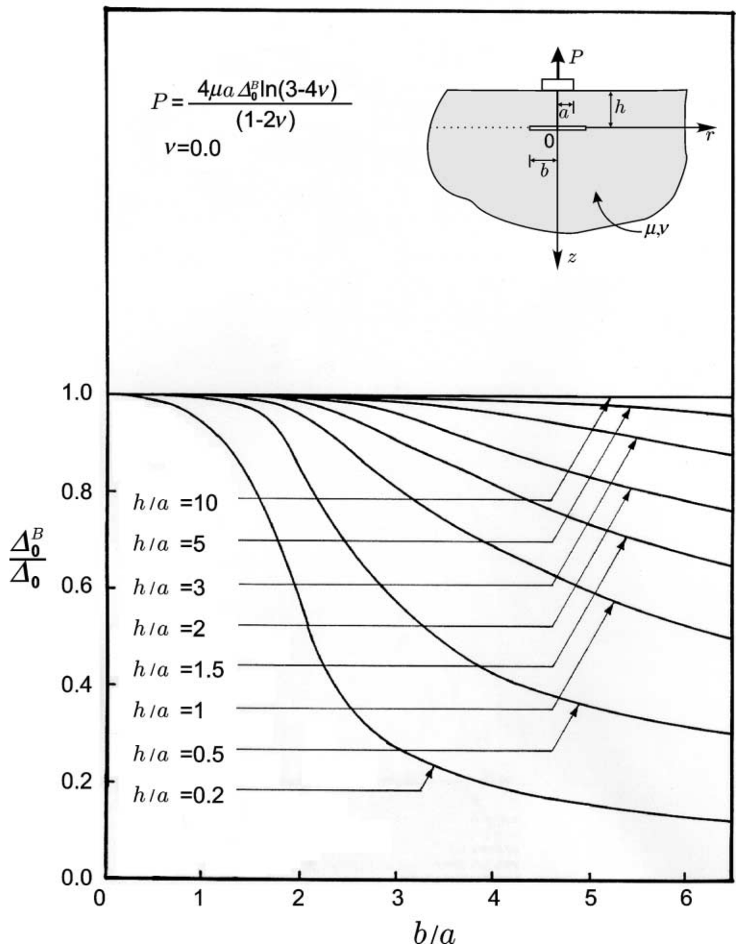

Fig. 3. Influence of a penny-shaped crack on the axial stiffness of the surface bonded rigid disc.

$$K_{\mathrm{I}}(b) = -\frac{2\mu\Omega_4(b)}{\sqrt{b}} \tag{74}$$

$$K_{\mathrm{II}}(b) = 2\mu\sqrt{\frac{2}{\pi}}\frac{\Omega_3(b)}{b} \tag{75}$$

4. Numerical procedures and results

The system of coupled Fredholm integral equations for $\Omega_i(t)$ defined by (50)–(53) can be rewritten in the form

$$\Omega_i(t) - \int_0^{\eta}\left\{\sum_{j=1,}^{4} K_{ij}(u,t)\Omega_j(t)\right\}\mathrm{d}u = f_i(t) \tag{76}$$

