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On the Displacements of an Elastic Half-Space Containing a Rigid Inhomogeneity*

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ABSTRACT. *This article examines the problem of an isotropic elastic half-space region that is reinforced with a rigid disc-shaped inhomogeneity and subjected to a uniform surface load of finite extent. The analysis of the axisymmetric problem is reduced to the solution of a set of integral equations that are solved numerically to obtain results of practical importance, particularly to geomechanics.*

I. Introduction

The problem of the surface loading of an isotropic elastic half-space due to a uniform circular load is one of considerable importance to geomechanics and contact mechanics. The results derived from this solution are used to estimate surface settlements of deep soil deposits, which are subjected to surcharge loads. Results for the displacements and stresses in the half-space region can be obtained through integration of the classical solution of Boussinesq[1] for the surface loading of a half-space by a concentrated force (see also Selvadurai[2]). An alternative procedure for the solution of this problem employs Hankel integral transform techniques. The basic problem has been extended to cover a variety of other modifications, including effects of elastic anisotropy, material nonhomogeneity, layering and the influence of basal rigid boundaries located at a finite depth below the loaded surface. The publications in this area are too numerous to be cited in their entirety; the reader is referred to the classic texts by Galin,[3] Ufliand,[4] Lur'e,[5] and those by Poulos and Davis,[6] Selvadurai,[7] Gladwell,[8] Johnson,[9] Mura[10] and Davis and Selvadurai[11] for comprehensive records of important contributions and developments in this area. The review articles by Hetenyi,[12] Goodman,[13] Gibson,[14] and Selvadurai[15,16] also contain relevant references to further work.

In this article we examine the problem of the uniform surface loading of a half-space that contains a rigid circular disc-shaped inhomogeneity or inclusion embedded in bonded contact and located at a finite depth below its surface (Figure 1). The problem can be regarded as a situation in geomechanics where either an anomaly in the form of a rigid inhomogeneity exists within the half-space or such a zone of relatively stiffer material can be created by a process such as injection grouting (Figure 2). The consideration of the reinforcing region as a rigid plate region with a circular boundary exhibiting axial symmetry is, of course, a gross idealization of a practical situation that can involve arbitrary orientation of the anomaly and possible deformations of the anomaly itself. Such situations can best be examined by recourse to numerical simulations based on finite element and boundary element techniques. The idealized problem, nonetheless, serves as a useful first approximation for examining features such as effectiveness of reinforcing regions in minimizing the surface displacements and, most importantly, in the case where the anomaly is created, provides certain guidelines with regard to its usefulness. A problem of related interest

* In Memoriam: Ian Naismith Sneddon, FRS (1919-2000)

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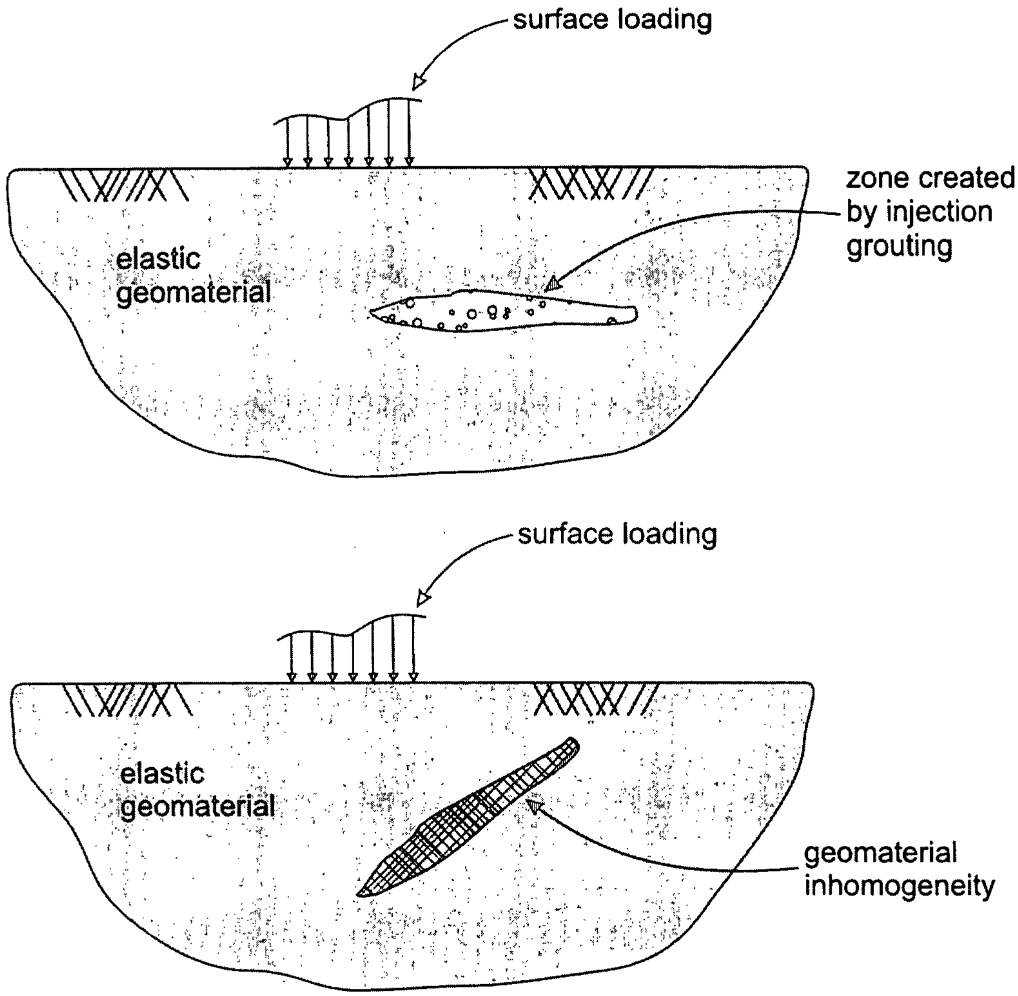

FIGURE 1 Embedded inhomogeneity in an elastic geomaterial half-space region.

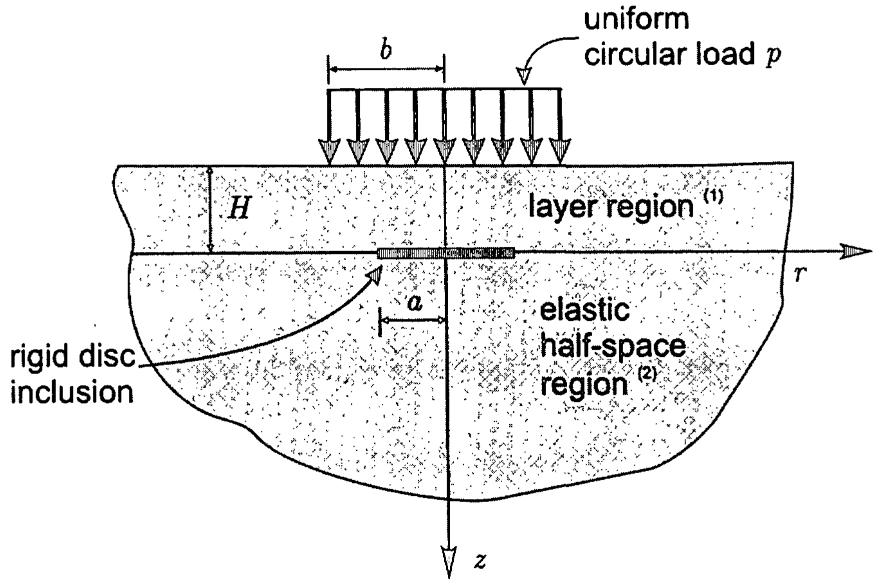

FIGURE 2 Idealized axisymmetric problem for a disc-shaped rigid inhomogeneity in a surface loaded half-space.

involving a disc-shaped anchor embedded in bonded contact in a half-space region was also examined by Selvadurai.[17]

The problem related to a half-space containing an inhomogeneity is treated as a classical mixed boundary value problem in the theory of elasticity, which takes into consideration the bonding between the inhomogeniety and the surrounding elastic medium, and the rigid body displacement of the inhomogeneity itself. The mathematical formulation of the problem is ultimately reduced to the solution of a system of integral equations of the Fredholm-type that are solved numerically to determine results of interest to geomechanics.

II. Governing Equations

The idealized axisymmetric problem refers to the uniform circular loading (radius 'b' and intensity p_0) of an isotropic elastic half-space region that contains a rigid disc-shaped inhomogeneity of radius 'a' that is located at a finite depth h below the surface of the half-space. The uniform loading is applied over a finite region $b > a$ or $b < a$ (Figure 1). The rigid disc-shaped inhomogeneity exhibits complete bonding with the surrounding elastic medium. Due to the application of the circular surface load the rigid inhomogeneity exhibits a rigid displacement Δ_0 in the axial direction. This displacement is an unknown in the problem and needs to be determined though the consideration of the null resultant force acting on the disc-shaped region. This class of axisymmetric problem in the classic theory of elasticity can be examined by appeal to either the biharmonic strain function approach proposed by Love[18] (see also Selvadurai[19]) or the Papkovich-Neuber (see, e.g., Westergaard[20]) approach based on harmonic functions. In this instance, it is convenient to employ the latter approach and to introduce the functions $\varphi(r,z)$ and $\chi(r,z)$, which, in the absence of body force fields, satisfy

$$\nabla^2\varphi(r,z) = 0 \quad , \quad \nabla^2\chi(r,z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system. The displacements and stress components derived from $\varphi(r,z)$ and $\chi(r,z)$ and referred to the cylindrical polar coordinate system are given by

$$u_r(r,z)=\frac{\partial\varphi}{\partial r}-z\frac{\partial\chi}{\partial r} \tag{3}$$

$$u(r,z)=(3-4v)\chi+\frac{\partial\varphi}{\partial z}-z\frac{\partial\chi}{\partial z} \tag{4}$$

and

$$\begin{aligned}
\sigma_{rr}(r,z)&=2\mu\left[\frac{\partial^2\varphi}{\partial r^2}+2v\frac{\partial\chi}{\partial z}-z\frac{\partial^2\chi}{\partial r^2}\right]\\
\sigma_{\theta\theta}(r,z)&=2\mu\left[\frac{1}{r}\frac{\partial\varphi}{\partial r}+2v\frac{\partial\chi}{\partial z}-\frac{z}{r}\frac{\partial\chi}{\partial r}\right]\\
\sigma_{zz}(r,z)&=2\mu\left[2(1-v)\frac{\partial\chi}{\partial z}+\frac{\partial^2\varphi}{\partial z^2}-z\frac{\partial^2\chi}{\partial z^2}\right]\\
\sigma_{rz}(r,z)&=2\mu\left[(1-2v)\frac{\partial\chi}{\partial r}+\frac{\partial^2\varphi}{\partial r\partial z}-z\frac{\partial^2\chi}{\partial r\partial z}\right]
\end{aligned} \tag{5}$$

where v is Poisson's ratio and μ is the linear elastic shear modulus. For the analysis of the mixed boundary value problem concerning the embedded inhomogeneity it is convenient to identify a half-space $(^{(2)})$ occupying the region $r\in(0,\infty)$; $z\in(0,\infty)$ and a layer $(^{(1)})$ occupying the region $r\in(0,\infty)$; $z\in(-h,0)$.

