Influence of Nonassociativity on the Bearing Capacity of a Strip Footing

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Abstract: This paper examines the ultimate bearing capacity of a strip footing located at the surface of a homogeneous soil. The approach adopted involves a numerical solution of the equations governing elastic-plastic soils with a nonassociative flow rule and makes use of the finite-difference code FLAC. This code is utilized to obtain the three bearing capacity factors for a wide range of values of the friction angle for four different values of the angle of dilation. The values of the bearing capacity factors obtained from the numerical approach are then compared with results derived from classical solutions modified to incorporate the nonassociative plastic flow of soil.

INTRODUCTION

The assessment of the ultimate bearing capacity is a requirement in the design of footings and other foundation units. The estimation of the ultimate bearing capacity is made invariably by reference to Terzaghi's (1943) classical result. Terzaghi proposed an equation for calculation of the bearing capacity of a strip footing (smooth and rough) that takes the form

$$q_u = c \cdot N_c + q \cdot N_q + \frac{1}{2} \gamma \cdot B \cdot N_\gamma \quad (1)$$

where c = cohesion; q = equivalent surcharge; γ = unit weight; and B = footing width. In (1) N_c, N_q, and N_γ are the bearing capacity factors dependent solely on the friction angle ϕ. Eq. (1) is valid for a situation where the shallow strip footing is subjected to a centrally located vertical load, which involves a symmetric failure pattern.

Many investigators have attempted to modify and extend Terzaghi's method for the calculation of bearing capacity. These methods may be classified into the following four categories: (1) the limit equilibrium method (Terzaghi 1943; Meyerhof 1951); (2) the method of characteristics (Prandtl 1920; Reissner 1924; Sokolovskii 1960; Hansen 1961; Bolton and Lau 1993); (3) the upper-bound plastic limit analysis (Shield 1954a,b; Chen 1975; Sarma 1979; Sarama and Iossifelis 1990; Drescher and Detournay 1993; Michalowski 1995, 1997; Soubra 1999); and (4) numerical methods based on either the finite-element technique or finite-difference method (Griffiths 1982; Frydman and Burd 1997).

In the case of dense granular materials, in particular, a key factor in its constitutive behavior is the presence of dilatancy, quantified by the dilation angle ψ. Dilatancy, in the context of a theory of plasticity, manifests itself as nonassociativity in the flow rule. An important aspect of the calculation of bearing capacity in dense granular materials relates to examining how this nonassociativity influences the bearing capacity of the footing. A number of researchers have used the finite-element method to calculate the bearing capacity of smooth (or rough) strip footings on soils regarded as nonassociative elastoplastic materials (Zienkiewicz et al. 1975; de Borst and Vermeer 1984; Mizuno and Chen 1990). The general observation is that the ultimate bearing capacity is influenced by the dilation angle ψ. It is noted that the variation of both friction angle ϕ and dilation angle ψ in these studies were limited to a small range. Recently, Frydman and Burd (1997) used finite-difference program FLAC (*FLAC* 1993) to calculate the bearing capacity factor N_γ of a strip footing (smooth and rough) for values of the friction angle values from 30° to 45° and the dilation angles that correspond to 0, ϕ/3, and 2ϕ/3 to ϕ.

The objective of this paper is to examine the effects of the nonassociativity in plasticity on the bearing capacity of a strip footing. A numerical solution is established using a finite-difference-based procedure in the FLAC (*FLAC* 1998) code.

FLAC (1998) is a commercially available two-dimensional (2D) finite-difference code. In this code, an explicit Lagrangian calculation scheme and a mixed discretization zoning technique are used. Such procedures ensure that plastic collapse load and continued plastic flow can be modeled accurately. The soil region is first divided into a finite-difference grid (or mesh) of quadrilateral elements. Internally, FLAC subdivides each element into two overlayed sets of constant-strain triangular elements.

In the modeling study, the width B of the footing is 6 m. Since the problem is symmetric, only half of the problem domain is considered. The half-domain has a depth of 15 m and extends 27 m beyond the edge of the footing. In the arrangement, the "boundary influence" on the estimation of the collapse load can be neglected. The domain is divided into 2,400 rectangular elements with a grid size of 0.5 m in the vertical direction and 0.375 m in the horizontal direction. Experience with a number of trial runs of the FLAC code indicates that uniform grids are convenient and result in high accuracy when compared with available analytical solutions.

