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On Boussinesq's problem

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Abstract

This note presents an elementary procedure for obtaining the solution to Boussinesq's problem for the loading of an isotropic elastic halfspace by a concentrated normal load.

1. Introduction

The problem of determining the state of stress in an isotropic elastic halfspace which is subjected to a concentrated force normal to a traction free surface was first considered by Boussinesq [1]. The solution to this problem can be obtained by several methods. The first approach consists of reducing the problem to a boundary value problem in potential theory. When the surface of the halfspace is subjected to *normal tractions only*, the elasticity problem is reduced to that of finding a single harmonic function with all the characteristic features of a single layer distributed over the plane region with an intensity proportional to the applied normal tractions. The solution to the concentrated force problem is recovered as a special case of the general normal loading. The second approach to the solution of Boussinesq's problem commences with Kelvin's solution for the point force acting at the interior of an infinite space and utilizes a distribution of combinations of dipoles, which are equivalent to a distribution of centers of compression along an axis, to eliminate the shear tractions occurring on the plane normal to the line of action of the Kelvin force, thereby recovering Boussinesq's solution. A third approach involves the application of integral transform techniques to the solution of a governing partial differential equation (e.g., for Love's strain function) which can then be used to explicitly satisfy the traction boundary con-

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dition's applicable directly to Boussinesq's problem. Details of this procedure are given by Sneddon [6]. These procedures are well documented in classical treaties and papers by Michell, Love, Westergaard, Sokolnikoff, Luré, and Timoshenko and Goodier [2–4,7–9]. While these approaches represent remarkably insightful procedures for obtaining a solution to Boussinesq's problem, there is the question as to whether there is a more direct approach via which Boussinesq's solution can be obtained. The objective of this note is to outline such a procedure which is relatively elementary, and as far as the author is aware, has not been presented in the literature (e.g. [4]).

2. Governing equations

We consider problems which are symmetric about the axis $\Theta = 0$ of a system of spherical polar coordinates (R, ϑ, Θ) with $R(0,\infty)$, $\vartheta(0, 2\pi)$ and $\Theta(0,\pi)$. The solution of such axisymmetric problems can be approached either via the Lamé strain potential $\varphi(R,\Theta)$ which satisfies

$$\nabla^2 \varphi(R,\Theta) = 0 \tag{1}$$

or a Love strain function $\Phi(R,\Theta)$ which satisfies

$$\nabla^2\nabla^2 \Phi(R,\Theta) = 0, \tag{2}$$

where ∇^2 is Laplace's operator in spherical polar coordinates; i.e.,

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R} + \frac{\cot\Theta}{R^2}\frac{\partial}{\partial\Theta} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}. \tag{3}$$

The displacement and stress components derived from $\varphi(R,\Theta)$ take the forms

$$2Gu_R = \frac{\partial\varphi}{\partial R}, \quad 2Gu_\Theta = \frac{1}{R}\frac{\partial\varphi}{\partial\Theta} \tag{4}$$

and

$$\begin{aligned} \sigma_{RR} &= \frac{\partial^2\varphi}{\partial R^2}, \quad \sigma_{\Theta\Theta} = \frac{1}{R}\frac{\partial\varphi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\varphi}{\partial\Theta^2}, \\ \sigma_{\vartheta\vartheta} &= \frac{1}{R}\frac{\partial\varphi}{\partial R} + \frac{\cot\Theta}{R^2}\frac{\partial\varphi}{\partial\Theta}, \quad \sigma_{R\Theta} = \frac{\partial^2}{\partial R\partial\Theta}\left(\frac{\varphi}{R}\right), \end{aligned} \tag{5}$$

respectively. The displacement and stress components derived from $\Phi(R,\Theta)$ are given by

$$\begin{aligned} 2Gu_R &= \cos\Theta\left[2(1-\nu)\nabla^2 - \frac{\partial^2}{\partial R^2}\right]\Phi + \frac{\sin\Theta}{R}\frac{\partial}{\partial\Theta}\left(\frac{\partial}{\partial R} - \frac{1}{R}\right)\Phi, \\ 2Gu_\Theta &= \sin\Theta\left[-2(1-\nu)\nabla^2 + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right]\Phi + \frac{\cos\Theta}{R}\frac{\partial}{\partial\Theta}\left(-\frac{\partial}{\partial R} + \frac{1}{R}\right)\Phi \end{aligned} \tag{6}$$

