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An application of Betti's reciprocal theorem for the analysis of an inclusion problem

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Abstract

This paper presents the application of Betti's reciprocal theorem for evaluation of displacements of an embedded rigid disc inclusion in an *exact closed form*.

Keywords: Betti's reciprocal theorem; Inclusion in elastic body; Axial stiffness of inclusion; Dual integral equations; Love's strain function; Axisymmetric inclusion problem; Exact closed form solution to inclusion problem

1. Introduction

In 1872 the Italian mathematician, Enrico Betti [1], proposed a reciprocal theorem that is recognized as one of the most significant results in the classical theory of elasticity. Although James Clerk Maxwell proposed a law of reciprocal displacements and rotations [2] in 1864, the contribution of Betti is acknowledged for its underlying formal mathematical basis and generality. The importance of the theorem to classical elasticity is quite evident. Firstly, for the reciprocal theorem to be applicable, a stored energy function must exist (see, e.g. Truesdell [3]) with the consequence that the elasticity matrix must be positive definite and symmetric. Conversely, if the elasticity matrix is symmetric and positive definite, the reciprocity property exists. The reciprocity property provides the facility to relate two states through their displacement fields. This can be of considerable advantage when one such solution can be obtained relatively conveniently. In connection with boundary element formulations of problems in elasticity, Betti's reciprocal relationship provides an unquestionable advantage in the development and application of the procedures. Betti's reciprocal theorem has been successfully applied particularly in situations where results of global interest are required. These can include, for example, the determination of the displacements that are generated in contact problems in situations where loads are not applied directly to the indentor (see, e.g. Shield [4] and Shield and Anderson [5]). Selvadurai [6–11], Selvadurai and Au [12] and Davis and Selvadurai [13] have presented a variety of problems where solutions to inclusion problems involving simplified concentrated loadings can be conveniently obtained by recourse to Betti's reciprocal theorem.

The objective of this paper is to demonstrate the effectiveness of Betti's reciprocal theorem in providing the solution to a problem related to a rigid disc inclusion that is embedded in bonded contact with an isotropic elastic medium of infinite extent. The direct integral equation based analysis of the problem is complicated by the fact that the embedded rigid disc inclusion is subjected to an externally placed loading. It is shown that Betti's reciprocal theorem in conjunction with the auxiliary solution that examines the problem of a rigid disc inclusion subjected to a concentrated force can be combined to obtain the required result in an *exact closed form*.

2. The inclusion problem

We restrict attention to the axisymmetric problem in classical elasticity theory where an isotropic elastic infinite space is bounded internally by a rigid disc inclusion. The interface between the elastic medium and the rigid disc inclusion exhibits complete bonding, leading to continuity of displacements and tractions. The infinite space region is subjected to an axisymmetric internal load of finite radius b and uniform intensity p_0 (Fig. 1). The analysis of the problem centres on the determination of the rigid displacement of the disc inclusion that is induced by the external loading p_0.

We first examine the problem of an isotropic elastic infinite space, void of the rigid disc inclusion, which is subjected to an internal load of radius b and uniform stress intensity p_0, applied at the plane $z = 0$. The analysis of this

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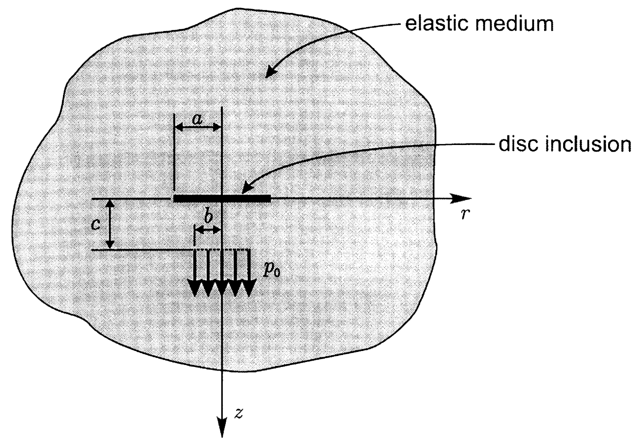