The left vertical side is the plane of symmetry and appropriate displacement constraints are imposed along this plane, i.e., free movement of the vertical direction and a zero-displacement constraint in the horizontal direction. The right vertical side is constrained in the horizontal direction only. The soil layer is assumed to adhere to the base support. Accordingly, on this bottom plane, the displacements are constrained in both vertical and horizontal directions. The loading of the rough rigid strip footing is simulated by imposing equal vertical velocities at the nine top nodes, which correspond to the footing region. Furthermore, to simulate the influence of perfect adhesion at the soil-footing interface, the nine boundary nodes are constrained in the horizontal direction. Frydman and Burd (1997) have used a similar approach to represent a rough rigid footing. The magnitude of vertical velocities used in the present study has been chosen as a result of a number of verification runs. The chosen final value of vertical velocity is 2.5×10^{-6} m/step downward.

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J. Geotech. Geoenviron. Eng., 2001, 127(11): 985-989

An elastic-plastic model with the Mohr-Coulomb failure criterion is used in the finite-difference modeling. The elastic moduli used are the shear modulus $G = 100$ MPa, and the bulk modulus $K = 133$ MPa (equivalent to Young's modulus $E = 240$ MPa and Poisson's ratio $\nu = 0.2$). The friction angle ϕ is varied from 0° to 45° in 5° increments. For each value of the friction angle ϕ, the dilation angle ψ is varied according to the following: $\psi = n\phi$; $n = 0$, 1/3, and 2/3 to 1.

The three equivalent-bearing capacity factors N_c, N_q, and N_γ are calculated individually. For example, in order to calculate N_c, the surcharge q and unit weight γ are set equal to zero and $c = 10$ kPa. Based on (1), the factor $N_c = q_u/c$. Similarly, to calculate N_q, the cohesion c and unit weight γ are assumed to be zero and $q = 10$ kPa, which gives $N_q = q_u/q$. To calculate N_γ, the cohesion c and surcharge q are assumed to be zero and the unit weight $\gamma = 7.5$ kN/m^3 is selected (the magnitude of γ does not affect the value of N_γ). From (1), $N_\gamma = 2q_u/\gamma B$. In all of the above calculations, the ultimate bearing pressure q_u is calculated by using the vertical components of the nine node forces divided by the half-width of the footing, i.e., $B/2 = 3$ m.

In the calculation of N_γ, first an initial gravity field is established. A ratio of $N_R = 2q/\gamma B$ is defined, where q is the average footing pressure computed by considering the resultant of the vertical reaction forces at the nine nodes. Fig. 1 shows typical curves of N_R versus vertical displacement for a friction angle $\phi = 35°$ and the dilation angle ψ varying from $\psi = 0$, $\psi = \phi/3 = 11.67°$, and $\psi = 2\phi/3 = 23.33°$ to $\psi = \phi = 35°$. The results in Fig. 1 indicate that N_R increases with the dilation angle. Also, N_R reaches a limiting value for vertical displacement ratio $\Delta d_v/B = 0.035$ for $\psi = 35°$, and $\Delta d_v/B = 0.04$ for $\psi = 0$. This limiting value corresponds to the bearing capacity factor N_γ. It may be noted that when $\psi = 0$, the curve of N_R versus $\Delta d_v/B$ exhibits oscillation or fluctuations, which can be attributed to the inherent numerical aspect of the FLAC code. When such fluctuations persist (see the curve for $\psi = 0$ in Fig. 1), the ultimate value of N_γ is estimated as a mean value within the range of the fluctuations. Similar phenomena can be observed in computations involving discrete element simulations of fragmented geomaterials [e.g., Selvadurai and Sepehr (1999)].

Fig. 2 shows the displacement vector field. It is noted that the value of the maximum magnitude d_{max} of the displacement vector varies with ψ; i.e., referring to Fig. 2(a), $d_{max} = 10.11$ m for $\psi = 35°$; Fig. 2(b), $d_{max} = 1.43$ m for $\psi = 23.33°$; Fig. 2(c), $d_{max} = 0.4603$ m for $\psi = 11.67°$; and Fig. 2(d), $d_{max} = 0.375$ m for $\psi = 0°$. The large displacement for $\phi = \psi$ is due to the large dilation angle used. The large displacement values that occurred in the region of the footing edge may not be realistic but shall not affect the calculation of N_γ. In general, when the dilation angle is large, significant displacements are restricted to the exterior region close to the edge of the footing, as shown in Fig. 2(a). The displacements in this decrease with decreasing dilation angle is shown in Figs. 2(b–d). Fig. 2 also