The boundary and continuity conditions governing the problem are as follows:

$$\sigma_{zz}^{(1)}(r,-h)=\begin{cases}-\sigma_0 & ; \quad 0\le r<b\\ 0 & ; \quad b<r<\infty\end{cases} \tag{6}$$

$$\sigma_{rz}^{(1)}(r,-h)=0 \quad ; \quad 0\le r\le\infty \tag{7}$$

$$u_r^{(1)}(r,0)=u_r^{(2)}(r,0)=0 \quad ; \quad 0\le r\le a \tag{8}$$

$$u_z^{(1)}(r,0)=u_z^{(2)}(r,0)=\Delta_0 \quad ; \quad 0\le r\le a \tag{9}$$

$$u_r^{(1)}(r,0)=u_r^{(2)}(r,0) \quad ; \quad a\le r\le\infty \tag{10}$$

$$u_z^{(1)}(r,0)=u_z^{(2)}(r,0) \quad ; \quad a\le r\le\infty \tag{11}$$

$$\sigma_{zz}^{(1)}(r,0)=\sigma_{zz}^{(2)}(r,0) \quad ; \quad a<r<\infty \tag{12}$$

$$\sigma_{rz}^{(1)}(r,0)=\sigma_{rz}^{(2)}(r,0) \quad ; \quad a<r<\infty \tag{13}$$

The displacements and stresses in the layer and the half-space regions should satisfy the appropriate regularity conditions as $r,z\to\infty$. The solutions of $\varphi(r,z)$ and $\chi(r,z)$ applicable to the regions $^{(1)}$ and $^{(2)}$ can be obtained through a Hankel transform development of the governing equations (1) (see, e.g., Sneddon[21]). We present here the expressions for u_r, u_z, σ_{zz}, and σ_{rz} applicable to these regions, because the boundary conditions and continuity conditions are posed directly in terms of these quantities. For the half-space region we have

$$u_r^{(2)}(r,z)=\int_0^\infty \left[A(\xi)+\xi z B(\xi)\right]e^{-\xi z}J_1(\xi r)d\xi \tag{14}$$

$$u_z^{(2)}(r,z)=\int_0^\infty \left[A(\xi)+\{(3-4\nu)+\xi z\}B(\xi)\right]e^{-\xi z}J_0(\xi r)d\xi \tag{15}$$

$$\sigma_{zz}^{(2)}(r,z)=-2\mu\int_0^\infty \xi\left[A(\xi)+\{2(1-\nu)+\xi z\}B(\xi)\right]e^{-\xi z}J_0(\xi r)d\xi \tag{16}$$

$$\sigma_{rz}^{(2)}(r,z)=-2\mu\int_0^\infty \xi\left[A(\xi)+\{(1-2\nu)+\xi z\}B(\xi)\right]e^{-\xi z}J_1(\xi r)d\xi \tag{17}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions. The solutions of (1) applicable to the layer region will include both positive and negative exponential terms in z. We can choose a form of the solution that identically satisfies the boundary conditions (6) and (7); the resulting solutions give the following expressions for the relevant displacement and stress components:

$$\begin{aligned}u_r^{(1)}(r,z)=\int_0^\infty &\Big\{\left[\xi(z+h)C(\xi)-2(1-\nu)D(\xi)\right]\cosh\{\xi(z+h)\}\\ &+\left[(1-2\nu)C(\xi)-\xi(z+h)D(\xi)\right]\sinh\{\xi(z+h)\}\\ &-\frac{\Omega(\xi)\cosh\{\xi(z+h)\}}{(1-2\nu)}\Big\}\frac{J_1(\xi r)}{\sinh(\xi h)}d\xi\end{aligned} \tag{18}$$

$$\begin{aligned}u_z^{(1)}(r,z)=\int_0^\infty &\Big\{\left[2(1-\nu)C(\xi)+\xi(z+h)D(\xi)\right]\cosh\{\xi(z+h)\}\\ &-\left[\xi(z+h)C(\xi)+(1-2\nu)D(\xi)\right]\sinh\{\xi(z+h)\}\\ &+\frac{\Omega(\xi)\sinh\{\xi(z+h)\}}{(1-2\nu)}\Big\}\frac{J_0(\xi r)}{\sinh(\xi h)}d\xi\end{aligned} \tag{19}$$

$$\begin{aligned}\sigma_{zz}^{(1)}(r,z)=2\mu\int_0^\infty &\Big\{\left[C(\xi)+\xi(z+h)D(\xi)\right]\sinh\{\xi(z+h)\}\\ &-\xi(z+h)C(\xi)\cosh\{\xi(z+h)\}\\ &+\frac{\Omega(\xi)\cosh\{\xi(z+h)\}}{(1-2\nu)}\Big\}\frac{\xi J_0(\xi r)}{\sinh(\xi h)}d\xi\end{aligned} \tag{20}$$

$$\begin{aligned}\sigma_{rz}^{(1)}(r,z)=-2\mu\int_0^\infty &\Big\{\left[D(\xi)-\xi(z+h)C(\xi)\right]\sinh\{\xi(z+h)\}\\ &+\xi(z+h)D(\xi)\cosh\{\xi(z+h)\}\\ &+\frac{\Omega(\xi)\sinh\{\xi(z+h)\}}{(1-2\nu)}\Big\}\frac{\xi J_1(\xi r)}{\sinh(\xi h)}d\xi\end{aligned} \tag{21}$$

where

$$\Omega(\xi) = -\frac{p_0 b(1-2\nu)}{2\mu}\frac{\sinh(\xi h)J_1(\xi b)}{\xi} \tag{22}$$

and

$C(\xi)$ and $D(\xi)$ are arbitrary constants.

III. The Inhomogeneity Problem

The representations for the displacements and stress componants (14) to (21) can now be used to formulate the remaining continuity conditions. Considering (8) to (10) we note that at the plane of the inhomogeneity

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0) \quad ; \quad r \in (0,\infty) \tag{23}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0) \quad ; \quad r \in (0,\infty) \tag{24}$$

We can use these conditions to obtain expressions for $A(\xi)$ and $B(\xi)$ in terms of $C(\xi)$ and $D(\xi)$. We can show that (23) and (24) are equivalent to the relationships

$$\begin{aligned} A(\xi) &= \left\{\xi h \coth(\xi h) + (1-2\nu)\right\} C(\xi) \\ &\quad - \left\{\xi h + 2(1-\nu)\coth(\xi h)\right\} D(\xi) - \frac{\Omega(\xi)\coth(\xi h)}{(1-2\nu)} \end{aligned} \tag{25}$$

$$\begin{aligned} B(\xi) = \frac{1}{(3-4\nu)} \Bigg[&\left\{\coth(\xi h)[2(1-\nu) - \xi h] - \xi h - (1-2\nu)\right\} C(\xi) \\ &+ \left\{\xi h[1 + \coth(\xi h)] - (1-2\nu) + 2(1-\nu)\coth(\xi h)\right\} D(\xi) \\ &+ \frac{\Omega(\xi)[1+\coth(\xi h)]}{(1-2\nu)} \Bigg] \end{aligned} \tag{26}$$

Hence, the solution of the problem is indeterminate to within the arbitrary functions $C(\xi)$ and $D(\xi)$. Considering the continuity of tractions defined by (12) and (13) we obtain

$$\begin{aligned} \sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0) = \frac{4\mu(1-\nu)}{(3-4\nu)} \int_0^\infty \xi\,[1+\coth(\xi h)] \Big\{ &(2(1-\nu) - \xi h)\,C(\xi) \\ &+ (1 - 2\nu + \xi h)\,D(\xi) + \frac{\Omega(\xi)}{(1-2\nu)} \Big\} J_0(\xi r)\,d\xi \end{aligned} \tag{27}$$

an·

$$\begin{aligned} \sigma_{rz}^{(1)}(r,0) - \sigma_{rz}^{(2)}(r,0) = \frac{4\mu(1-\nu)}{(3-4\nu)} \int_0^\infty \xi\,[1+\coth(\xi h)] \Big\{ &-(2(1-\nu) + \xi h)\,D(\xi) \\ &+ (1 - 2\nu + \xi h)\,C(\xi) - \frac{\Omega(\xi)}{(1-2\nu)} \Big\} J_1(\xi r)\,d\xi \end{aligned} \tag{28}$$

Introducing the substitutions

$$R_1(\xi) = C(\xi)\{1 - 2\nu + \xi h\} - D(\xi)\{2(1-\nu) + \xi h\} \tag{29}$$

$$R_2(\xi) = C(\xi)\{2(1-\nu) - \xi h\} - D(\xi)\{1 - 2\nu - \xi h\} \tag{30}$$

we obtain the following set of integral equations representing the continuity conditions (8) to (13) at the plane of inhomogeniety

$$\int_0^\infty [R_1(\xi)\eta_{11}(\xi) + R_2(\xi)\eta_{12}(\xi)] J_0(\xi r)\, d\xi = \frac{(3-4\nu)}{(1-2\nu)}\Big[\int_0^\infty \Omega(\xi) J_0(\xi r)\, d\xi - (1-2\nu)\Delta_0\Big] \quad ; \quad 0 \le r \le a \tag{31}$$

$$\int_0^\infty [R_1(\xi)\eta_{21}(\xi) + R_2(\xi)\eta_{22}(\xi)] J_1(\xi r)\, d\xi = -\frac{(3-4\nu)}{(1-2\nu)}\int_0^\infty \Omega(\xi)\coth(\xi h) J_1(\xi r)\, d\xi \quad ; \quad 0 \le r \le a \tag{32}$$

$$\int_0^\infty \xi\Big[1+\coth(\xi h)\Big]\Big[R_1(\xi) - \frac{\Omega(\xi)}{(1-2\nu)}\Big] J_1(\xi r)\, d\xi = 0 \quad ; \quad a < r < \infty \tag{33}$$

$$\int_0^\infty \xi\Big[1+\coth(\xi h)\Big]\Big[R_2(\xi) + \frac{\Omega(\xi)}{(1-2\nu)}\Big] J_0(\xi r)\, d\xi = 0 \quad ; \quad a < r < \infty \tag{34}$$

where

$$\begin{aligned}
\eta_{11}(\xi) &= \{2(1-\nu) - \xi h\}\{\xi h \coth(\xi h) - (1-2\nu)\} \\
&\quad + \{(1-2\nu) - \xi h\}\{2(1-\nu)\coth(\xi h) - \xi h\} \\
\eta_{12}(\xi) &= \{\xi h + 2(1-\nu)\}\{2(1-\nu)\coth(\xi h) - \xi h\} \\
&\quad + \{\xi h \coth(\xi h) - (1-2\nu)\}\{\xi h + (1-2\nu)\} \\
\eta_{21}(\xi) &= \{(1-2\nu) - \xi h\}\{\xi h \coth(\xi h) + (1-2\nu)\} \\
&\quad + \{\xi h - 2(1-\nu)\}\{2(1-\nu)\coth(\xi h) - \xi h\} \\
\eta_{22}(\xi) &= \{\xi h + (1-2\nu)\}\{2(1-\nu)\coth(\xi h) + \xi h\} \\
&\quad - \{\xi h + 2(1-\nu)\}\{\xi h \coth(\xi h) + (1-2\nu)\}
\end{aligned} \tag{35}$$