Fig. 1. A rigid disc inclusion subjected to an external distributed loading.

problem can be approached in a variety of ways, including formulations based on Hankel transform techniques and techniques based on the integration of Kelvin's solution for the internal loading of an isotropic elastic infinite space. Consider Kelvin's solution for the problem of a concentrated force acting at the interior of an infinite space region: since the problem is axisymmetric we note that the displacements in the elastic infinite space region due to a concentrated force of magnitude P acting at the origin in the positive z-direction are given by

$$u_r^c(r,z) = \frac{Prz}{16\pi\mu(1-\nu)(r^2+z^2)^{3/2}} \tag{1}$$

and

$$u_z^c(r,z) = \frac{P}{16\pi\mu(1-\nu)} \times \left[\frac{2(1-\nu)}{(r^2+z^2)^{1/2}} + \frac{1}{(r^2+z^2)^{1/2}} + \frac{z^2}{(r^2+z^2)^{3/2}}\right] \tag{2}$$

where μ is the linear elastic shear modulus and υ is Poisson's ratio. We can make use of this result to determine the displacement field in the elastic infinite space when the concentrated force P acts at the location $(r, (\theta+\theta^*), c)$. This result can in turn be integrated over the area of application of p_0 to determine the displacement field in the elastic infinite space due to circular distributed load of intensity p_0 and radius b located on the plane $z = c$. Avoiding details of calculations, it can be shown that

$$u_r^p(r,c) = \frac{-cp_0}{16\pi\mu(1-\nu)} \int_0^{2\pi}\int_0^b \frac{r^*}{s^{1/2}}(r - r^*\cos\theta^*)\mathrm{d}r^*\,\mathrm{d}\theta^* \tag{3}$$

$$u_z^p(r,c) = \frac{p_0}{16\pi\mu(1-\nu)} \int_0^{2\pi}\int_0^b \left[\frac{(3-4\nu)}{s^{1/2}} + \frac{c^2}{s^{3/2}}\right] r^*\,\mathrm{d}r^*\,\mathrm{d}\theta^* \tag{4}$$

where

$$s^2 = r^2 + (r^*)^2 - 2rr^*\cos\theta^* + c^2$$
$$s\cos\phi = r - r^*\cos\theta^* \tag{5}$$

and r^*, θ^* are integration variables.

The integrals in Eqs. (3) and (4) can be further reduced to representations in terms of elliptic functions. The exact nature of these representations is not important to the discussion that follows.

For the formulation of the elastostatic boundary value problem governing the axisymmetric loading of the embedded rigid disc inclusion problem, we can use a variety of representations ranging from the Neuber–Papkovich formulation (see, e.g. Gurtin [14]) to Love's strain function approach [15]. It is convenient to adopt Love's formulation, which states that, in the absence of body forces, the solution to the axisymmetric problem for an isotropic elastic material can be represented in terms of a *single* function $\Phi(r,z)$, which is biharmonic, i.e.

$$\nabla^2\nabla^2\Phi(r,z) = 0 \tag{6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{7}$$

is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system. The displacement and stress components, also referred to the cylindrical polar coordinate system, can be uniquely represented in terms of $\Phi(r,z)$. The results of interest to the present discussion are as follows:

$$2\mu u_r(r,z) = -\frac{\partial^2\Phi}{\partial r\,\partial z}$$
$$2\mu u_z(r,z) = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{8}$$

and

$$\sigma_{zz}(r,z) = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right]$$
$$\sigma_{rz}(r,z) = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right]\ldots \tag{9}$$

and by virtue of axial symmetry, $u_\theta = 0$ and $\sigma_{r\theta}$ and $\sigma_{\theta z}$ are also zero.