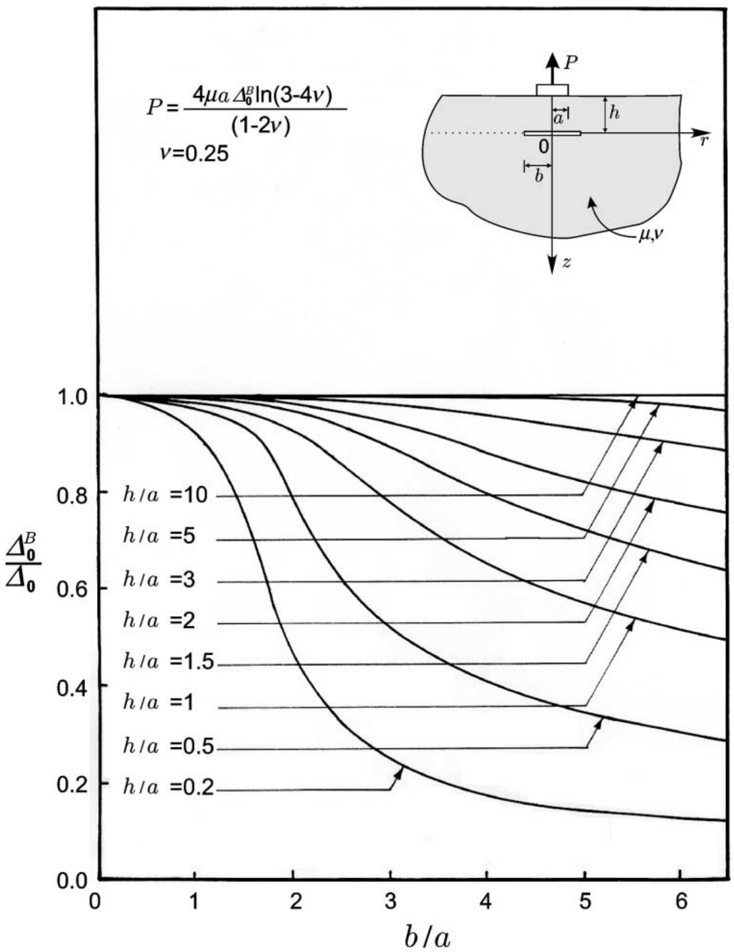

Fig. 4. Influence of a penny-shaped crack on the axial stiffness of the surface bonded rigid disc.

where

$$\eta = \begin{cases} a; & \text{for } i = 1,2 \\ b; & \text{for } i = 3,4 \end{cases} \tag{77}$$

Also, the integral equations are defined over the following intervals:

$$\begin{array}{ll} 0 < t < a; & \text{for } i = 1,2 \\ 0 < t < b; & \text{for } i = 3,4 \end{array} \tag{78}$$

and

$$f_i(t) = \begin{cases} 2\Delta_0/\pi; & \text{for } i = 1 \\ 0; & \text{for } i = 2,3,4 \end{cases} \tag{79}$$

An examination of the system of coupled Fredholm integral equations of the second-kind, (76), for the unknown functions $\Omega_i(t)$ $(i = 1,2,3,4)$, indicates that these equations are not amenable to solution in an exact closed form. The number of integral equations may be further reduced by a process of elimination;