The procedure for the solution of the integral equations (31) to (34) is straightforward; we introduce the finite Fourier cosine transformation such that

$$\Big[1+\coth(\xi h)\Big]\Big[R_1(\xi) - \frac{\Omega(\xi)}{(1-2\nu)}\Big] = \xi\int_0^a \Phi(t)\cos(\xi t)\, dt \tag{36}$$

$$\Big[1+\coth(\xi h)\Big]\Big[R_2(\xi) + \frac{\Omega(\xi)}{(1-2\nu)}\Big] = \xi\int_0^a \Psi(t)\cos(\xi t)\, dt \tag{37}$$

where $\Phi(t)$ and $\Psi(t)$ are arbitrary functions, such that the integral equations (33) and (34) are identically satisfied. The remaining equations (31) and (32) can now be expressed in terms of $\Phi(t)$ and $\Psi(t)$ in the forms

$$\Psi(t) + \int_0^a \Psi(u) K_{11}(u,t)\, dt + \int_0^a \Phi(u) K_{12}(u,t)\, du = \gamma_1(t) \quad ; \quad 0 \le t \le a \tag{38}$$

$$\Phi(t) + \int_0^a \Phi(u) K_{22}(u,t)\, dt + \int_0^a \Psi(u) K_{21}(u,t)\, du = \gamma_2(t) \quad ; \quad 0 \le t \le a \tag{39}$$

where

$$K_{11}(u,t) = \frac{4}{\pi(3-4\nu)}\int_0^\infty \frac{\xi\cos(\xi u)\cos(\xi t)}{[1+\coth(\xi h)]}\Big[-\frac{(3-4\nu)}{2}\{1+\coth(\xi h)\} + \{\xi h + 2(1-\nu)\}\{2(1-\nu)\coth(\xi h) - \xi h\} + \{\xi h\coth(\xi h) - (1-2\nu)\}\{\xi h + (1-2\nu)\}\Big]d\xi \tag{40}$$

$$K_{12}(u,t) = -\frac{4}{\pi(3-4\nu)}\int_0^\infty \frac{\cos(\xi u)\cos(\xi t)}{[1+\coth(\xi h)]}\Big[\{\xi h\coth(\xi h) - (1-2\nu)\}\{2(1-\nu) - \xi h\} - \{2(1-\nu)\coth(\xi h) - \xi h\}\{\xi h - (1-2\nu)\}\Big]d\xi \tag{41}$$

$$K_{21}(u,t) = \frac{4t}{\pi(3-4\nu)}\int_0^\infty \frac{\cos(\xi u)[1-\cos(\xi t)]}{\xi[1+\coth(\xi h)]}\Big[-\{\xi h\coth(\xi h) + (1-2\nu)\}\{\xi h + 2(1-\nu)\} + \{\xi h + (1-2\nu)\}\{2(1-\nu)\coth(\xi h) + \xi h\}\Big]d\xi \tag{42}$$

$$K_{22}(u,t) = -\frac{4t}{\pi(3-4\nu)}\int_0^\infty \frac{\cos(\xi u)[1-\cos(\xi t)]}{[1+\coth(\xi h)]}\Big[-\frac{(3-4\nu)}{2}\{1+\coth(\xi h)\} + \{\xi h - (1-2\nu)\}\{\xi h\coth(\xi h) + (1-2\nu)\} + \{2(1-\nu) - \xi h\}\{2(1-\nu)\coth(\xi h) + \xi h\}\Big]d\xi \tag{43}$$

and

$$\gamma_1(t) = \frac{4}{\pi}\Big\{\Delta_0 - \frac{1}{(1-2\nu)(3-4\nu)}\int_0^\infty \Omega(\xi)\cos(\xi t)\Big[-(3-4\nu) + \{2(1-\nu) - \xi h\}\{\xi h\coth(\xi h) - (1-2\nu)\} + (3-4\nu)\{2(1-\nu)\coth(\xi h) - \xi h\} + \{\xi h + (1-2\nu)\}\{\xi h\coth(\xi h) - (1-2\nu)\}\Big]d\xi\Big\} \tag{44}$$

$$\gamma_2(t) = -\frac{4t}{\pi(3-4\nu)(1-2\nu)}\int_0^\infty \frac{\Omega(\xi)[1-\cos(\xi t)]}{\xi[1+\coth(\xi h)]}\Big[(3-4\nu)\coth(\xi h) + \{2(1-\nu) + \xi h\}\{\xi h\coth(\xi h) + (1-2\nu)\} - (3-4\nu)\{2(1-\nu)\coth(\xi h) + \xi h\} + \{(1-2\nu) - \xi h\}\{\xi h\coth(\xi h) + (1-2\nu)\}\Big]d\xi \tag{45}$$

The axisymmetric problem related to the distributed surface loading of an elastic half-space region containing a rigid disc inhomogeneity is now reduced to the solution of the pair of simultaneous Fredholm integral equations of the second-kind for the functions $\Phi(t)$ and $\Psi(t)$. These integral equations are amenable to only numerical solution (see, e.g., Baker,[22]) Delves and Mohamed,[23] and Atkinson[24] and the numerical procedures are described in Section IV. Two specific results are of importance in connection with engineering applications; the first relates to the evaluation of the rigid translation of the rigid disc inhomogeneity due to the surface loading and the second relates to the assessment of the influence of the rigid disc-shaped inhomogeneity on the surface displacements of the half-space region.

The rigid displacement Δ_0 of the embedded rigid inhomogeneity can be evaluated by using the condition that the net force on the inhomogeneity in the axial direction is zero. The zero net axial force condition gives

$$\int_0^{2\pi}\int_0^a \left[\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0)\right] r\,dr\,d\theta = 0 \tag{46}$$

Using the results (27), (30), and (38) we can show that (46) is equivalent to the contstraint

$$\int_0^a \Psi(t)\,dt = 0 \tag{47}$$

which can be used to evaluate the rigid displacement Δ_o.

Avoiding details of calculations, it can be shown that the surface displacement $u_z(r, -h)$ can be evaluated from the expression

$$\begin{aligned} u_z^{(1)}(r,-h) = \frac{2(1-\nu)}{(3-4\nu)} &\left[\int_0^a \Phi(t)\,dt \int_0^\infty \frac{\xi\{\xi h-(1-2\nu)\}\, J_0(\xi r)\cos(\xi t)}{\sinh(\xi h)\{1+\coth(\xi h)\}}d\xi \right. \\ &\left. + \int_0^a \Psi(t)\,dt \int_0^\infty \frac{\{\xi h+2(1-\nu)\}\, J_0(\xi r)\cos(\xi t)}{\sinh(\xi h)\{1+\coth(\xi h)\}}d\xi\right] \\ &- \frac{(3-4\nu)}{(1-2\nu)}\int_0^\infty \frac{\Omega(\xi)\, J_0(\xi r)}{\sinh(\xi h)}d\xi \quad ; \quad 0<r<\infty \end{aligned} \tag{48}$$

In the limit as the radius of the surface loading $b \to 0$, with the proviso that $pb^2\pi \to P$, where P is a concentrated force acting at the origin

$$\Omega(\xi) = -\frac{P(1-2\nu)\sinh(\xi h)}{2\pi\mu} \tag{49}$$

IV. Numerical Solution of the Integral Equations

The system of coupled Fredholm integral equations of the second kind (38) and (39) governing the embedded inhomogeneity are not amenable to solution in an exact closed form. In this study we adopt a numerical technique for their solution. The kernel functions of these integral equations given by (40) to (43) can be simplified as follows:

$$\begin{aligned} K_{11}(u,t) = \frac{4}{\pi(3-4\nu)}\int_0^\infty e^{-2\xi h}&\left[(\xi h)^2+(3-4\nu)\xi h+4(1-\nu)^2\right. \\ &\left. -\frac{(3-4\nu)}{2}\right]\cos(\xi u)\cos(\xi t)\,d\xi \end{aligned} \tag{50}$$

$$K_{12}(u,t) = \frac{4}{\pi(3-4\nu)}\int_0^\infty e^{-2\xi h}\left[(\xi h)^2-2(1-\nu)(1-2\nu)\right]\cos(\xi u)\cos(\xi t)\,d\xi \tag{51}$$

$$\begin{aligned} K_{21}(u,t) = \frac{-4t}{\pi(3-4\nu)}\int_0^\infty e^{-2\xi h}&\left[(\xi h)^2-2(1-\nu)(1-2\nu)\right]\cos(\xi u) \\ &\cdot\frac{[1-\cos(\xi t)]}{\xi}d\xi \end{aligned} \tag{52}$$

$$\begin{aligned} K_{22}(u,t) = \frac{-4t}{\pi(3-4\nu)}\int_0^\infty e^{-2\xi h}&\left[(\xi h)^2-(3-4\nu)\xi h+4(1-\nu)^2\right. \\ &\left. -\frac{(3-4\nu)}{2}\right]\cos(\xi u)[1-\cos(\xi t)]\,d\xi \end{aligned} \tag{53}$$

Considering the result (22) for $\Omega(\xi)$, the expression (44) for $\gamma_1(t)$ can be expressed in the form

$$\gamma_1(t) = \frac{4}{\pi}\left\{\Delta_0 - \frac{pb}{2\mu}\int_0^\infty e^{-\xi h}[(\xi h)+2(1-\nu)]\,J_1(b\xi)\frac{\cos(\xi t)}{\xi}d\xi\right\} \tag{54}$$

and similarly $\gamma_2(t)$ can be expressed in the form

$$\gamma_2(t) = -\frac{pb}{2\mu}\left(\frac{4t}{\pi}\right)\int_0^\infty e^{-\xi h}\left[-(\xi h) + (1-2\nu)\right] J_1(b\xi)\frac{[1-\cos(\xi t)]}{\xi^2}d\xi \tag{55}$$