Let us now consider the problem of the embedded rigid disc inclusion of radius a, which is embedded at the plane $z = 0$ and loaded by the axisymmetric external loading of uniform intensity p_0 and radius b located at the plane $z = c$. Since the disc inclusion is located on the plane $z = 0$, we can formulate the resulting elastostatic boundary value problem by selecting solutions of Eq. (6) which are applicable to the regions $z \geq 0$ and $z \leq 0$. The solutions of Eq. (6)

chosen should satisfy the regularity conditions associated with the embedded inclusion problem, in that, the displacement and stress fields should vanish as $(r, z) \longrightarrow \infty$. Since the problem is axisymmetric, we seek solutions of Eq. (6) that are based on a Hankel transform development of the governing partial differential equation (see, e.g. Sneddon [16] and Selvadurai [17]). The appropriate solutions of Eq. (6) which satisfy the regularity conditions are given by the following:

1. for the region $z \geq 0$

$$\Phi^{(1)}(r,z) = \int_0^\infty \xi[A(\xi) + B(\xi)z]\mathrm{e}^{-\xi z} J_0(\xi r)\mathrm{d}\xi \tag{10}$$

2. for the region $z \leq 0$

$$\Phi^{(2)}(r,z) = \int_0^\infty \xi[C(\xi) + D(\xi)z]\mathrm{e}^{\xi z} J_0(\xi r)\mathrm{d}\xi \tag{11}$$

where $A(\xi)$, $B(\xi)$, etc. are arbitrary functions and $J_0(\xi r)$ is the zeroth-order Bessel function of the first kind. The superscripts (1) and (2) will refer to quantities associated with the region $z \geq 0$ and $z \leq 0$, respectively. The embedded inclusion problem can now be formulated as follows: on the plane containing the embedded rigid disc inclusion

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0) = -u_r^p(r); \qquad 0 \leq r \leq a \tag{12}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0) = \Delta_0 - u_z^p(r); \qquad 0 \leq r \leq a \tag{13}$$

$$u_r^{(1)}(r,0) - u_r^{(2)}(r,0) = 0; \qquad a \leq r \leq \infty \tag{14}$$

$$u_z^{(1)}(r,0) - u_z^{(2)}(r,0) = 0; \qquad a \leq r \leq \infty \tag{15}$$

$$\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0) = 0; \qquad a < r < \infty \tag{16}$$

$$\sigma_{rz}^{(1)}(r,0) - \sigma_{rz}^{(2)}(r,0) = 0; \qquad a < r < \infty \tag{17}$$

where $u_r^p(r)$ and $u_z^p(r)$ are the displacement fields due to the exterior load p_0 applied at $z = c$ defined by Eqs. (3) and (4), respectively, and Δ_0 is the axisymmetric rigid translation of the rigid disc inclusion which is to be determined by evaluating the net resultant force on the rigid disc inclusion, i.e.

$$\int_0^{2\pi} \int_0^a [\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0)] r \,\mathrm{d}r\,\mathrm{d}\theta = 0 \tag{18}$$

The mixed boundary conditions (12)–(17) yield the following set of integral equations for the solution of the rigid disc inclusion problem:

$$\int_0^\infty \xi[-\xi A(\xi) + B(\xi)] J_1(\xi)\mathrm{d}\xi = -\frac{u_r^p(r)}{2\mu}; \qquad 0 \leq r \leq a \tag{19}$$

$$\int_0^\infty \xi[-\xi A(\xi) - 2(1-2\nu)B(\xi)] J_0(\xi r)\mathrm{d}\xi = \frac{1}{2\mu}[\Delta_0 - u_z^p(r)]; \qquad 0 \leq r \leq a \tag{20}$$

$$\int_0^\infty \xi[-\xi\{A(\xi)c + C(\xi)\} + B(\xi) - D(\xi)] J_1(\xi r)\mathrm{d}\xi = 0; \quad 0 \leq r < \infty \tag{21}$$

$$\int_0^\infty \xi[-\xi\{A(\xi) + C(\xi)\} - 2(1-2\nu)\{B(\xi) - D(\xi)\}] J_0(\xi r)\mathrm{d}\xi = 0; \quad 0 \leq r < \infty \tag{22}$$

$$\int_0^\infty \xi^2[\xi\{A(\xi) + C(\xi)\} + (1-2\nu)\{B(\xi) - D(\xi)\}] J_0(\xi r)\mathrm{d}\xi = 0; \quad a < r < \infty \tag{23}$$

$$\int_0^\infty \xi^2[\xi\{A(\xi) - C(\xi)\} - 2\nu\{B(\xi) + D(\xi)\}] J_1(\xi r)\mathrm{d}\xi = 0; \quad a < r < \infty \tag{24}$$

where, $J_1(\xi r)$ is the first-order Bessel function of the first kind.