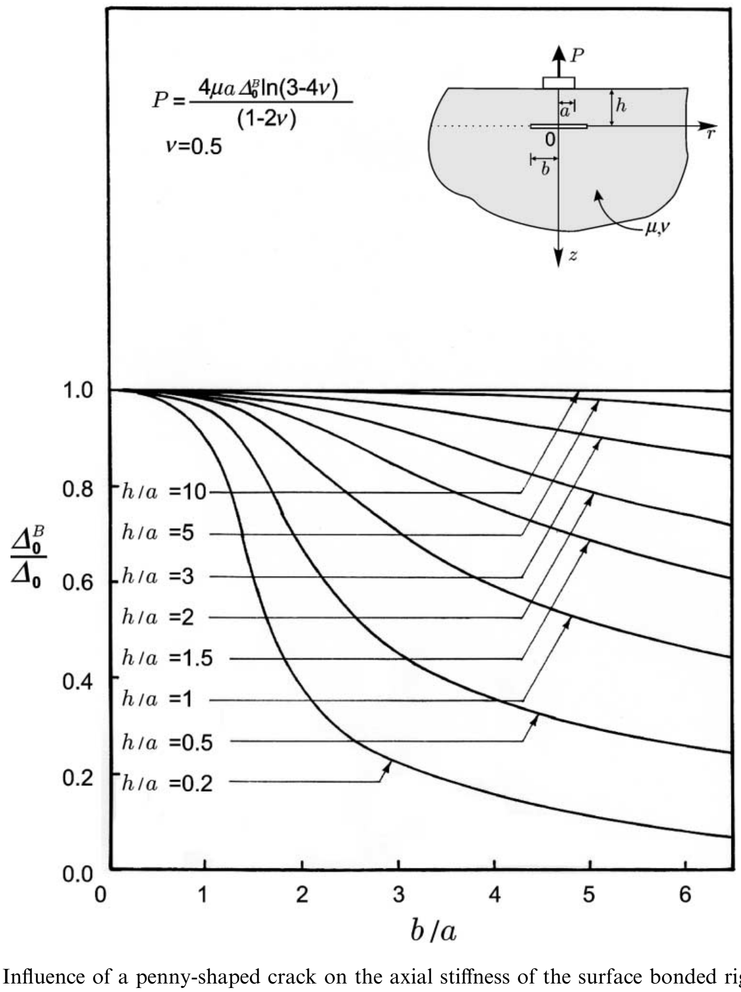

Fig. 5. Influence of a penny-shaped crack on the axial stiffness of the surface bonded rigid disc.

there is, however, no distinct advantage to be gained by adopting such a procedure, which results in a different set of integral equations for the reduced set of unknown functions for $\Omega_i(t)$. The procedure adopted here is to retain the Fredholm character of the system of coupled integral equations so that advantage can be taken of the extensive range of numerical techniques that have been developed for the numerical solution of Fredholm integral equations of the second-kind (see e.g. Baker, 1977; Delves and Mohamed, 1985). In this paper we adopt a collocation scheme. Briefly, the intervals $[0, a]$ and $[0, b]$ are divided into N_1 and N_2 segments respectively. The collocation points for the set of governing integral Eq. (76), are mid-points of the segments. The integral equations can be reduced to a matrix equation for the discrete values of $\Omega_i(t)$ at the collocation points. The resulting matrix equation can be written in form

$$[\mathbf{K}]\{\underline{\boldsymbol{\Omega}}\} = \{\mathbf{f}\} \tag{80}$$

where $[\mathbf{K}]$ is the matrix of coefficients obtained from the integration of the kernel functions $K_{ij}(u, t)$; $\{\underline{\boldsymbol{\Omega}}\}$ is the vector of values for $\Omega_i(t)$ applicable to the mid-points of the segments and $\{\mathbf{f}\}$ is the vector derived from $f_i(t)$ defined by (79). The accuracy of the general procedure has been verified by appeal to comparisons with known exact solutions and other numerical schemes based on series approximation techniques. The ele-