The governing Fredholm integral equations of the second kind can now be written in the form

$$\begin{bmatrix}\Psi(t)\\ \Phi(t)\end{bmatrix} + \int_0^a \begin{bmatrix}K_{11}(u,t) & K_{12}(u,t)\\ K_{21}(u,t) & K_{22}(u,t)\end{bmatrix}\begin{bmatrix}\Psi(u)\\ \Phi(u)\end{bmatrix}du = \begin{bmatrix}\frac{4\Delta_0}{\pi}\\ 0\end{bmatrix} + \begin{bmatrix}\tilde{\gamma}_1(t)\\ \gamma_2(t)\end{bmatrix} \tag{56}$$

where

$$\tilde{\gamma}_1(t) = \gamma_1(t) - \frac{4\Delta_0}{\pi} \tag{57}$$

The interval [0, a] is divided into N segments such that the discrete values of u and t are defined by

$$u_i = (i-1)\frac{a}{N} \quad \text{with} \quad i = 1,2,\dots(N+1) \tag{58}$$

and

$$t_i = \frac{1}{2}(u_i + u_{i+1}) \quad \text{with} \quad i = 1,2,\dots(N) \tag{59}$$

The interval equations (56) can be discretized into a matrix equation of $(2N+1)$ unknowns for $\Phi(t)$ and $\Psi(t)$ (1 = 1,2,..N) and Δ_0. The extra equation is furnished by (47). The system of equations can be expressed as

$$[\mathbf{A}]\{\mathbf{X}\} = \{\mathbf{B}\} \tag{60}$$

or order (2N + 1). The coefficients of the matrix [**A**] are given by

$$\begin{aligned}
A_{2l-1,2m-1} &= \delta_{lm} + K_{11}(t_m,t_l)\frac{a}{N}\\
A_{2l-1,2m} &= K_{12}(t_m,t_l)\frac{a}{N}\\
A_{2l,2m-1} &= K_{21}(t_m,t_l)\frac{a}{N}\\
A_{2l,2m} &= \delta_{lm} + K_{22}(t_m,t_l)\frac{a}{N}\\
A_{2l-1,2N+1} &= -\frac{4}{\pi}\\
A_{2l,2N+1} &= A_{2N+1,2l} = 0\\
A_{2N+1,2l-1} &= 1
\end{aligned} \tag{61}$$

where l, m = 1,2,…N. The right-hand side vector is defined as

$$B_{2l-1} = \tilde{\gamma}_1(t_l) \quad ; \quad B_l = \gamma_2(t_l) \tag{62}$$

with l = 1,2,…N. The vector of unknowns {**X**} consists of

$$\begin{aligned}
X_{2l-1} &= \Psi(t_l)\\
X_{2l} &= \Phi(t_l)\\
X_{2N+1} &= \Delta_0
\end{aligned} \tag{63}$$

Upon solution of the matrix equation (60), the rigid displacement of the inhomogeneity is given by X_{2N+l}. The surface displacement of the half-space region, given by (48) can be written as

$$u_z^{(1)}(r,-h) = \frac{2(1-\nu)}{(3-4\nu)}\left[\sum_{i=1}^{2N} F_i(r)\,X_i\right] - 2(1-\nu)\,\overline{U}(r) \tag{64}$$

where

$$F_{2l-1}(r) = \int_0^\infty e^{-\xi h}\left[\xi h + 2(1-\nu)\right] J_0(\xi r)\cos(\xi t_l)\,d\xi$$

$$F_{2l}(r) = \int_0^\infty e^{-\xi h}\left[\xi h - (1-2\nu)\right] J_0(\xi r)\cos(\xi t_l)\,d\xi \tag{65}$$

and

$$\overline{U}(r) = -\frac{pb}{2\mu}\int_0^\infty \frac{J_1(\xi b)\,J_0(\xi r)}{\xi}\,d\xi \tag{66}$$

with $l = 1,2,\ldots N$.

V. Numerical Results

Prior to the presentation of numerical results it is instructive to note that the solution to the problem of a concentrated force acting at the surface of a half-space containing the embedded rigid inhomogeneity can be obtained as a special case of the results derived for the uniform circular load of stress intensity p. In the limit as $b \to 0$ (54) and (55) give

$$\gamma_1(t) = \frac{4}{\pi}\left\{\Delta_0 - \frac{P}{4\pi\mu}\int_0^\infty e^{-\xi h}\left[(\xi h) + 2(1-\nu)\right]\cos(\xi t)\,d\xi\right\} \tag{67}$$

and

$$\gamma_2(t) = -\frac{Pt}{\pi^2\mu}\int_0^\infty e^{-\xi h}\left[-(\xi h) + (1-2\nu)\right]\frac{\{1-\cos(\xi t)\}}{\xi}\,d\xi \tag{68}$$

respectively. Considering (67), we can rewrite the equation in the form.

$$\gamma_1(t) = \frac{4}{\pi}\left\{\Delta_0 - \frac{P}{4\pi\mu}\frac{d}{dt}\left[\int_0^t \frac{r\,dr}{(t^2-r^2)^{1/2}}\int_0^\infty e^{-\xi h}\left[(\xi h) + 2(1-\nu)\right] J_0(\xi r)\,d\xi\right]\right\} \tag{69}$$

Considering the explicit forms for the finite integrals

$$\int_0^\infty e^{-\xi h}\left[\xi\ ;\ 1\right] J_0(\xi r)\,d\xi = \left[\frac{h}{(r^2+h^2)^{3/2}}\ ;\ \frac{1}{(r^2+h^2)^{1/2}}\right] \tag{70}$$

we can write (69) as

$$\gamma_1(t) = \frac{4}{\pi}\left\{\Delta_0 - \frac{d}{dt}\int_0^t \frac{r\,l(r)\,dr}{(t^2-r^2)^{1/2}}\right\} \tag{71}$$

where

$$l(r) = \frac{P}{4\pi\mu}\left[\frac{h^2}{(r^2+h^2)^{3/2}} + \frac{2(1-\nu)}{(r^2+h^2)^{1/2}}\right] \tag{72}$$

which can be identified as the displacement u_z $(r,0)$ of the half-space region, $r \in (0,\infty)$; $z \in (-h, \infty)$, which is subjected to a concentrated force P acting at the location $(0, -h)$ normal to $z = -h$.

Considering (68) we can rewrite this equation in the form

$$\gamma_2(t) = -\frac{Pt}{\pi^2\mu}\int_0^t \frac{dr}{(t^2-r^2)^{1/2}}\int_0^\infty e^{-\xi h}\left[-(\xi h) + (1-2\nu)\right] J_1(\xi r)\, d\xi \tag{73}$$

Considering the explicit forms for the infinite integrals in (73) we have

$$\int_0^\infty e^{-\xi h}\left[\xi\ ;\ 1\right] J_1(\xi r)\, d\xi = \left[\frac{r}{(r^2+h^2)^{3/2}}\ ;\ \frac{(r^2+h^2)^{1/2}-h}{r\,(r^2+h^2)^{1/2}}\right] \tag{74}$$

We can write (73) as

$$\gamma_2(t) = \frac{4t}{\pi}\int_0^t \frac{m(r)\, dr}{(t^2-r^2)^{1/2}} \tag{75}$$

where

$$m(r) = \frac{P}{4\mu\pi}\left[\frac{rh}{(r^2+h^2)^{3/2}} - \frac{r(1-2\nu)}{\left[(r^2+h^2)+h(r^2+h^2)^{1/2}\right]}\right] \tag{76}$$

which can be identified as the displacement u_r $(r,0)$ of the half-space region, $r \in (0,\infty)$; $z \in$ (– h, ∞), which is subjected to a concentrated force P acting at the location $(0, -h)$, normal to $z = -h$.

Also, $\overline{U}$ (r) in (66) reduces to

$$\overline{U}(r) = -\frac{P}{2\pi\mu}\int_0^\infty \frac{J_0(\xi r)}{2}\, d\xi = -\frac{P}{4\pi\mu r} \tag{77}$$

Substituting (77) in (64), the last term on the right-hand side of (64) gives the surface displacement $u_z^{(1)}$ $(r, -h)$ of a half-space region $r \in (0,\infty)$; $z \in (-h, \infty)$, which is subjected to a concentrated force P acting normal to the surface $z = -h$.

The numerical procedure outlined in Section IV was used to determine results of engineering interest. The accuracy of numerical procedure has been established, in the previous investigations, by comparison with exact closed form solutions derived for disc-shaped anchors embedded in elastic media of infinite extent (see, e.g., Selvadurai[15,16,25]). In the limiting case when $(h/a) \to 0$ and when $(b/a) \leq 1$, the problem of the embedded inhomogeneity reduces to that of the axial loading of a surface punch of radius a, which is in adhesive contact with a half-space region and subjected to an axisymmetric load P $(=p\pi b^2)$. The exact solution to this problem was given by Mossakovskii[26] and Ufliand[27] (see also Gladwell[8]), and the Hilbert transform formulation of the problem also takes into account the oscillatory form of the stress singularly at the boundary of the bonded circular punch for arbitrary values of ν. The exact solution for the load (P)-displacement(Δ_0^B) relationship for the bonded circular rigid punch is given by

$$P = \frac{4\mu a \Delta_0^B \ln(3-4\nu)}{(1-2\nu)} \tag{78}$$

The solution to the axisymmetric surface punch problem that incorporates only a regular $1/\sqrt{r}$ type stress singularity at the boundary of the rigid punch was also presented by Selvadurai.[25] It can be shown that the maximum discrepancy between the exact solution (78) and the approximate result does not exceed 0.5% when $v = 0$ and the results of the two approaches coincide when $v = 0.5$.

Figures 3 to 7 illustrate the influence of Poisson's ratio (v), the ratio of the radius of the loaded region to the radius of the circular inhomogeneity (b/a), and the relative depth of embedment (h/a) on the axisymmetric rigid displacement (Δ_0) of the inhomogeneity. The displacement Δ_0 is normalized with respect to the rigid displacement of a punch bonded to the surface of a half-space. It is evident that for $b/a \leq 1$, the value of Δ_0/Δ_0^B approaches unity as $(h/a) \rightarrow 0$. Hence Δ_0/Δ_0^B gives a value lower than that obtained for situations where $b/a \geq 1$. Figure 8 illustrates the influence of v and (h/a) on the normalized displacement Δ_0/Δ_0^B for the case where a concentrated load P acts on the surface of the half-space. These results closely resemble the case when $(b/a) = 0.40$. Also for $b/a \leq 1$ and as (h/a) becomes large, the displacement ratio approaches a similar asymptotic value, indicating that the manner of application of the loading is less important to the overall displacement of the rigid inhomogeneity. Ths Figures 9 to 14 illustrate the radial variation in the axial surface displacement of the half-space containing the embedded inhomogeneity for selected values of h/a, b/a, and ν. These displacements are normalized with respect to the maximum surface displacement of the inhomogeneity-free half-space which is subjected to a uniform circular load of radius b and stress intensity $p = P/\pi b^2$. These results indicate that the presence of the rigid inhomogeneity has an influence on the surface displacement profile particularly when $b/a < 1$ and $h/a < 1$.