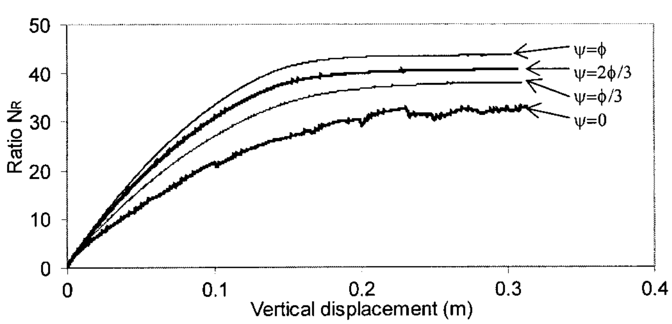

FIG. 1. Ratio N_R versus Vertical Displacement for Different Dilation Angle $\psi = \phi$, $2\phi/3$, $\phi/3$, and 0

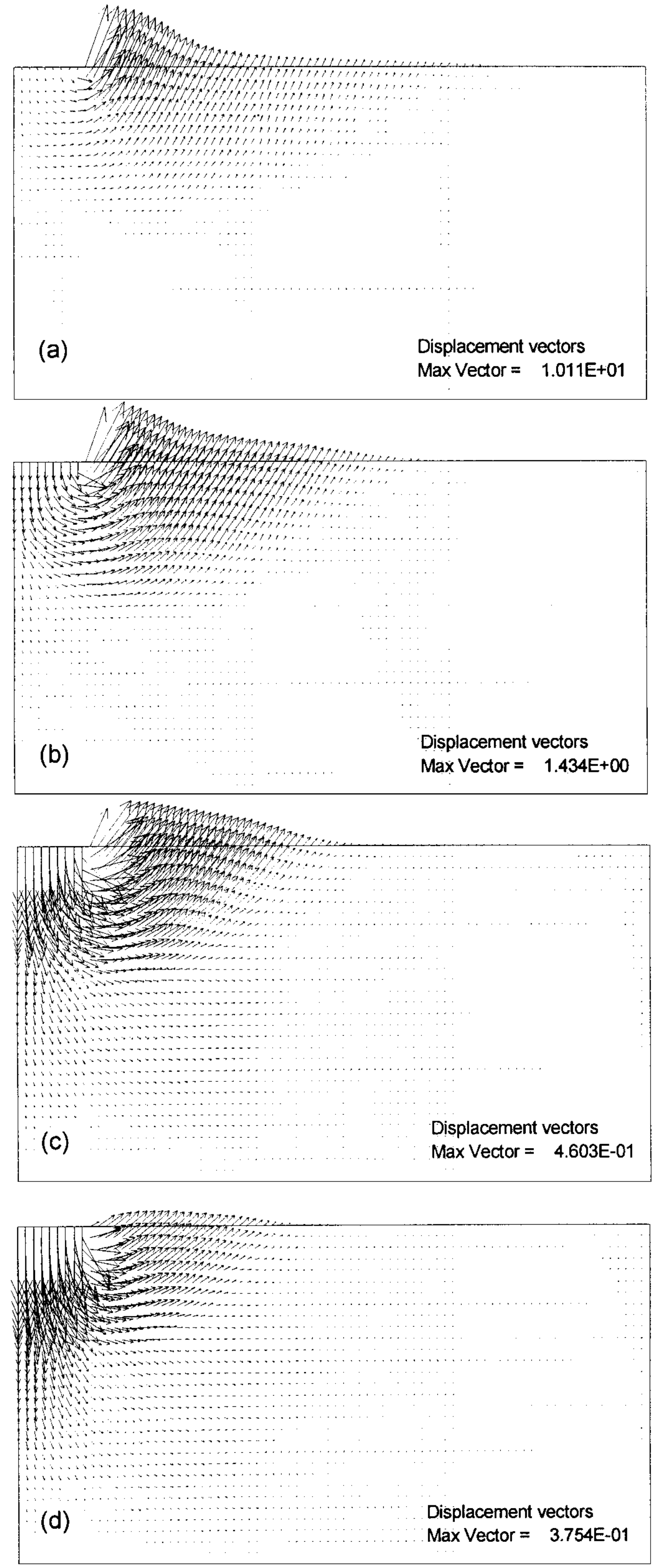

FIG. 2. Displacement Vector for N_R: (a) $\psi = \phi$; (b) $\psi = 2\phi/3$; (c) $\psi = \phi/3$; (d) $\psi = 0$

indicates that the dimensions of the half-domain used to present the semi-infinite region in the FLAC modeling are satisfactory, since no significant displacements occur near the extreme right boundary.