Through the introduction of substitutions and appropriate finite Fourier cosine transforms, the systems of integral Eqs. (19)–(24) can be further reduced to a pair of coupled Fredholm integral equations of the second kind of the general form

$$\Psi_1(t) + \int_0^a \Psi_1(u) K_{11}(u,t)\mathrm{d}u + \int_0^a \Psi_2(u) K_{12}(u,t)\mathrm{d}u = F_1(t); \quad 0 < r < a \tag{25}$$

$$\Psi_2(t) + \int_0^a \Psi_1(u) K_{21}(u,t)\mathrm{d}u + \int_0^a \Psi_2(u) K_{22}(u,t)\mathrm{d}u = F_2(t); \quad 0 < r < a \tag{26}$$

for the unknown functions $\Psi_1(t)$ and $\Psi_2(t)$ and the unknown displacement Δ_0, where $K_{11}(u,t)$, $K_{12}(u,t)$, etc. are kernel functions and $F_1(t)$ and $F_2(t)$ depend on the solutions $u_r^p(r)$, $u_z^p(r)$ and Δ_0. A further equation is provided through the equilibrium Eq. (18), which now completely defines the integral equations of the disc inclusion problem. Examples, which illustrate the reduction procedures applicable to this type of problem can be found in the articles by Selvadurai [18,19]. Once the solutions for $\Psi_1(t)$ and $\Psi_2(t)$ are obtained, the solution of the problem is complete and all the displacements and stress fields in the elastic medium, including the displacement of the rigid inclusion Δ_0, are determined. These coupled integral equations, however, are not amenable to solution in any exact closed form; recourse must be made to solution of the system of coupled integral equations of the type (25) and (26) in a numerical fashion, which implies that the result for Δ_0 can be obtained only through a numerical solution. Hence, the direct solution of the externally loaded rigid disc inclusion problem invariably results in the development of only a numerical solution to the primary result of engineering interest.

3. Application of Betti's reciprocal theorem

As an alternative to the formal procedure outlined in the previous section, we examine the embedded externally loaded rigid disc inclusion problem by adopting Betti's reciprocal theorem. We consider the auxiliary problem involving axisymmetric direct loading of the rigid disc inclusion that is embedded in bonded contact with the isotropic elastic medium of infinite extent. The nature of the direct loading of the embedded rigid disc inclusion is immaterial; the only requirement is that the resulting displacement of the rigid disc inclusion should induce a state of axial symmetry in the deformation of the elastic medium. For the present, let us assume that, due to the direct axisymmetric loading of the embedded rigid disc inclusion, the disc experiences a rigid translation Δ^* in the z-direction. The resulting state of deformation in the elastic infinite space exhibits asymmetry in u_r and σ_{zz} about the plane $z = 0$. The embedded rigid disc inclusion problem can thus be posed as a mixed boundary value problem related to a single halfspace region ($z \geq 0$) where the plane $z = 0$ is subjected to the following mixed boundary conditions:

$$u_z^{(1)}(r,0) = \Delta^*; \qquad 0 \leq r \leq a \tag{27}$$

$$u_r^{(1)}(r,0) = 0; \qquad 0 \leq r < \infty \tag{28}$$

$$\sigma_{zz}^{(1)}(r,0) = 0; \qquad a < r < \infty \tag{29}$$

For ease of a non-dimensional presentation of the results we select a Hankel integral representation of $\Phi(r,z)$ in the form

$$\Phi(r,z) = \frac{1}{a^2}\int_0^\infty \zeta\left[A(\zeta) + B(\zeta)\frac{z}{a}\right]\mathrm{e}^{-\zeta z/a}J_0(\zeta r/a)\mathrm{d}\zeta \tag{30}$$

Using Eq. (30) and the relationships (8) and (9) the mixed boundary conditions (27)–(29) yield the following system of dual integral equations for the unknown function $A(\zeta)$, i.e.