VI. Conclusions

The axisymmetric problem in the classic theory of elasticity for a half-space containing a rigid disc-shaped inhomogeneity can be effectively reduced to the solution of a pair of coupled Fredholm integral equations of the second kind. A numerical solution of these integral equations can be used to determine the displacement of the rigid disc-shaped inhomogeneity due to the circular uniform load applied at the surface of the half-space region. Numerical results can also be developed to illustrate the influence of the embedded rigid disc inhomogeneity on the surface displacements of the half-space region. The numerical results illustrate the influence of the surface loading on the displacements of the rigid inhomogeneity and the extent to which the rigid inhomogeneity becomes effective in mitigating the surface displacement of the half-space region. The numerical results presented in the paper are sufficient to make some general comments relating to *trends* in the behavior of the half-space region containing the rigid disc inhomogeneity. It is evident that a rigid inhomogeneity that is located at a depth ratio $h/a > 3$ contributes little to mitigating the surface displacement of the half-space region. This comment is valid for all values of Poisson's ratio $v \in (0, 1/2)$. When the disc inhomogeneity is located at a depth ratio $h/a < {}^{1/2}$, the surface displacement closely resembles the surface displacement profile associated with a rigid disc that is bonded to the surface of an elastic half-space region. In this instance, the simpler elasticity solution associated with the displacement of a half-space region due to the surface indentation by a bonded rigid disc can be used to examine the effects of an embedded rigid inhomogeneity. Also, for material incompressibility, the solution to the problem of a rigid circular disc which is bonded to a half-space region is identical to the case of Boussinesq's classical solution involving the smooth indentation of the surface of a half-space region by a rigid indentor.

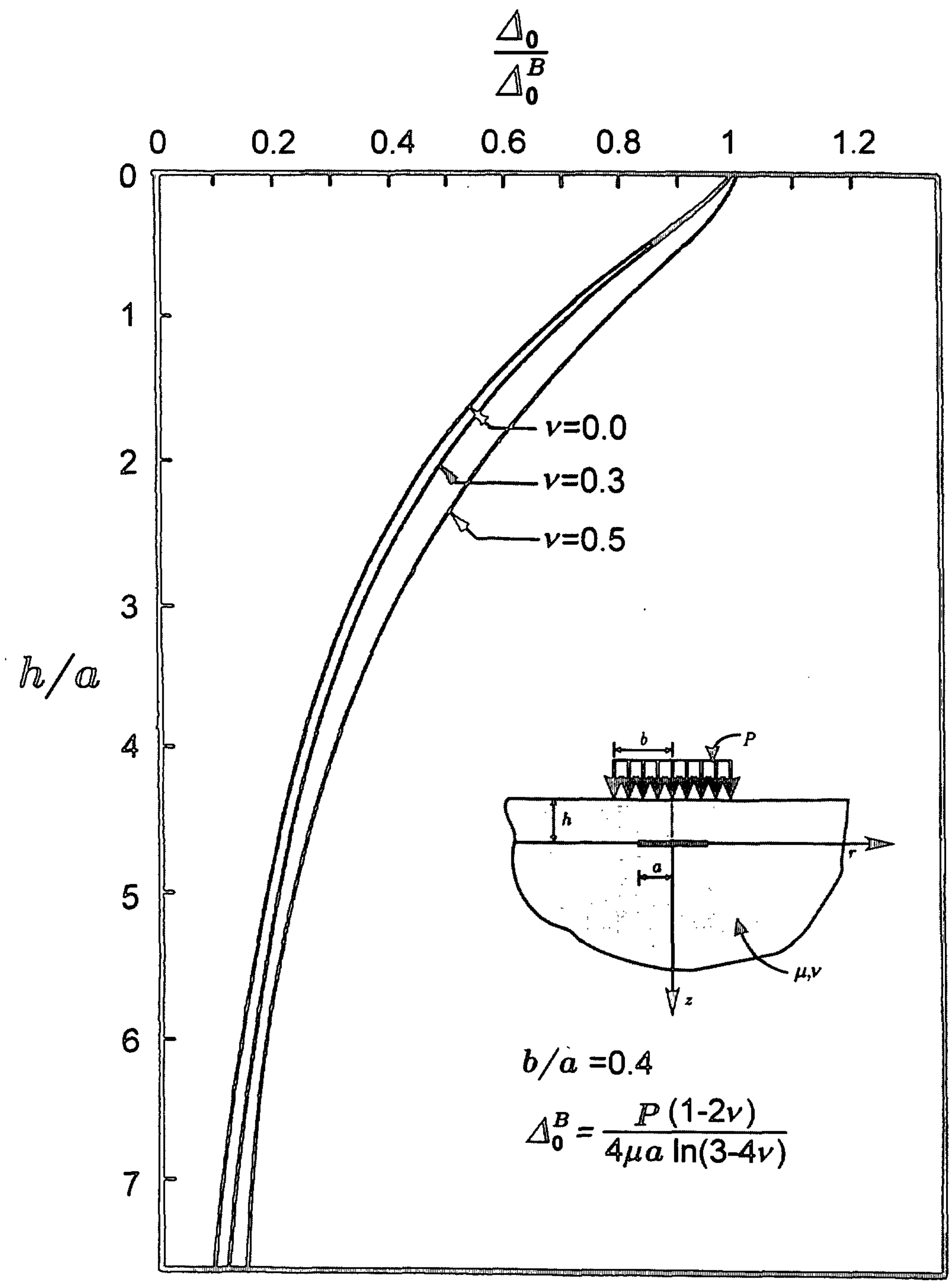

FIGURE 3 Axial displacement of the embedded rigid disc due to distributed surface loading of the half-space.

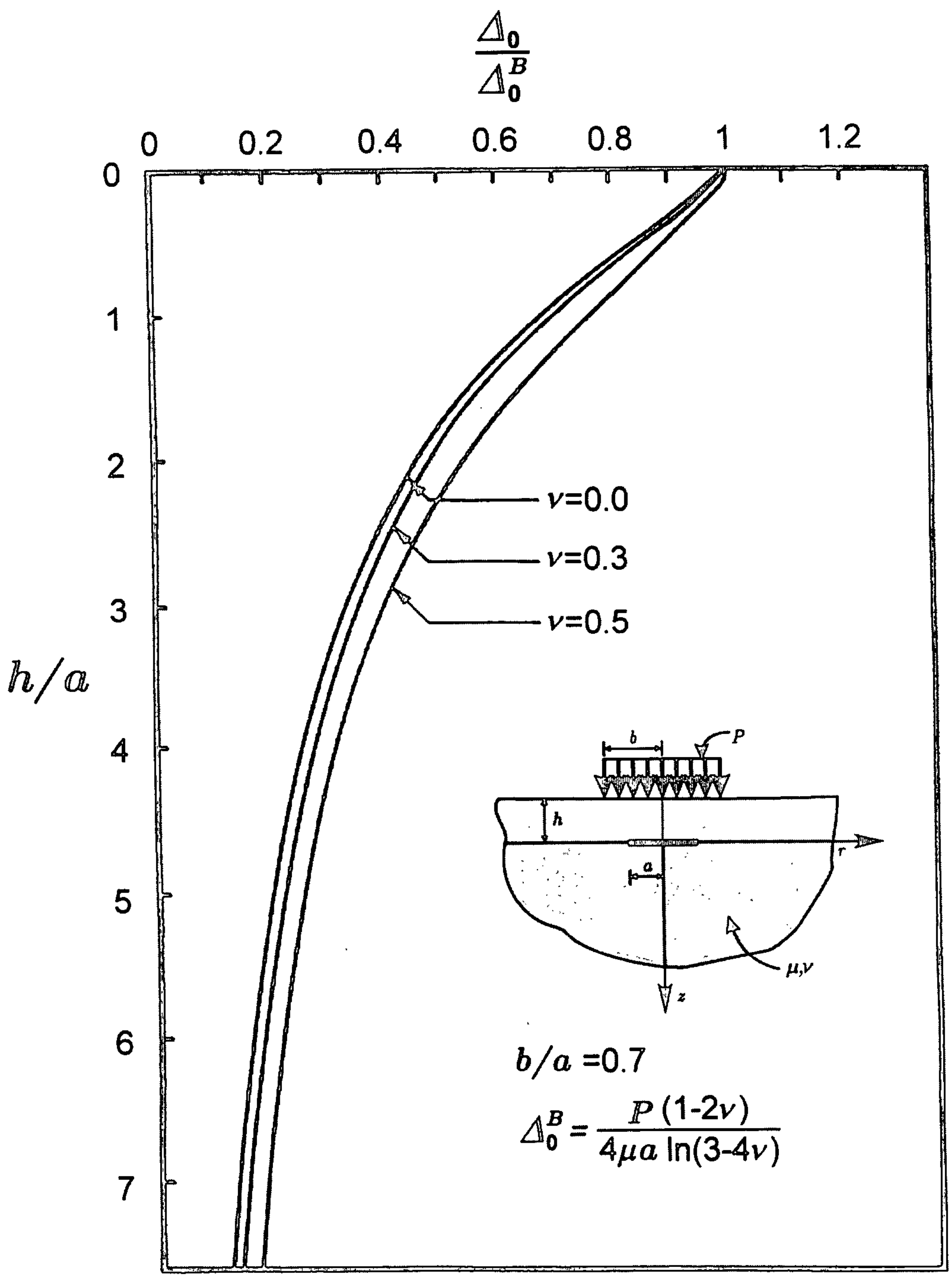

FIGURE 4 Axial displacement of the embedded rigid disc due to distributed surface loading of the half-space.