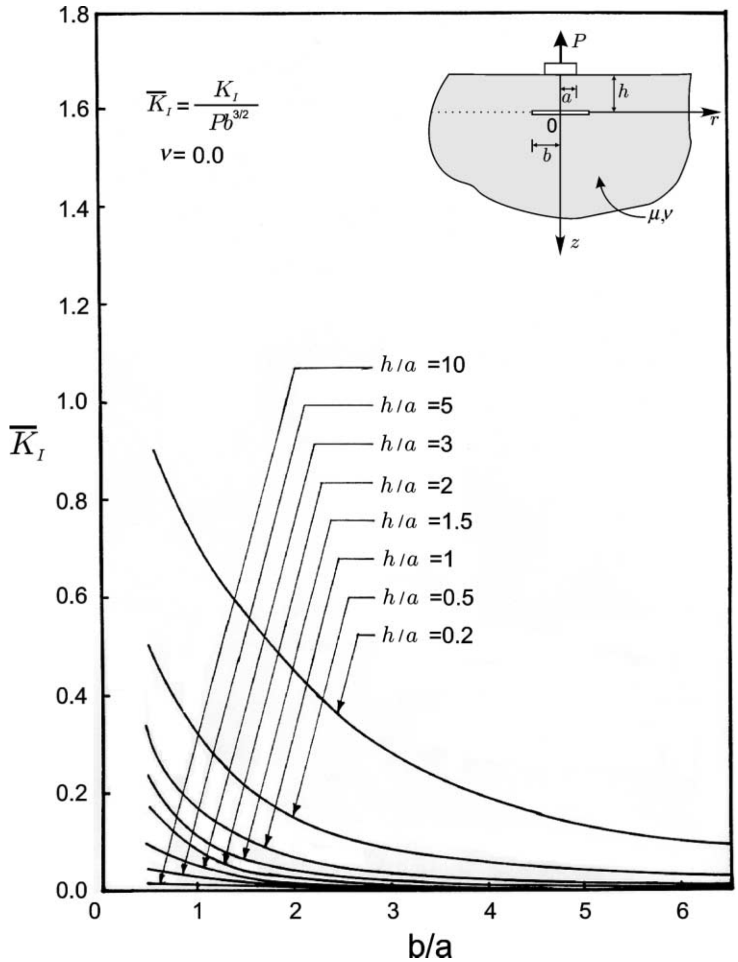

Fig. 6. Mode I stress intensity factor at the crack tip.

ments of $\{\underline{\boldsymbol{\Omega}}\}$ can be used to evaluate the results (71), (74) and (75). In the limiting case when $b \to 0$, the problem reduces to that of the axisymmetric loading of a rigid circular disc which is bonded to the surface of a half-space region. The exact solution to this problem was given by Mossakovskii (1954) and Ufliand (1956) and the details of the analyses are also given by Gladwell (1980). These solution procedures are based on the Hilbert problem and takes into consideration the oscillatory form of the stress singularity at the boundary of the bonded circular disc for arbitrary values of Poisson's ratio. The exact closed form solution for the load (P)–displacement (Δ_0^B) relationship is given by

$$P = \frac{4\mu a \Delta_0^B \ln(3-4\nu)}{(1-2\nu)} \tag{81}$$

The same problem was also examined by Selvadurai (1993) where the oscillatory form of the stress singularity is replaced by a regular $1/\sqrt{r}$-type singularity in the stress field at the boundary of the bonded disc. In this case, the analysis of the problem yields a single Fredholm integral equation of the second-kind which can be solved numerically to obtain the load–displacement relationship. It is found that the maximum

