Pergamon

Mechanics Research Communications, Vol. 28, No. 2, pp 157–164, 2001

0093-6413/01/$–see front matter

PII: S0093-6413(01)00157-4

Mindlin's Problem for a Halfspace with a Bonded Flexural Surface Constraint

A.P.S. Selvadurai
Department of Civil Engineering and Applied Mechanics
McGill University,
817 Sherbrooke Street West, Montreal, QC, Canada H3A 2K6

Email: apss@civil.lan.mcgill.ca

(Received 8 January 2001; accepted for 21 February 2001)

1 Introduction

The classical problem of the internal loading of an isotropic elastic infinite space by a concentrated force acting at its interior was first presented by Kelvin [1]. This result essentially forms the basis for obtaining solutions to a variety of problems involving nuclei of strain and presents an alternative technique for determining the solution of Boussinesq's [2] classical result for the localized loading of an elastic halfspace by a concentrated force acting normal to its boundary (See also Selvadurai [3]). The celebrated work of Mindlin [4] generalized the problem in such a way that both Kelvin's solution and Boussinesq's solution occur as limiting cases of a *single* solution. To achieve this, Mindlin examined the problem of the loading of a halfspace region by a localized force that acts at a finite distance from a traction free boundary and directed normal to it. Mindlin's problem is recognized not only for its elegance of analysis but also for the utility of the solution in examining problems related to load transfer from embedded objects such as anchoring devices, pile foundations, mechanics of fibre reinforcement, etc. The generalization of Mindlin's problem to include bi-material regions, material orthotropy, etc. is documented in literature in solid mechanics and materials science (see, e.g., Selvadurai [5], Gladwell [6] and Mura [7]). The examination of Mindlin's problem to include effects of surface constraints was first presented by Selvadurai [8]. This study represents the treatment of Boussinesq's classical problem for the indentation of a halfspace by a smooth rigid indentor in the presence of a Mindlin force. In subsequent studies, the condition related to a *bi-material smooth contact*, along with the shape of the indentor profile, was relaxed. All these studies have culminated in the development of *exact closed form solutions* that describe the indentor behaviour [9,10].

In this paper we extend Mindlin's classical problem to include the case where the surface of the halfspace is reinforced by a bonded thin plate, which satisfies the classical Germain-Poisson-Kirchhoff theory. The analysis of the problem of interaction between a thin plate of infinite extent and an underlying elastic halfspace was presented independently by Hogg [11] and Holl [12]. According to Korenev [13], the analysis of this problem was also presented by Shekhter [14] and Leonev [15]. Sneddon et al. [16] and Selvadurai [17] give further studies related to the unbonded interaction problem. Korenev [13], Hetenyi [18], Selvadurai [5] and Gladwell [6] give comprehensive reviews of the mechanics of plates resting on elastic foundations. In many of the problems dealing with the contact between an infinite elastic plate and an elastic halfspace, the contact is assumed to be bilateral and smooth. This implies that the interface between the plate and the halfspace is smooth but capable of sustaining tensile tractions. This interaction model is, of course, valid for situations where the entire plate is subjected to either a precompression or the self weight of the infinite plate is included to maintain the interface stresses compressive over the entire central region. When such precompression is absent, the contact invariably involves separation and the problem needs to be formulated as a unilateral contact problem where the extent of the contact zone now becomes an unknown in the problem. This class of problems has been investigated in the literature (see, e.g., Gladwell [19], Laermann [20]) and references to further work can be found in references [5], [6] and [18]. The alternative to both bilateral and unilateral contact conditions assumes that the infinite plate is completely bonded to the interface. In this case the zero shear traction boundary condition is replaced by a zero radial displacement boundary condition, in instances where the bonded plate exhibits inextensibility constraints in its plane. This is perhaps a more realistic contact condition, which is applicable to structural elements that are invariably bonded to the underlying elastic substrate to generate a greater interactive stiffness under transverse loads. This paper presents the solution to the problem of a halfspace the surface of which is reinforced with a bonded flexible thin plate and internally loaded by a Mindlin force (Figure 1). The objective of the exercise is to generate integral expressions for the deflections and flexural moments in the bonded plate due to the action of the internal Mindlin force.