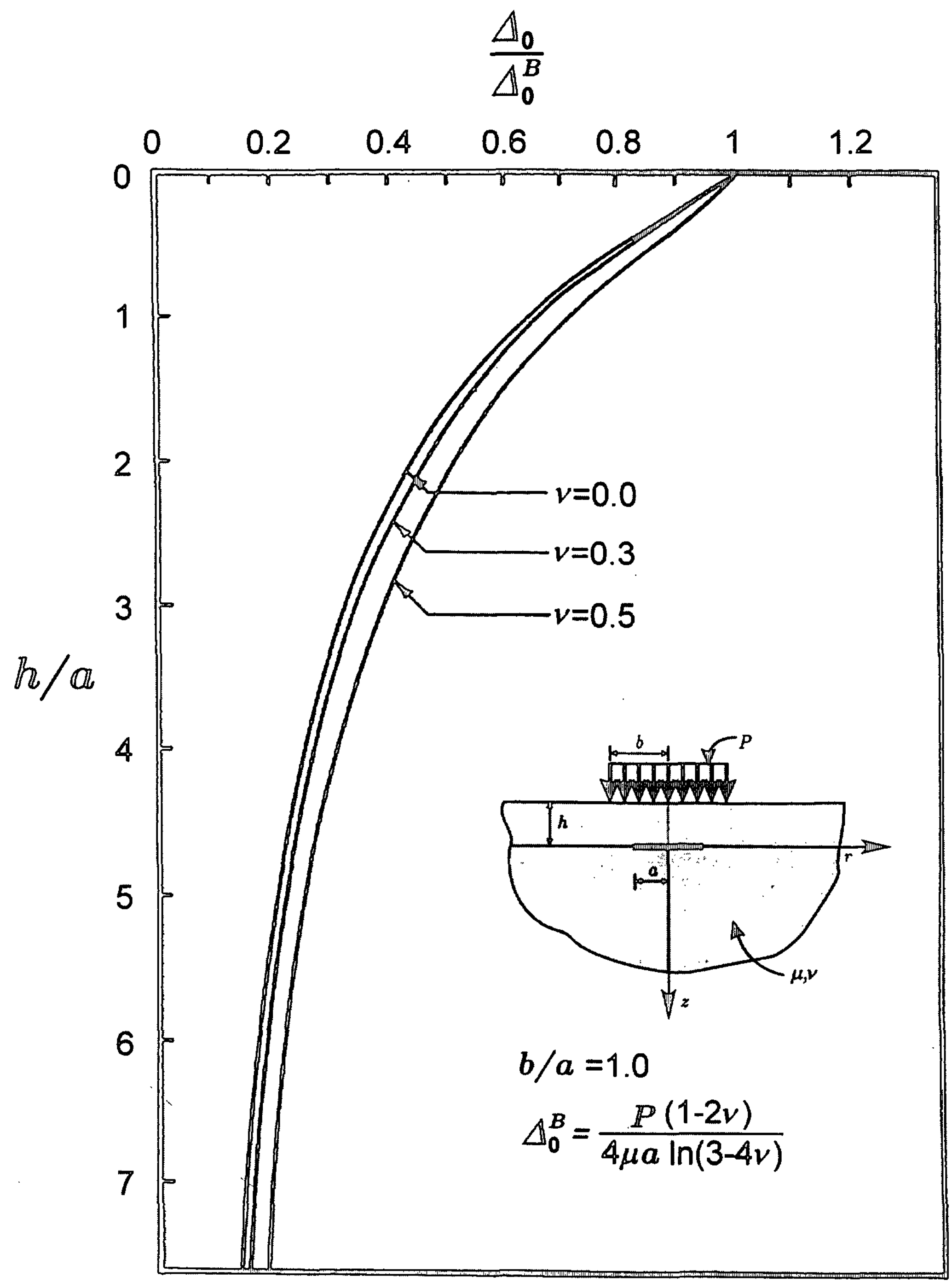

FIGURE 5 Axial displacement of the embedded rigid disc due to distributed surface loading of the half-space.

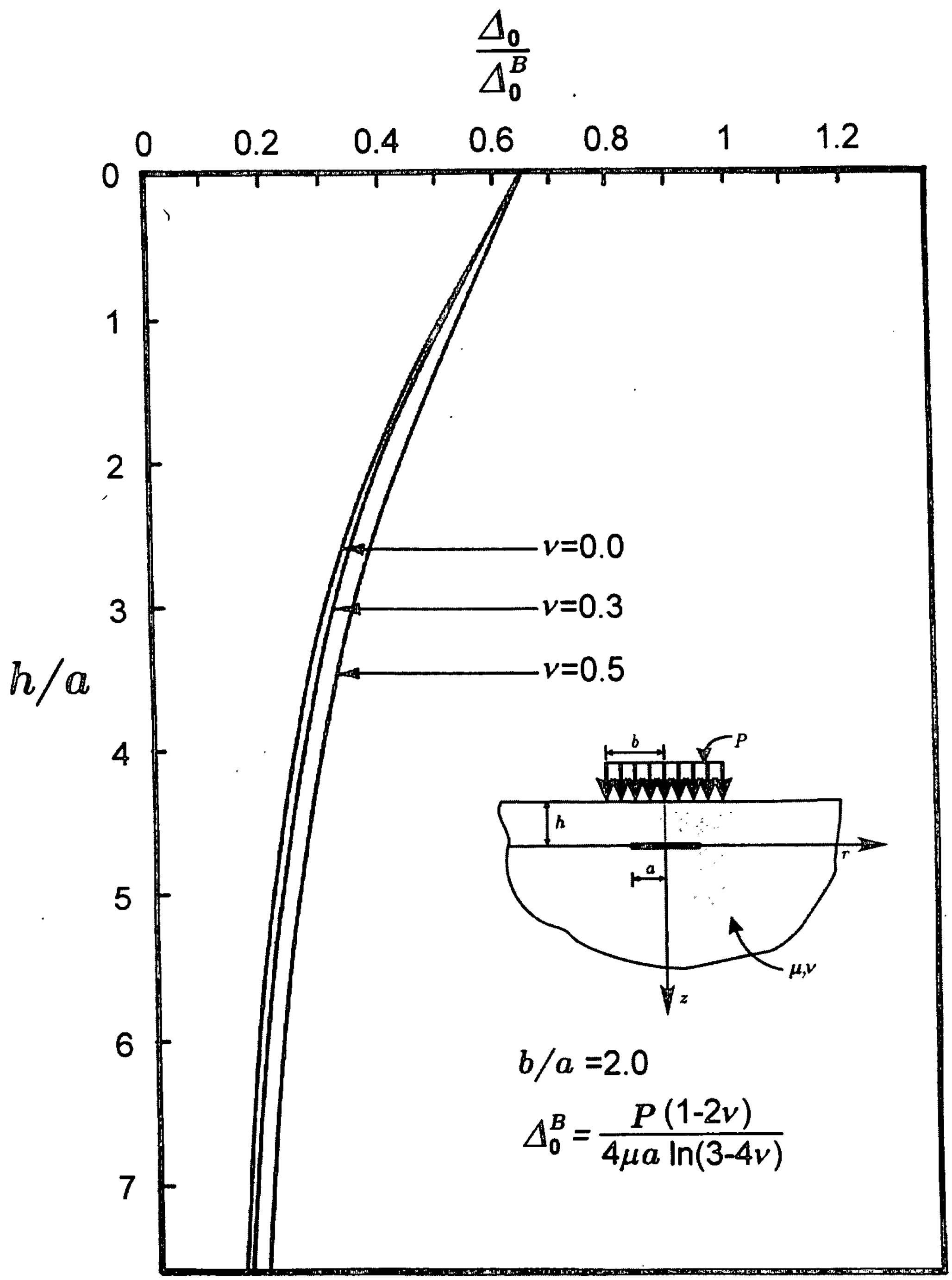

FIGURE 6 Axial displacement of the embedded rigid disc due to distributed surface loading of the half-space.

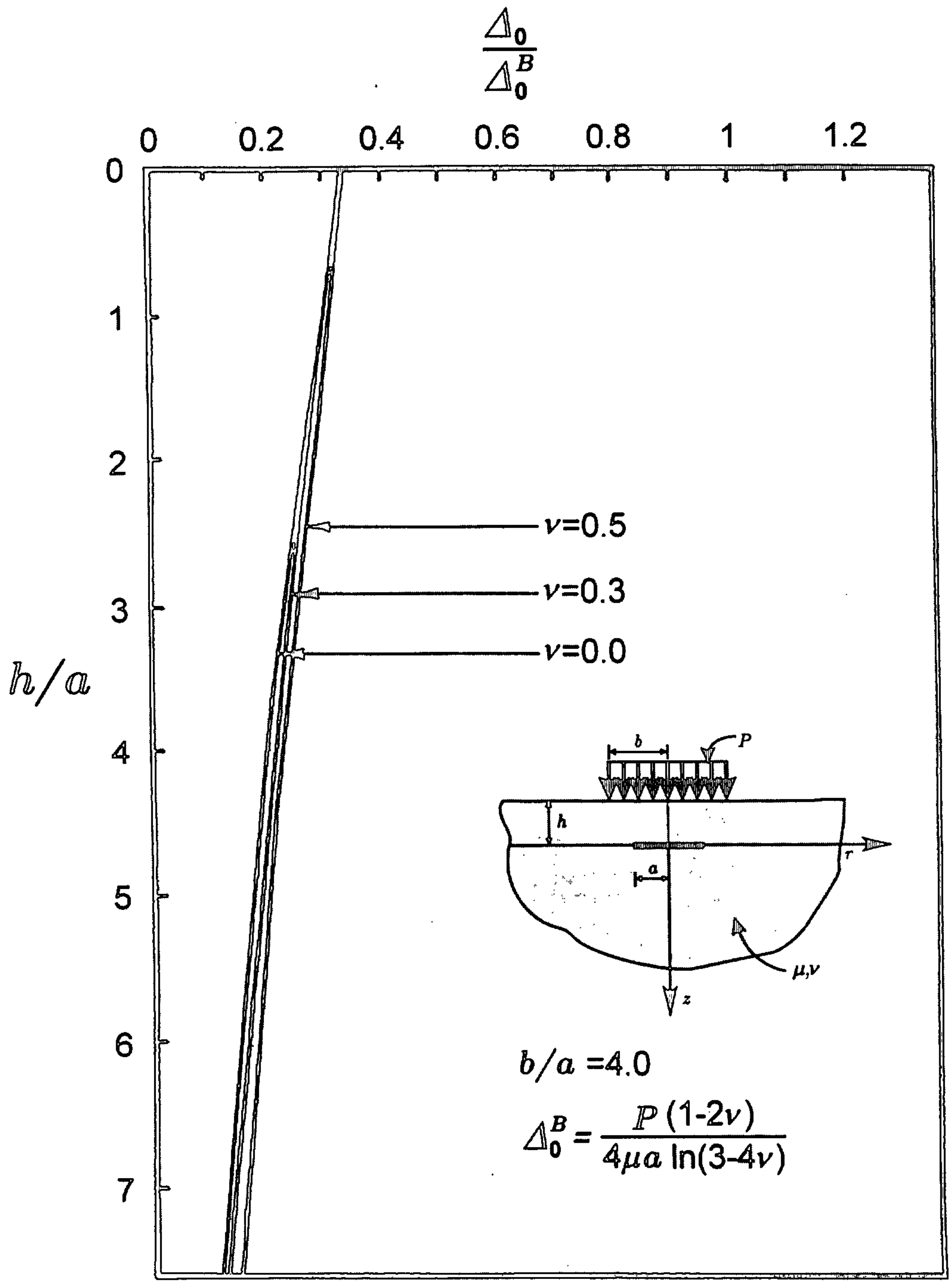

FIGURE 7 Axial displacement of the embedded rigid disc due to distributed surface loading of the half-space.