Fig. 3 depicts contours of the maximum shear strain rate corresponding to the displacement vector field shown in Fig. 2. Since the contour lines are very close to each other, no

J. Geotech. Geoenviron. Eng., 2001, 127(11): 985-989

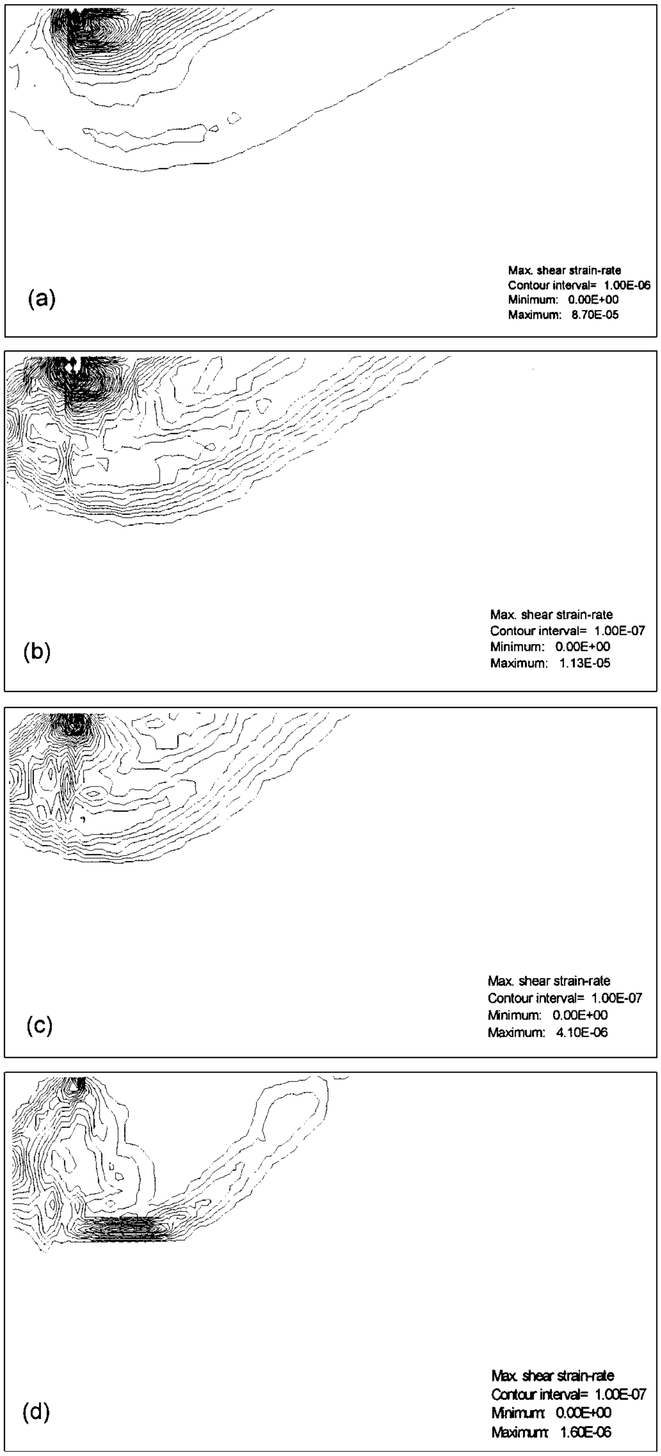

FIG. 3. Contours of Maximum Shear Strain Rate for N_R: (a) $\psi = \phi$; (b) $\psi = 2\phi/3$; (c) $\psi = \phi/3$; (d) $\psi = 0$

attempt is made to label the magnitudes. Fig. 3 is indicative of the shear strains developed in the soil mass subjected to the loading from the rough rigid footing. It is seen that the size of the shear zone decreases with decreasing value of the dilation angle. For $\phi = \psi$, as shown in Fig. 3(a), the shear zone is similar to the failure mechanism used by Prandtl (1920) and Terzaghi (1943) in the analysis of bearing capacity of a strip footing, i.e., a rigid triangular wedge immediately underneath the footing, a radial shear zone, and an emerging passive wedge exterior to the footing boundary. When the dilation angle is zero, however, as shown in Fig. 3(d), the overall shear zone is smaller, and both the triangular wedge and the radial shear zone are different from that employed by Prandtl (1920) and Terzaghi (1943).