and

$$
\begin{aligned}
\sigma_{RR} &= \cos\Theta\frac{\partial}{\partial R}\left[(2-\nu)\nabla^2-\frac{\partial^2}{\partial R^2}\right]\Phi+\frac{\sin\Theta}{R}\frac{\partial}{\partial\Theta}\left[-\nu\nabla^2+\frac{\partial^2}{\partial R^2}-\frac{2}{R}\frac{\partial}{\partial R}+\frac{2}{R^2}\right]\Phi,\\
\sigma_{\Theta\Theta} &= \cos\Theta\frac{\partial}{\partial R}\left[\nu\nabla^2-\frac{1}{R}\frac{\partial}{\partial R}-\frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right]\Phi\\
&\quad+\frac{\sin\Theta}{R}\frac{\partial}{\partial\Theta}\left[-(2-\nu)\nabla^2+\frac{3}{R}\frac{\partial}{\partial R}-\frac{2}{R^2}+\frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right]\Phi,\\
\sigma_{\vartheta\vartheta} &= \left\{\cos\Theta\frac{\partial}{\partial R}-\frac{\sin\Theta}{R}\frac{\partial}{\partial\Theta}\right\}\left[-(1-\nu)\nabla^2+\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right]\Phi,\\
\sigma_{R\Theta} &= \frac{\cos\Theta}{R}\frac{\partial}{\partial\Theta}\left[(1-\nu)\nabla^2-\frac{\partial^2}{\partial R^2}+\frac{2}{R}\frac{\partial}{\partial R}-\frac{2}{R}\right]\Phi\\
&\quad+\sin\Theta\frac{\partial}{\partial R}\left[-(1-\nu)\nabla^2+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\Theta^2}\right],
\end{aligned}
\tag{7}
$$

respectively. In (4)–(7), G and ν are, respectively, the shear modulus and Poisson's ratio. We further note that $R^2\varphi(R,\Theta)$ is biharmonic.

3. Boussinesq's problem

We start with Kelvin's problem for the point force of magnitude P_K acting at the interior of an isotropic elastic infinite space. A basic observation is that since P_K is a point force and since the medium is of infinite extent, there is no natural length scale associated with Kelvin's problem. Yet the use of either a Lamé potential function or a Love strain function should yield, through appropriate differentiations with respect to R, expressions for stresses which are of order $1/R^2$ to generate the correct dimensions for stress (i.e., differentiation of $\varphi(R,0)$ twice with respect to R and the differentiation of $\Phi(R,\Theta)$ thrice with respect to R). Also the choice of an 'exterior' solution should be such that the stresses are finite within the region (excluding the origin) and should reduce to zero as $R\to\infty$. The axial component of tractions acting on any closed surface which encloses the point of application of the Kelvin force (or includes it on the boundary of the surface) should be identically equal to P_K. This invariance requirement also point to the fact that the dimensions of the stresses should be of order $1/R^2$ (Michell [4] correctly makes this observation; see also [5]). The exterior solution for $\varphi(R,\Theta)$ is C/R, where C is a constant. From (5) it is clear that the 'exterior' Lamé solution will not yield the required order $1/R^2$ for the stress distribution. Love's strain function derived from this exterior solution

$$\Phi(R,\Theta)=CR \tag{8}$$

will provide the correct order in R for the stress components. Avoiding details, the displacements and stresses applicable to Kelvin's solution take the forms

$$2Gu_R = \frac{4C(1-\nu)\cos\Theta}{R}, \quad 2Gu_\Theta = -\frac{C(3-4\nu)\sin\Theta}{R} \tag{9}$$

and

$$\begin{aligned} \sigma_{RR} &= -\frac{2C(2-\nu)\cos\Theta}{R^2}, \\ \sigma_{\Theta\Theta} &= \sigma_{\vartheta\vartheta} = \frac{C(1-2\nu)\cos\Theta}{R^2}, \quad \sigma_{R\Theta} = \frac{C(1-2\nu)}{R^2}\sin\Theta, \end{aligned} \tag{10}$$

where

$$C = \frac{P_{\mathrm{K}}}{8\pi}(1-\nu). \tag{11}$$

If we consider the halfspace region $z \geqslant 0$ associated with the solution to Kelvin's problem, it is clear that the plane $z = 0$ is subjected to the stresses

$$\sigma_{\Theta\Theta}\left(R, \frac{\pi}{2}\right) = 0, \quad \sigma_{R\Theta}\left(R, \frac{\pi}{2}\right) = \frac{C(1-2\nu)\sin\Theta}{R^2}. \tag{12}$$