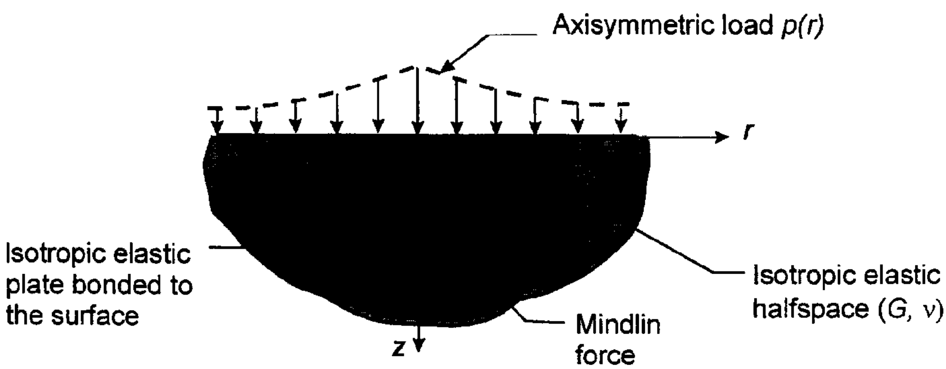

Figure 1. Internally loaded elastic halfspace with a bonded flexural surface constraint

2 Axisymmetric problem for a surface constrained halfspace

We consider the axisymmetric problem of a thin plate of thickness t and infinite extent which is bonded to the surface of a halfspace and loaded by an external axisymmetric load $p(r)$ and a Mindlin force of magnitude P_M, which is located at a distance h from the bonded plate (Figure 1). The plate is assumed to be *inextensible* in its plane; this introduces a zero radial displacement constraint at the surface of the halfspace. The objective of the preliminary analysis is to develop the relationship between an applied axisymmetric surface normal stress and the corresponding axisymmetric surface displacement in the axial direction. The solution of this class of problem can be approached by appeal to Love's [21] strain function ($\Phi(r, z)$)(See also Selvadurai [22]) formulation, where the governing partial differential equation is

$$\nabla^2\nabla^2\Phi(r,z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is Laplace's operator in axisymmetric cylindrical polar coordinates. By adopting a Hankel integral transform solution of (1) such that the zeroth-order Hankel transform of ($\Phi(r,z)$) is defined by (Sneddon [23])

$$\overline{\Phi}^0(\xi,z) = H_0\{\Phi(r,z);\xi\} = \int_0^\infty r\Phi(r,z)J_0(\xi r)dr \tag{3}$$

it can be shown that the relationship between the transformed values of the surface displacement and the applied normal contact stress for a halfspace with a *zero radial displacement constraint* can be evaluated in the form

$$\overline{w}_I^0(\xi) = \frac{(3-4\nu)}{4G(1-\nu)\xi}\overline{q}^0(\xi) \tag{4}$$

where ν and G are Poisson's ratio and the linear elastic shear modulus, respectively, and

$$\overline{u}_z(\xi,0) = \overline{w}_I^0(\xi) = \int_0^\infty r u_z(r,0)J_0(\xi r)dr \tag{5}$$

$$\overline{\sigma}_{zz}(\xi,0) = \overline{q}^0(\xi) = \int_0^\infty r q(r)J_0(\xi r)dr \tag{6}$$

We can also develop a similar result for the problem involving the action of a Mindlin force P_M at the interior of the halfspace, with the revised boundary conditions

$$u_r^s(r,0) = 0$$
$$\sigma_{zz}^s(r,0) = 0 \qquad (7)$$

The solution to this problem can be obtained by simply superposing two Kelvin Forces (P_M) (associated with the *infinite space*) both of which act in the negative z-direction but placed at $z = \pm h$. For this combination of Kelvin forces the boundary conditions (7) are exactly satisfied and

$$u_z^s(r,0) = \frac{-P_M}{8\pi G(1-\nu)}\left[\frac{(3-4\nu)}{(r^2+h^2)^{1/2}} + \frac{h^2}{(r^2+h^2)^{3/2}}\right] \qquad (8)$$