The values for the three bearing capacity factors, N_c, N_q, and N_γ derived from the FLAC computation are summarized in Figs. 4–6. Values obtained by Prandtl (1920), Reissner (1924), Vesic (1973), and Michalowski (1997) were extended to consider nonassociative flow, and are presented in the figures as well. Discussion of the extended Prandtl (1920), Reissner (1924), and Vesic (1973) methods and results are presented in the following section.

MODIFIED ANALYTICAL SOLUTIONS ACCOUNTING FOR NONASSOCIATIVITY OF PLASTIC FLOW

Prandtl (1920) and Reissner (1924) derived the following expressions for N_c and N_q for a rough footing:

$$N_c = [N_q - 1]\cot\phi \tag{2}$$

$$N_q = \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) e^{\pi\tan\phi} \tag{3}$$

Vesic (1973) suggested the following expression for N_γ:

$$N_\gamma = 2(N_q + 1)\tan\phi \tag{4}$$

The above solutions are considered to be upper-bound estimates based on limit force equilibrium and an associative flow rule, i.e., $\psi = \phi$. When $\psi \neq \phi$, the plastic potential surface is not identical with the yield surface. In the case of plane-strain condition, velocity characteristics are not identical with the

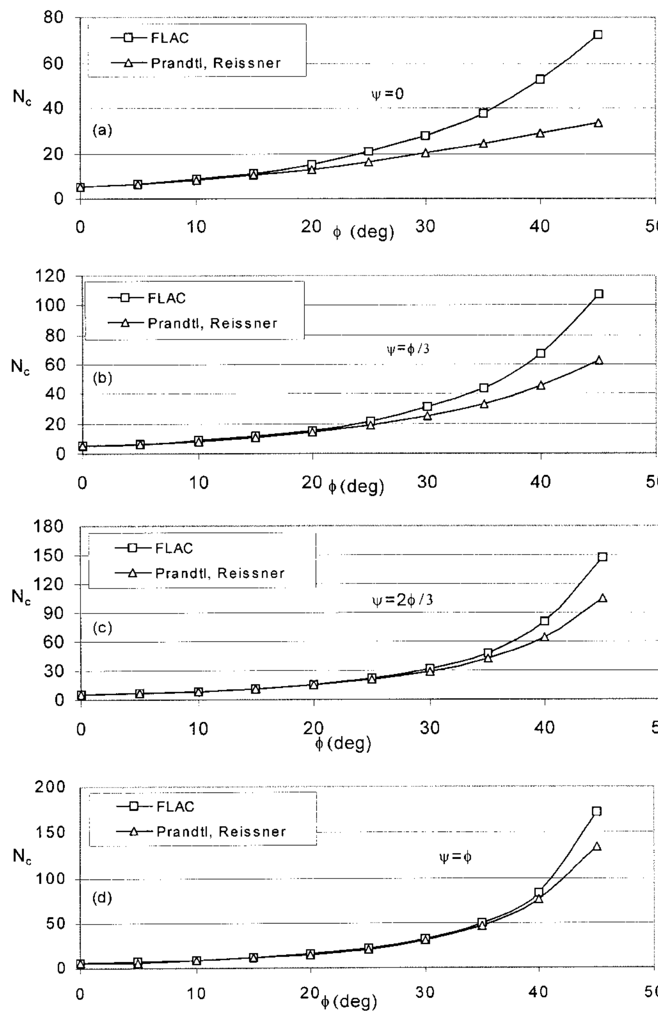

FIG. 4. Comparison of N_c: (a) $\psi = 0$; (b) $\psi = \phi/3$; (c) $\psi = 2\phi/3$; (d) $\psi = \phi$