Consider Boussinesq's problem where the surface of the halfspace is subjected to a concentrated normal force P_{B} at $R = 0$. Here again, there is no natural length parameter associated with the problem and the solutions derived from either the Lamé potential $\varphi(R,\Theta)$ or Love's strain function $\Phi(R,\Theta)$ should yield the correct form of the order $1/R^2$ in the appropriate derivatives to provide a dimensionally consistent measure of the stresses. We have already employed the exterior solution for $\varphi(R,\Theta)$ to generate the Love strain function for Kelvin's problem. Consequently a biharmonic solution cannot be expected to provide a solution with the correct order $1/R^2$ for the variation in stress. We therefore seek a solution of the Lamé strain potential which should be of a form such that when differentiated twice with respect to R, the resulting expression should be of order $1/R^2$. The required solution should thus be of the form

$$\varphi(R,\Theta) = A\ln[Rf(\Theta)], \tag{13}$$

where A is a constant and $f(\Theta)$ is an arbitrary function. Substituting (13) in (1) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\Theta}\left\{\frac{\sin\Theta}{f}\frac{\mathrm{d}f}{\mathrm{d}\Theta}\right\} + \sin\Theta = 0. \tag{14}$$

The solution of (14), obtained via successive integrations has the following form:

$$f(\Theta) = \exp\left\{\int_0^\Theta \left(\frac{\cos\phi - 1}{\sin\phi}\right)\mathrm{d}\phi\right\} = (1+\cos\Theta). \tag{15}$$

The Lamé strain potential

$$\varphi(R,\Theta) = A\ln(R + R\cos\Theta) \tag{16}$$

gives the stress components

$$\sigma_{\Theta\Theta}(R,\Theta) = \frac{A\cos\Theta}{R^2(1+\cos\Theta)}, \quad \sigma_{R\Theta}(R,\Theta) = \frac{A\sin\Theta}{R^2(1+\cos\Theta)}. \tag{17}$$

The result (17) can now be combined with the stresses derived for the Kelvin problem, (10), to satisfy the zero shear traction boundary condition required for Boussinesq's solution. This gives

$$A = -C(1-2v) \tag{18}$$

and C can be determined by evaluating the resultant of axial tractions acting on a hemispherical surface, of arbitrary radius a, centered about the origin, i.e.,

$$P_{\mathrm{B}} + 2\pi\int_0^{\pi/2} [\sigma^*_{RR}\cos\Theta - \sigma^*_{R\Theta}\sin\Theta]a^2\sin\Theta \; \mathrm{d}\Theta = 0, \tag{19}$$

where σ^*_{RR} and $\sigma^*_{R\Theta}$ refer to the stress state obtained by combining (10) and (17). This gives

$$C = \frac{P_{\mathrm{B}}}{2\pi}. \tag{20}$$

The displacement and stress components take forms

$$\begin{aligned} 2Gu_R &= \frac{P_{\mathrm{B}}}{2\pi R}[4(1-v)\cos\Theta - (1-2v)] \\ 2Gu_\Theta &= \frac{P_{\mathrm{B}}\sin\Theta}{2\pi R}\left[-(3-4v) + \frac{(1-2v)}{(1+\cos\Theta)}\right] \end{aligned} \tag{21}$$

and

$$\begin{aligned} \sigma_{RR} &= \frac{P_{\mathrm{B}}}{2\pi R^2}[1-2v-2(2-v)\cos\Theta], \\ \sigma_{\Theta\Theta} &= \frac{P_{\mathrm{B}}(1-2v)\cos^2\Theta}{2\pi R^2(1+\cos\Theta)}, \quad \sigma_{\vartheta\vartheta} = \frac{P_{\mathrm{B}}(1-2v)}{2\pi R^2}\left(\frac{\cos\Theta - \sin^2\Theta}{1+\cos\Theta}\right), \\ \sigma_{R\Theta} &= \frac{P_{\mathrm{B}}(1-2v)}{2\pi R^2}\frac{\sin\Theta\cos\Theta}{(1+\cos\Theta)}. \end{aligned} \tag{22}$$

As is evident, when $v = \frac{1}{2}$, both Boussinesq's solution for the normal loading of the surface of a halfspace and Kelvin's solution for the interior loading of an infinite space by a concentrated force reduce to the same result, where the state of stress is purely radial.

4. Concluding remarks

The expositions of the derivation of the solution to Boussinesq's classical problem concerning the surface loading of a halfspace by a concentrated normal force given in the literature range from the use of results of potential theory, superposition schemes involving Kelvin's solution and the application of integral transform techniques. The former two procedures are largely based on familiarity with the appropriate mathematical analogy and the ingenious choice of superposition of concentrated force solutions associated with the infinite space. The Hankel integral transform procedure is more formal and readily yields the solution to Boussinesq's problem. It is shown that the solution to Boussinesq's problem can also be obtained through the use of a Lamé potential, the form of the solution of which is guided by dimensional considerations.

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