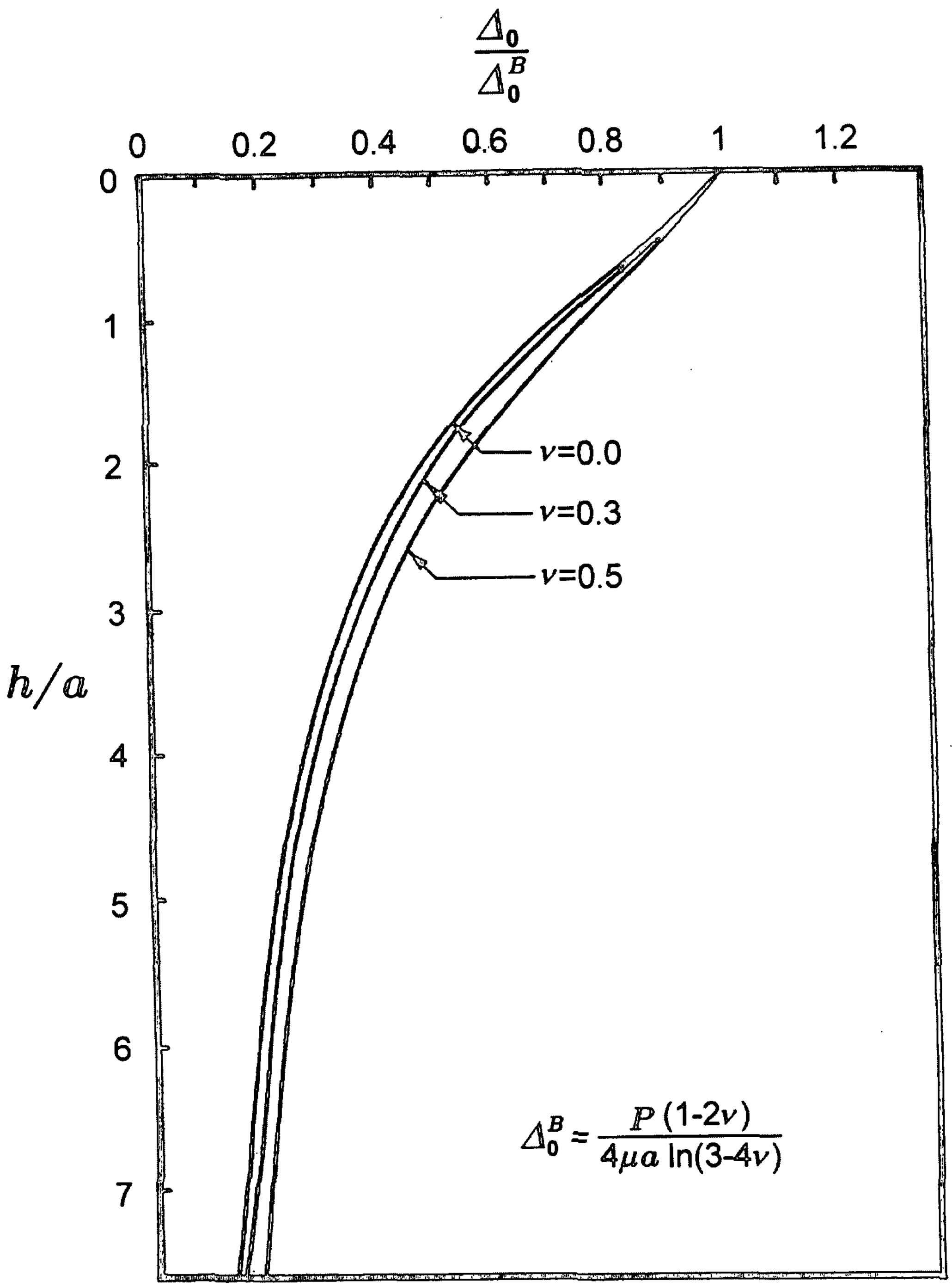

FIGURE 8 Axial displacement of the embedded rigid disc due to Boussinesq force.

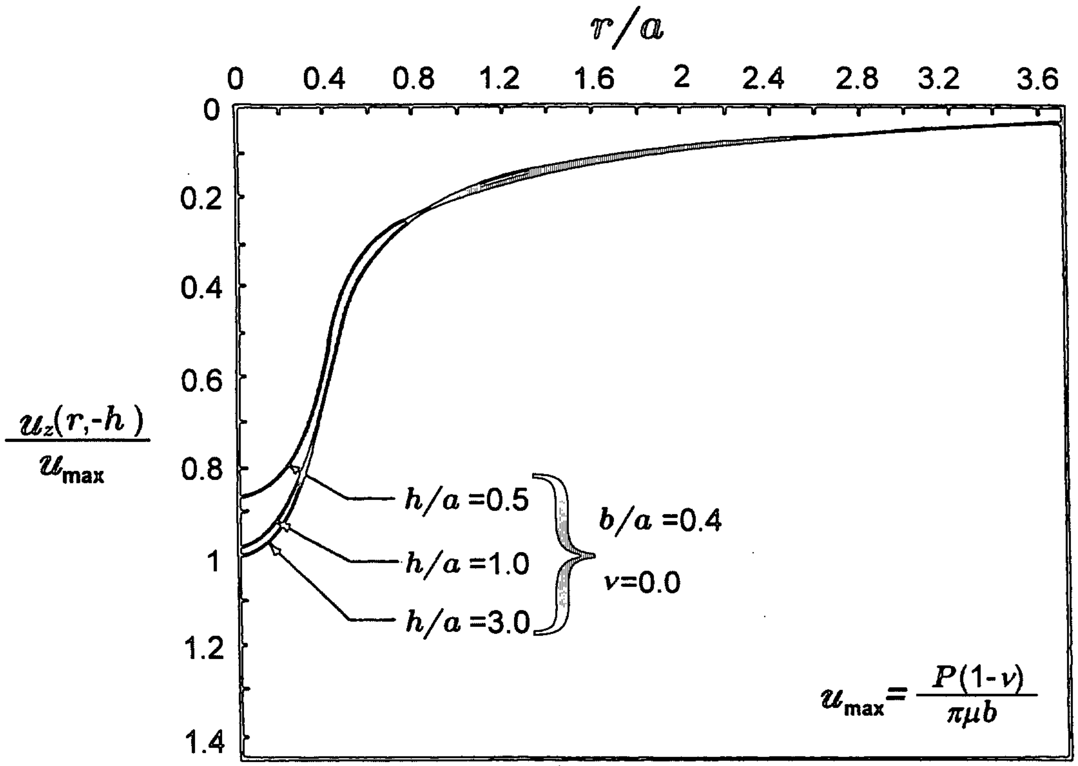

FIGURE 9 Variation of axial displacement of the surface of the inclusion-reinforced half-space region.

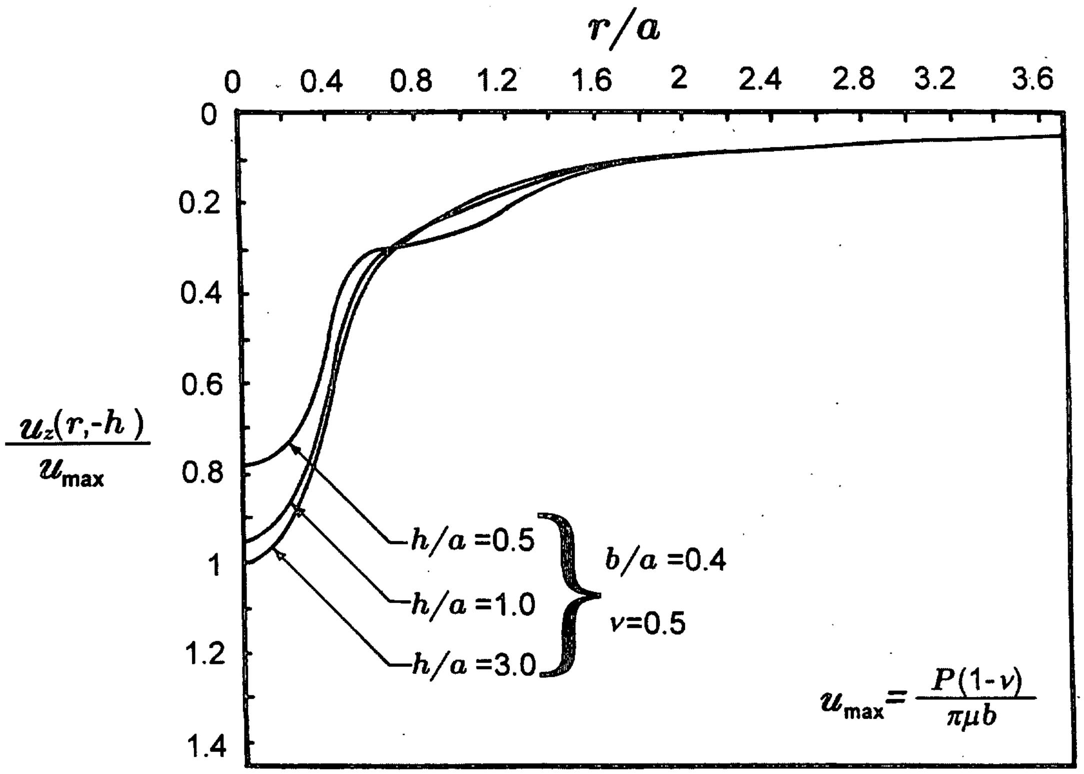

FIGURE 10 Variation of axial displacement of the surface of the inclusion-reinforced half-space region.