J. Geotech. Geoenviron. Eng., 2001, 127(11): 985-989

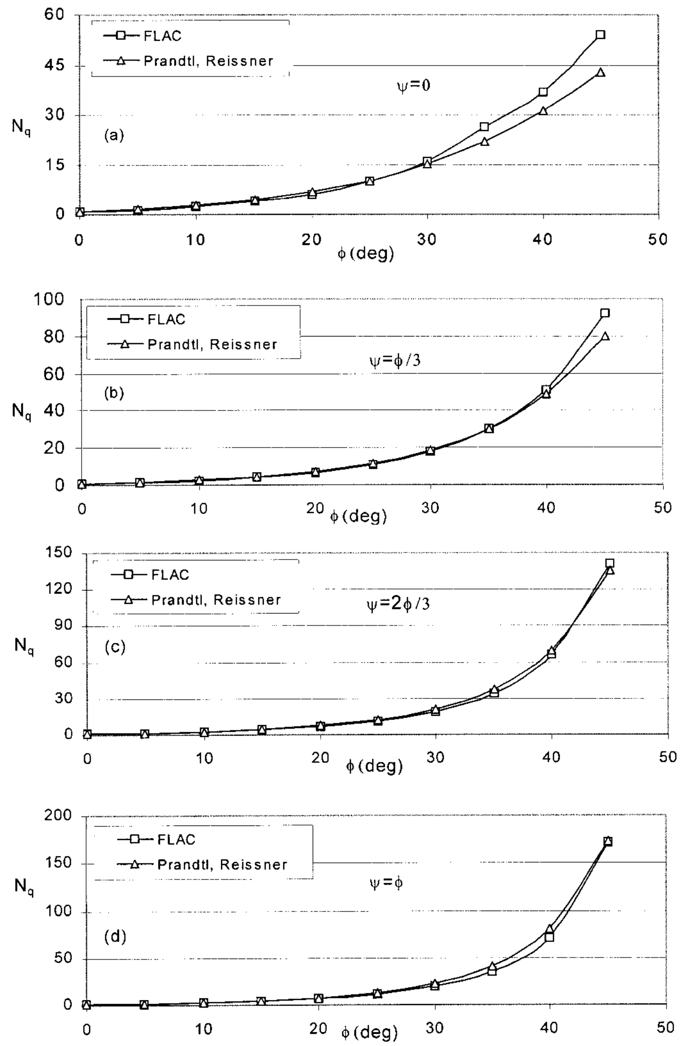

FIG. 5. Comparison of N_q: (a) $\psi = 0$; (b) $\psi = \phi/3$; (c) $\psi = 2\phi/3$; (d) $\psi = \phi$