The zeroth-order Hankel transform of the surface displacement (8) due to the action of P_M in a radially surface constrained, normal traction free halfspace is given by

$$\bar{u}_z^{s0}(\xi,0) = \bar{w}_{II}^{0}(\xi) = -\frac{(3-4\nu)}{4G(1-\nu)\xi}\bar{s}^{0}(\xi) \qquad (9)$$

where

$$\bar{s}^{0}(\xi) = \frac{P_M}{2\pi}\left[1 + \frac{\xi h}{(3-4\nu)}\right]e^{-\xi h} \qquad (10)$$

3 The halfspace with a flexural surface constraint

We now consider the axisymmetric problem of the halfspace, the surface of which is constrained by a bonded thin flexible plate. The surface of the plate is subjected to an external stress $p(r)$ and the internal Mindlin force P_M. Since there is no loss of contact at the bonded interface, the surface displacement of the halfspace corresponds to the flexural deflection of the thin plate $w(r)$. The differential equation governing the flexure of the thin plate is

$$D\tilde{\nabla}^2\tilde{\nabla}^2 w(r) + q_c(r) = p(r) \qquad (11)$$

where $D=E_p t^3/12(1-\nu_p^2)$, is the flexural rigidity of the plate, E_p and ν_p are elastic constants of the plate material, $q(r)$ is the contact normal stress at the bonded surface and

$$\tilde{\nabla}^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \qquad (12)$$

Operating on (12) with the zeroth-order Hankel transform we obtain

$$D\xi^4 \bar{w}^0(\xi) + \bar{q}_c^0(\xi) = \bar{p}^0(\xi) \tag{13}$$

The relationship between the surface displacement of the halfspace due to the combined action of $q(r)$ and the Mindlin force can be obtained by combining (4) and (9), i.e.,

$$\bar{w}^0(\xi) = \frac{(3-4\nu)}{4G(1-\nu)\xi}\left[\bar{q}_c^0(\xi) - \bar{s}^0(\xi)\right] \tag{14}$$

We can eliminate $\bar{q}^0(\xi)$ between (13) and (14) to obtain an expression for $\bar{w}^0(\xi)$. Inverting the result we obtain

$$w(r) = \frac{(3-4\nu)}{4G(1-\nu)}\int_0^\infty \left[\bar{p}^0(\xi) - \frac{P_M}{2\pi}\left\{1 + \frac{\xi h}{(3-4\nu)}\right\}e^{-\xi h}\right]\frac{J_0(\xi r)d\xi}{\left[1 + R^*\xi^3\right]} \tag{15}$$

where

$$R^* = \frac{D(3-4\nu)}{4G(1-\nu)} \tag{16}$$

Expressions for the flexural moments and shear force in the plate can be obtained from the results

$$M_r = -D\left[\frac{d^2 w}{dr^2} + \frac{\nu_p}{r}\frac{dw}{dr}\right] \tag{17}$$

$$M_\theta = -D\left[\frac{1}{r}\frac{dw}{dr} + \nu_p \frac{d^2 w}{dr^2}\right] \tag{18}$$

$$Q_r = D\left[\frac{d}{dr}\left(\tilde{\nabla}^2 w(r)\right)\right] \tag{19}$$

This formally completes the analysis of the Mindlin problem for a halfspace with a bonded flexural surface constraint. The expressions for the displacements and stresses within the halfspace region can be obtained by superposing on the Kelvin doublet solution the displacement and stress fields derived from the contact stress $q_c(r)$ which is defined through (14). In the case where $p(r)$ has an arbitrary axisymmetric form the resulting integral expressions for $w(r)$, $q_c(r)$, $M_r(r)$, etc., can be evaluated only through numerical techniques.