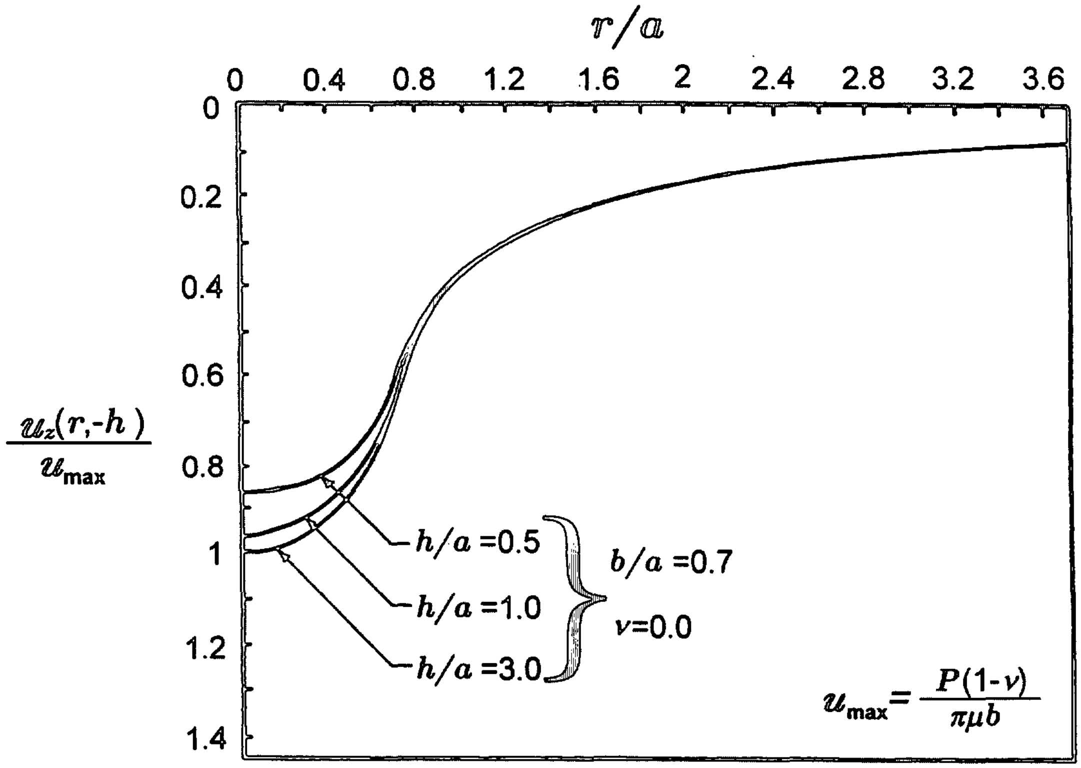

FIGURE 11 Variation of axial displacement of the surface of the inclusion-reinforced half-space region.

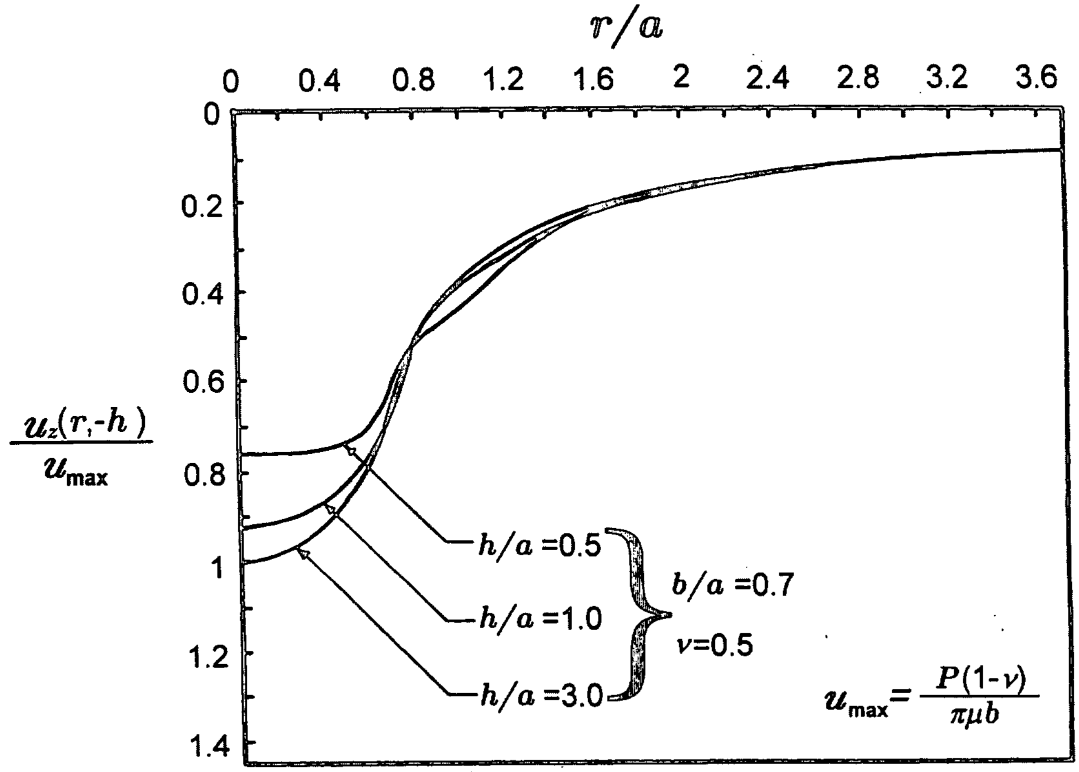

FIGURE 12 Variation of axial displacement of the surface of the inclusion-reinforced half-space region.

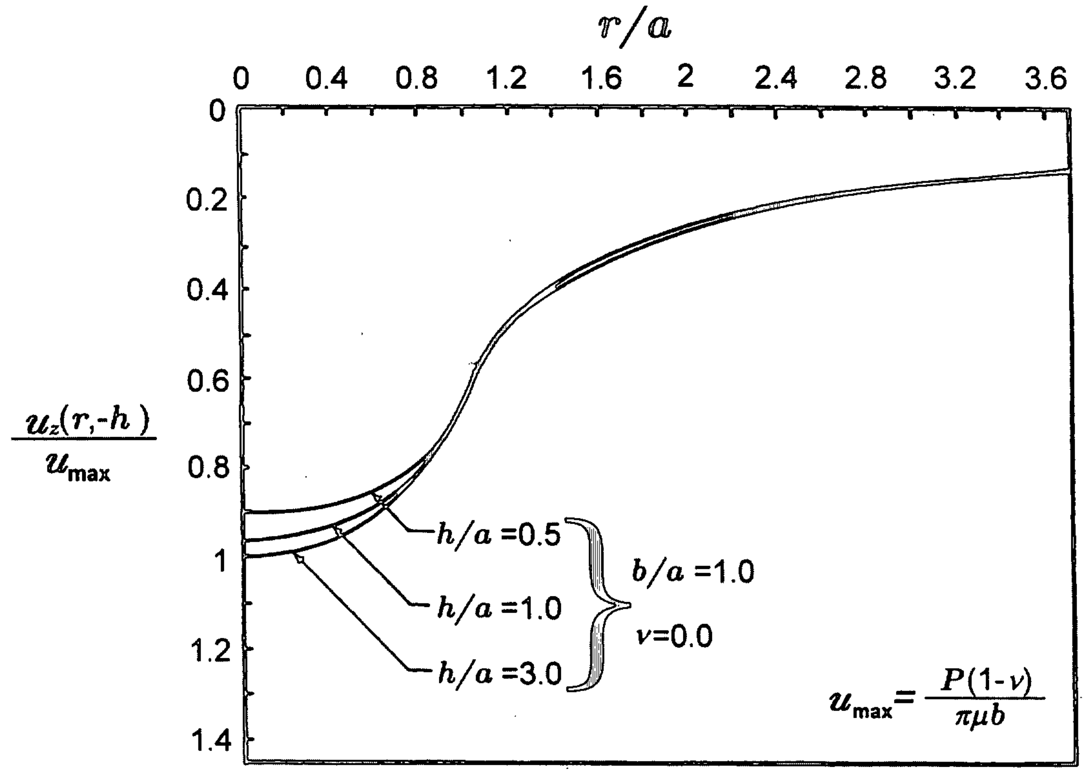

FIGURE 13 Variation of axial displacement of the surface of the inclusion-reinforced half-space region.

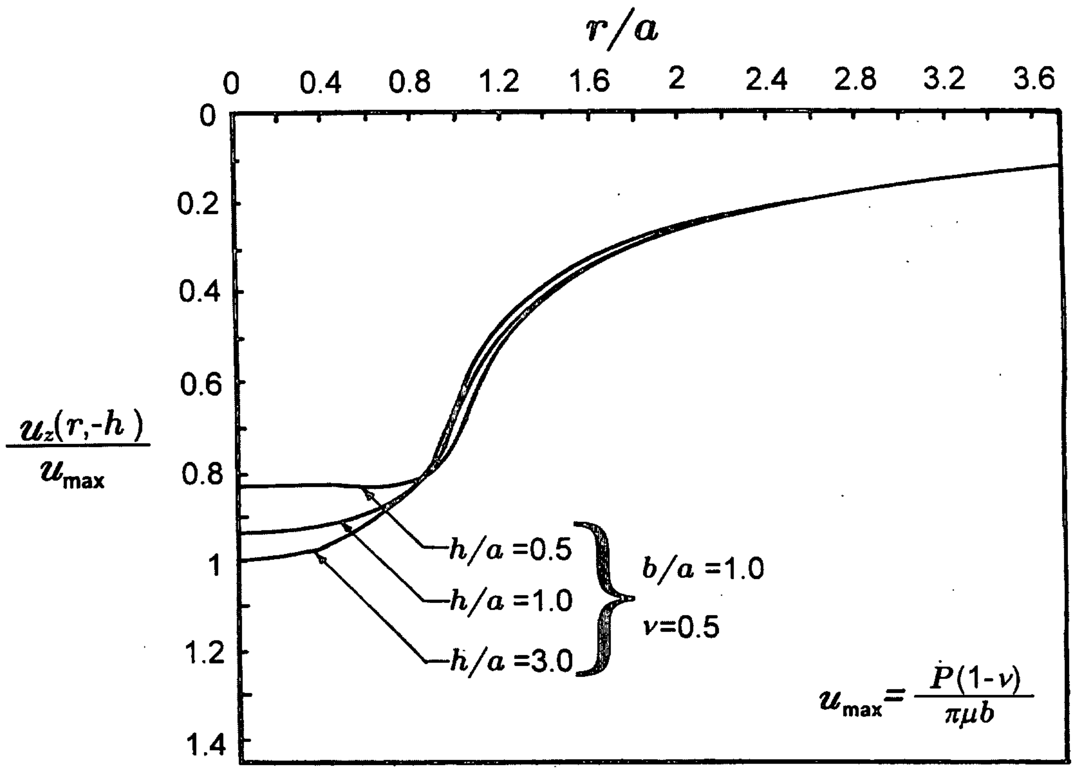

FIGURE 14 Variation of axial displacement of the surface of the inclusion-reinforced half-space region.

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