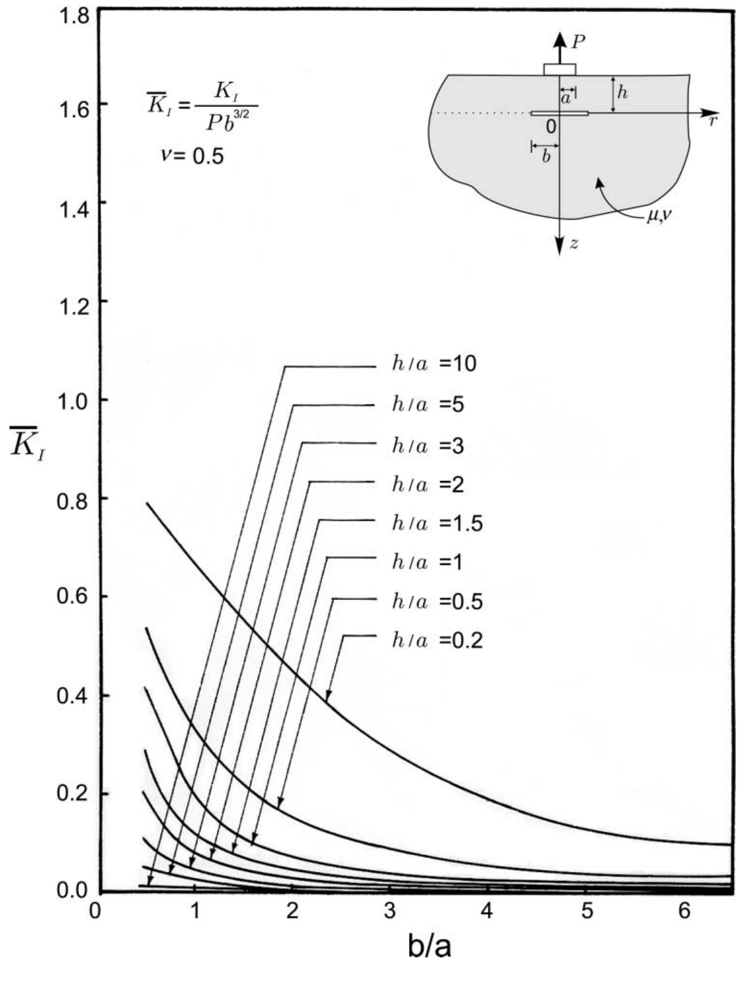

Fig. 7. Mode I stress intensity factor at the crack tip.

discrepancy between the exact closed form solution (81) and the numerical result occurs when $v = 0$ and this does not exceed 0.5%. The solutions converge to the exact result when $v \to 1/2$. The result (81) can be used to normalize the load–displacement relationship for the disc which is bonded to the surface of a half-space region containing a crack. The result (71) for the load–displacement relationship can be written in the discrete form

$$\frac{\Delta_0^B}{\Delta_0} = \frac{\pi(1-2v)}{\ln(3-4v)} \left\{ \frac{1}{N_1} \sum_{\alpha=1,}^{N_1} \Omega_1(\xi_\alpha) \right\} \tag{82}$$

where Δ_0^B is defined by (81) and $\xi_\alpha = (t_\alpha/a)$.

The Figs. 3–5 illustrate the variation of the displacement ratio Δ_0^B/Δ_0 as a function of the radii ratio b/a. The displacement ratio Δ_0^B/Δ_0 can be interpreted as the inverse of the *stiffness ratio*; i.e., the stiffness of a disc bonded to a half-space containing a crack, normalized with respect to the stiffness of a disc bonded to an intact half-space. It is clear that as the radius of the cracked region increases in relation to the radius of