For example, in order to evaluate the displacement $w(0)$ of the stiffening surface plate we need to evaluate integrals of the type

$$I = \int_0^\infty \frac{\xi\, e^{-\lambda\xi}}{\left[1+\xi^3\right]} d\xi \tag{20}$$

Although symbolic manipulations through the use of software such as MAPLE™ and MATHEMATICA™ can be used to represent the integral (20) in a compact form as follows

$$I = -\frac{1}{3} e^{\lambda} Ei(1,\lambda) + \frac{e^{-\lambda/2}}{6\sqrt{C_1}} \left\{ C_1 C_2 Ei(1,\overline{C}_3) + \overline{C}_2 Ei(1, C_3) \right\} \tag{21}$$

where

$$\begin{aligned} C_1 &= (-1)^{\lambda\sqrt{3}/\pi}; \\ C_2 &= 1 + i\sqrt{3}; \\ C_3 &= -\frac{\lambda}{2} C_2 \end{aligned} \tag{22}$$

$\overline{C}_2$ and $\overline{C}_3$ are complex conjugates and $Ei(n,x)$ is the exponential integral defined by

$$Ei(n,x) = \int_1^\infty e^{-xt} t^{-n} dt \tag{23}$$

these integrals themselves need to be evaluated separately or recourse must be made to the tabulated values of the integrals given in the literature (see e.g. Abramowitz and Stegun [24]. In these circumstances it may be more convenient to evaluate the integral expressions for the displacement, flexural stresses etc., by directly using a quadrature scheme.

4 A Limiting case

It is instructive to record results for a certain limiting case to establish the influence of the adhesive nature of the contact between the elastic plate and the halfspace. Consider the specific case when the plate is subjected purely to an external concentrated force P^* such that $\bar{p}^0(\xi) = P^*/2\pi$. The expression for $w(r)$ given by (15) (with $P_M = 0$) reduces to

$$w(r) = \frac{(3-4\nu)P^*}{8\pi G(1-\nu)} \int_0^\infty \frac{J_0(\xi r)d\xi}{(1+R^*\xi^3)} \tag{24}$$

We can evaluate $w(0)$ as

$$[w(0)]_{bonded} = \frac{\sqrt{3}}{9}\left[\frac{P^*}{D^{1/3}}\right]\left[\frac{(3-4\nu)}{4G(1-\nu)}\right]^{2/3} \tag{25}$$

Consider the case where the plate is in unbonded but bilateral contact with the elastic halfspace and subjected to purely a localized concentrated force P^*. The maximum deflection at the point of application of P^* is (Hogg [11])

$$[w(0)]_{bilateral} = \frac{\sqrt{3}}{9}\left[\frac{P^*}{D^{1/3}}\right]\left[\frac{(1-\nu)}{G}\right]^{2/3} \tag{26}$$

The ratio of the maximum deflection due to the bonded and bilateral contact between the plate and the elastic halfspace can be written as

$$W^* = \frac{[w(0)]_{bonded}}{[w(0)]_{bilateral}} = \left[\frac{(3-4\nu)}{4(1-\nu)^2}\right]^{2/3} \tag{27}$$

As $\nu \to 1/2$, $W^* \to 1$, indicating that the displacement of the plate is uninfluenced by the nature of the adhesive/bilateral contact conditions. This is consistent with responses obtained for other classes of adhesive contact problems in elastostatics. The primary reason for this coincidence of results is that, in the case of material incompressibility, the radial displacements at the surface of the halfspace due to the application of any form of axisymmetric normal contact stress is zero. In exactly the same way, when $\nu = 1/2$, Kelvin's solution for the interior loading of an infinite space also furnishes the solution to Boussinesq's problem for the surface normal loading of a halfspace with both radial traction and radial displacement being zero at the surface of the halfspace.

When $\nu = 0$

$$W^* = \sqrt[3]{\frac{9}{16}} = 0.825 \tag{28}$$

which indicates that the displacement of an infinite plate, which is bonded to a halfspace, is less than that for the infinite plate in smooth contact with an elastic halfspace.