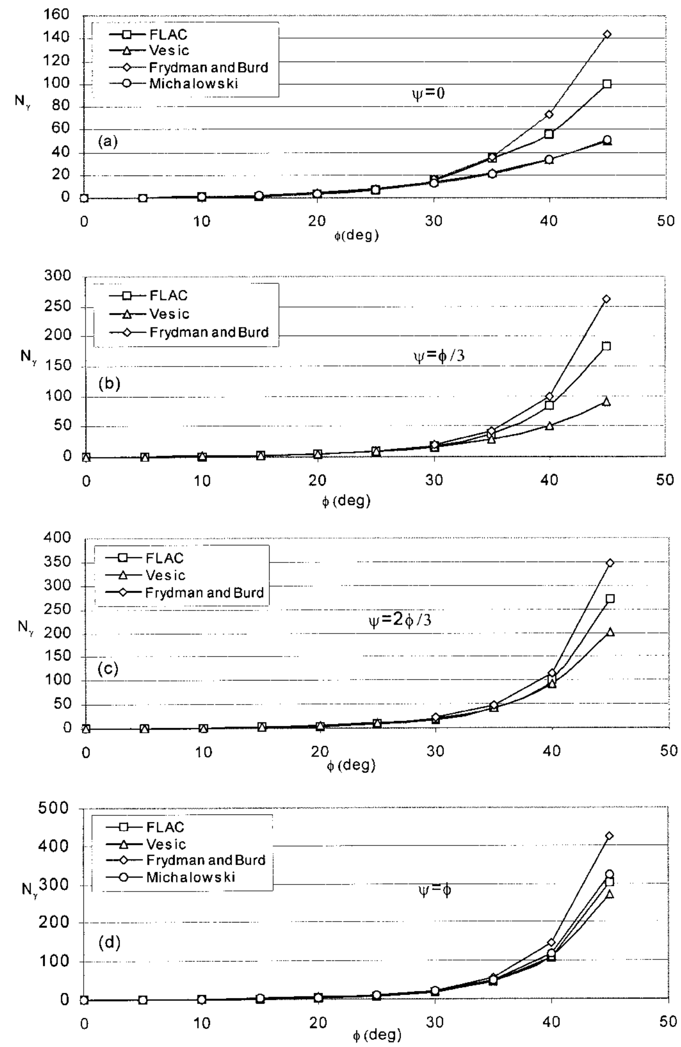

FIG. 6. Comparison of N_γ: (a) $\psi = 0$; (b) $\psi = \phi/3$; (c) $\psi = 2\phi/3$; (d) $\psi = \phi$

stress characteristics (Davis 1968). Davis found that, on a velocity characteristic, the shear stress τ^* and the effective normal stress σ^* satisfy a Mohr-Coulomb-type equation

$$\tau^* = c^* + \sigma_n'^* \tan \phi^* \tag{5}$$

where

$$\frac{\tan \phi^*}{\tan \phi} = \frac{c^*}{c} = \mu; \quad \mu = \frac{\cos \psi \cos \phi}{1 - \sin \phi \sin \psi} \tag{6a,b}$$

Only when $\psi = \phi$, are c^* and ϕ^* the same as Coulomb's c and ϕ. For $\psi < \phi$, both c^* and ϕ^* are $<c$ and ϕ, respectively.

For soils following a nonassociative flow rule, as suggested by Drescher and Detournay (1993), and applied further by Michalowski and Shi (1995) and Michalowski (1997), the modified cohesion c^* and friction angle ϕ^* given by (6) can be used in the expressions for N_c and N_q in (2) and (3), and N_γ in (4). The results calculated using this ''modification'' to the classic solutions (Michalowski and Shi 1995; Michalowski 1997) are presented in Figs. 4–6.

ANALYSIS AND DISCUSSION OF RESULTS

As seen from Figs. 4–6, for $\psi = \phi$ (associative flow case), values of N_c, N_q, and N_γ from the present FLAC modeling are, in general, slightly larger than those from the classic solutions (except for N_q). As ψ decreases from $\psi = \phi$ to $2\phi/3$, $\phi/3$, and 0, the values of N_c, N_q, and N_γ from FLAC become gradually larger than those from the modified solutions. The results from FLAC modeling for $\psi = \phi$ are smaller than the values obtained by Frydman and Burd (1997). The FLAC results are smaller than the values for $\psi = \phi$, but larger than the values for $\psi = 0$ obtained from Michalowski (1997). Values from Vesic's (1973) solution are smaller than the FLAC results. From the results in Figs. 1–6, it is evident that the dilation angle ψ has a significant influence on the three bearing capacity factors. The larger the dilation angle, the larger the values of three bearing capacity factors. Relative differences for N_c, N_q, and N_γ may be defined as $(N_{c(\psi=\varphi)} - N_{c(\psi=0)})/N_{c(\psi=\varphi)}$, $(N_{q(\psi=\varphi)} - N_{q(\psi=0)})/N_{q(\psi=\varphi)}$, and $(N_{\gamma(\psi=\varphi)} - N_{\gamma(\psi=0)})/N_{\gamma(\psi=\varphi)}$. For $\phi = 15°$, the relative differences for 2.6% and N_c, 4.3% for N_q, and 8.9% for N_γ. And for $\phi = 45°$, the relative differences are 57.9% for N_c, 68.4% for N_q, and 66.8% for N_γ. The results from Michalowski (1997) show a similar influence of the dilation angle on N_γ (Fig. 6).

CONCLUSIONS

From the results of numerical calculations using FLAC, the following conclusions may be drawn:

- The dilation angle ψ has significant influences on the values of the three bearing capacity factors N_c, N_q, and N_γ. The relative differences increase with the values of friction angle and the dilation angle. The relative differences vary from zero for friction angle $\phi = 0$ to 57.9, 68.4, and 66.8% for N_c, N_q, and N_γ, respectively, for $\phi = 45°$.

J. Geotech. Geoenviron. Eng., 2001, 127(11): 985-989

- Since the dilation angle greatly affects the bearing capacity factors, it is advisable to use a proper value of the dilation angle to find the corresponding values of the three factors for foundation designs.

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J. Geotech. Geoenviron. Eng., 2001, 127(11): 985-989