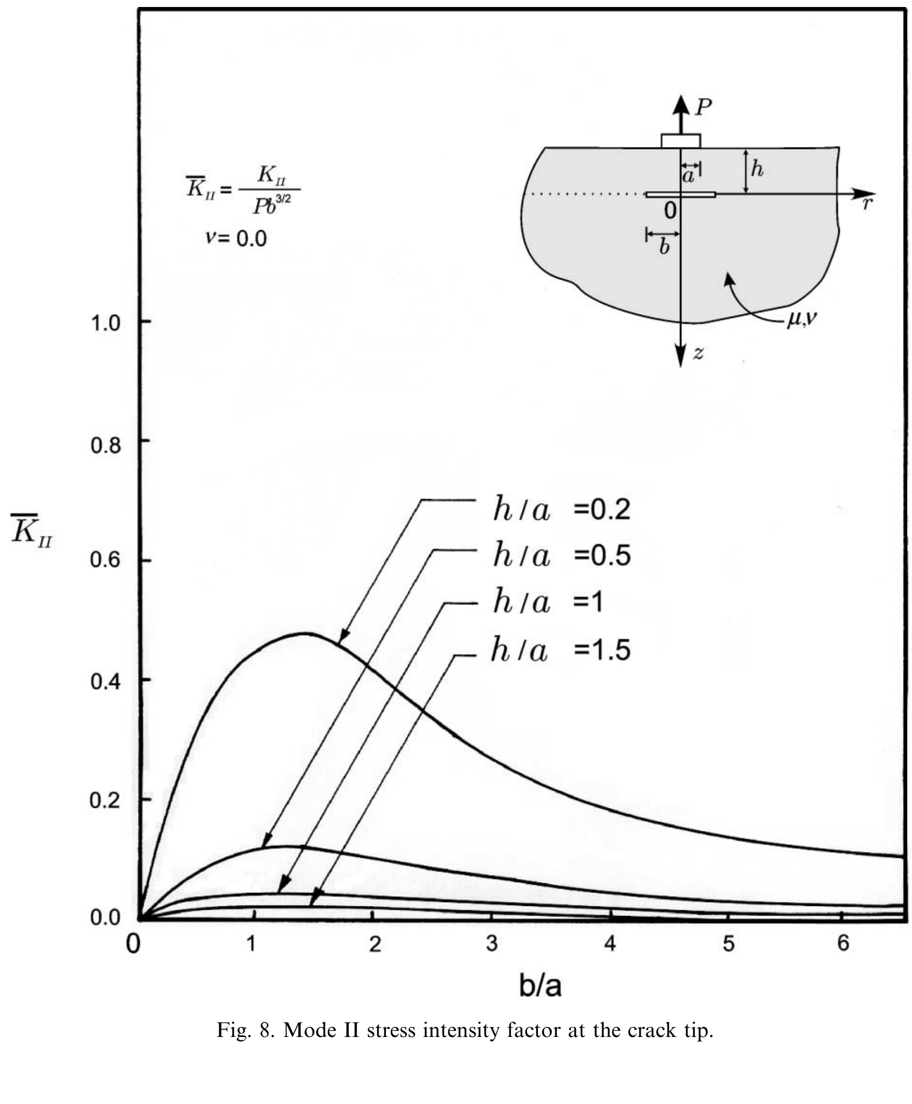

Fig. 8. Mode II stress intensity factor at the crack tip.

the cracked region increases in relation to the radius of the disc, the indentational flexibility of the disc increases. Also, as the depth of location of the crack increases in relation to the radius of the bonded disc (i.e. $h/a \rightarrow$ large) the influence of the crack on the stiffness of the disc decreases. The numerical results indicate that for $(b/a) \in (0, 6)$, the axial stiffness of the disc in uninfluenced provided $(h/a) > 10$.

The stress intensity factors at the tip of the penny-shaped crack can be expressed in the following forms:

$$\overline{K}_{\mathrm{I}}(b) = \frac{K_{\mathrm{I}}(b)}{P/b^{3/2}} = -\frac{b\Omega_4(b)}{2\pi C} \tag{83}$$

$$\overline{K}_{\mathrm{II}}(b) = \frac{K_{\mathrm{II}}(b)}{P/b^{3/2}} = \frac{\sqrt{b}\Omega_3(b)}{2\pi C} \tag{84}$$

where

$$C = \int_o^a \Omega_1(t)\,\mathrm{d}t \tag{85}$$

Figs. 6 and 7 illustrate the variation of $\overline{K}_{\mathrm{I}}(b)$ as a function of the radii ratio b/a and the depth of the embedment factor h/a. It is evident that $\overline{K}_{\mathrm{I}}(b)$ reduces to a negligible value when h/a exceeds 10, for $\nu \in (0, 1/2)$. Similar results for the variation of $\overline{K}_{\mathrm{II}}(b)$ with b/a and h/a are shown in Figs. 8 and 9. In this case, $\overline{K}_{\mathrm{II}}(b)$ reduces to a negligible value when h/a exceeds 2 for $\nu \in (0, 1/2)$. Numerical values of the stress intensity factors can be used in conjunction with their critical values $K_{\mathrm{I}}^{\mathrm{C}}$ and $K_{\mathrm{II}}^{\mathrm{C}}$ to evaluate the magnitude