References

[1] Lord Kelvin, Note on the integrations of the equations of equilibrium of an elastic solid, *Camb. and Dublin Math. J.*, **3** (1848).

[2] J. Boussinesq, *Applications des Potentials à l'étude de l'équilibre et du mouvement des solides élastique,* Gauthier-Villars, Paris (1885).

[3] A.P.S. Selvadurai, On Boussinesq's problem, *Int. J. Engng. Sci.*, **39**:319-323 (2001).

[4] R.D. Mindlin, Force at a point in the interior of a semi-infinite solid, *Physics,* **7**, 195-202 (1936).

[5] A.P.S. Selvadurai, *Elastic Analysis of Soil-Foundation Interaction,* Elsevier Sci. Publ., Amsterdam (1979).

[6] G.M.L. Gladwell, *Contact Problems in the Classical Theory of Elasticity,* Sijthoff and Noordhoff, Dordrecht (1980).

[7] T. Mura, *Micromechanics of Defects in Solids,* Sijthoff and Noordhoff, The Netherlands (1987).

[8] A.P.S. Selvadurai, The interaction between a rigid circular punch on an elastic halfspace and a Mindlin force, *Mech. Res. Comm.*, **5**, 57-64 (1978).

[9] A.P.S. Selvadurai, Unilateral contact between a rigid circular punch on a halfspace and a Mindlin force, *Mech. Res. Comm.*, **17**, 181-187 (1990).

[10] A.P.S. Selvadurai, Hertzian contact in the presence of a Mindlin force, *J. Appl. Math. Phys. (ZAMP),* **41**, 865-874 (1990).

[11] A.H.A. Hogg, Equilibrium of a thin plate symmetrically loaded and resting on an elastic foundation, *Phil. Mag. Ser 7,* **25**, 576-582 (1938).

[12] D.L. Holl, Thin plates on elastic foundations, *Proc. 5th Int. Congr. Appl. Mech.*, Cambridge, Mass., Wiley, New York, 71-74 (1938).

[13] B.G. Korenev, Structures resting on elastic foundations, in: *Structural Mechanics in the USSR* (I.M. Rabinovich, Ed), Pergamon Press, 160-190 (1960).

[14] O. Ia. Shekhter, Analysis of an infinite plate resting on an elastic foundation of finite and infinite thickness and loaded by a concentrated force, *Sb. Tr. Nauk. Iss. Sekt. Fund.*, **10** (1939).

[15] M.Ya. Leonev, Consideration of horizontal forces when calculating an infinite plate lying on an elastic halfspace, *Prikl. Math. Mech,*,. **3**, (3) (1939).

[16] I.N. Sneddon, G.M.L. Gladwell and S. Coen, Bonded contact of an infinite plate and an elastic foundation, *Letters in Appl. and Engng Sci.*, **3**, 1-13 (1975).

[17] A.P.S. Selvadurai, The influence of a Mindlin force on the axisymmetric interaction between an infinite plate and an elastic halfspace, *Letters in Appl. and Engng Sci.*, **5**, 379-388 (1977).

[18] M. Hetenyi, Beams and plates on elastic foundations, *Appl. Mech. Rev.*, **19**, 95-102 (1966).

[19] G.M.L. Gladwell, Unbonded contact between a circular plate and an elastic foundation, in: *The Mechanics of Contact Between Deformable Bodies,* Proc. IUTAM Symp., Enschede, (A.D. de Pater and J.J. Kalker, Eds), Delft University Press, 99-109 (1975).

[20] K.H. Laermann, Plates on elastic foundation considering friction and real contact area, *Proc. 6th CANCAM,* Vancouver, 31-32 (1977).

[21] A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity,* Cambridge University Press, Cambridge (1927).

[22] A.P.S.Selvadurai, *Partial Differential Equations in Mechanics. Vol. 2; The Biharmonic Equation, Poisson's equation*, Springer-Verlag, Berlin (2000).

[23] I.N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York (1951).

[24] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Applied Mathematics Series, 55, National Bureau of Standards, Wasington, D.C. (1970).