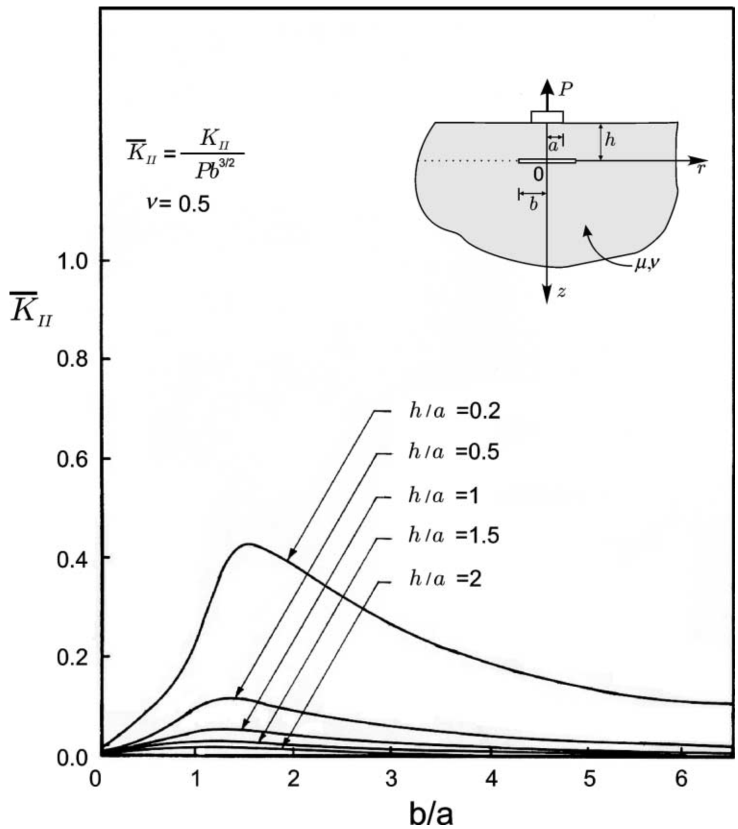

Fig. 9. Mode II stress intensity factor at the crack tip.

of loading required for crack extension. For most brittle elastic materials $K_{\mathrm{I}}^{\mathrm{C}} \ll K_{\mathrm{II}}^{\mathrm{C}}$ (see e.g. Sih, 1991) as such, the tensile loading is expected to induce self-similar extension of the crack in its plane.

5. Concluding remarks

An axisymmetric mixed boundary value problem associated with the loading of an elastic half-space region, containing a penny-shaped crack, by a bonded rigid circular disc is examined. The loading of the bonded circular disc is assumed to be such that the surfaces of the penny-shaped crack are maintained traction free. The analysis of the crack-bonded disc interaction problem can be reduced to a system of four coupled Fredholm integral equations of the second-kind. These integral equations can be solved numerically to evaluate the load–displacement response for the bonded rigid disc and the stress intensity factors at the tip of the penny-shaped crack. In the study of this problem, the stress singularity at the boundary of the bonded rigid disc is represented by the conventional $1/\sqrt{r}$-type singularity in contrast to the oscillatory form of the stress singularity which is characteristic of the result derived by adopting a Hilbert problem approach for bonded contact problems. Within the context of a numerical solution of the governing integral equations, the distinction between the two types of stress singularities become relatively minor, particularly in the evaluation of global results related to the load–displacement behaviour. The numerical results indicate that the load–displacement behaviour of the bonded disc is unaffected by the presence of a penny-shaped crack for relative depths of embedment $(h/a) > 10$ and for relative crack geometries $(b/a) \in (0, 6)$. At such depths of location of the penny-shaped crack, the load–displacement behaviour of the bonded circular disc can be evaluated by using the classical Mossakovski–Ufliand result applicable for the intact half-space. Also, for values of $(h/a) > 10$ and $(b/a) > 6$, the problem can be simplified by treating the influence of the loaded, bonded disc by a concentrated force P acting at the origin. In this case the results of importance pertains to the evaluation of the stress intensity factors at the crack tip. The problem essentially reduces the solution to a pair of Fredholm integral equations of the second-kind which can be solved in an analogous fashion. When the bonded rigid disc is subjected to a load in the z-direction (i.e., a compressive force) a more pertinent analysis should consider contact between the faces of the penny-shaped crack. Depending on the relative values of (h/a) and (b/a), the crack faces can experience contact and separation in the appropriate region. The analysis of this type of unilateral contact problem for a crack will require the identification of the regions of separation and/or contact through the integral equation formulation of the appropriate mixed boundary value problem.

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