$$\int_0^\infty \zeta^3 A(\zeta)J_0(\zeta r/a)\mathrm{d}\zeta = -\frac{2\mu\Delta^* a^4}{(3-4\nu)}; \qquad 0 \leq r \leq a \tag{31}$$

$$\int_0^\infty \zeta^4 A(\zeta)J_0(\zeta r/a)\mathrm{d}\zeta = 0; \qquad a < r < \infty \tag{32}$$

The solution of the dual systems (31) and (32) is described in a number of texts on integral equations, elasticity and contact mechanics and will not be discussed here in detail (see, e.g. Sneddon [16,20], Little [21], Gladwell [22] and Selvadurai [17]). It is sufficient to note that by introducing a finite Fourier cosine transform in terms of an unknown function, Eq. (32) can be identically satisfied and Eq. (31) can be reduced to an Abel-type integral equation for this unknown function. The resulting solution for $A(\zeta)$ can be written as

$$A(\zeta) = -\frac{4\mu\Delta^* a^4}{\pi\zeta^4} \tag{33}$$

The contact stresses acting on the interface between the disc inclusion and the halfspace region ($z \geq 0$) is given by

$$\sigma_{zz}(r,0) = -\frac{8\Delta^*\mu(1-\nu)}{\pi(3-4\nu)}\frac{1}{\sqrt{a^2-r^2}}; \qquad 0 < r < a \tag{34}$$

the negative sign indicating a compressive stress state. From the asymmetry of the problem, we note that $\sigma_{zz}(r,0)$ $\sigma_{zz}(r,0)$ acting on the face of the inclusion in contact with the region $z \leq 0$ is the tensile equivalent of Eq. (34). If we assume that the axisymmetric loading on the inclusion corresponds to a uniform load of radius b and intensity p^*, we can use the equation of equilibrium for the inclusion to relate p^* to Δ^*, i.e.

$$\pi p^* b^2 = \frac{32\Delta^*\mu a(1-\nu)}{(3-4\nu)} \tag{35}$$

where we assume that Δ^* occurs in the direction of application of p^*.

Avoiding details of calculations, it can be shown that the displacement field in the region $z \geq 0$ due to the axisymmetric direct loading of the rigid disc inclusion is given by

$$u_z(r,z) = \frac{p^* b^2}{16\mu(1-\nu)a} \times \int_0^\infty \frac{\sin\zeta}{\zeta}\left\{(3-4\nu) + \zeta\frac{z}{a}\right\}\mathrm{e}^{-\zeta z/a}J_0(\zeta r/a)\mathrm{d}\zeta \tag{36}$$

We can now use Eqs. (35) and (36) in conjunction with Betti's reciprocal theorem to determine the displacement Δ_0 induced on the rigid disc inclusion due to a uniform circular load of intensity p_0 and radius b located at the plane $z = c$. From the reciprocal theorem, we have

$$\int_0^b p^*\Delta_0 r\,\mathrm{d}r = \int_0^b p_0 u_z(r,c) r\,\mathrm{d}r \tag{37}$$

If we assume the intensity of loading is identical for both states (i.e. $p_0 = p^*$), then we obtain from Eq. (37)

$$\Delta_0 = \frac{p_0}{8\mu(1-\nu)} \times \int_0^b r\left[\int_0^\infty \frac{\sin\zeta}{\zeta}\left\{(3-4\nu) + \frac{\zeta c}{a}\right\}\mathrm{e}^{-\zeta c/a}J_0(\zeta r/a)\mathrm{d}\zeta\right]\mathrm{d}r \tag{38}$$

The purpose of this paper is also to demonstrate that, although non-routine, the integrals occurring in Eq. (38) can be evaluated in explicit form (see Appendix A). The final results for the displacement of the rigid disc inclusion due to the externally placed axisymmetric circular load of radius b located at the plane $z = c$and stress intensity p_0 can be obtained in the form

$$\frac{\Delta_0}{\Delta^*} = \frac{4\lambda}{\pi\eta^2(3-4\nu)}\left\{1 - R\sin\left(\frac{\theta_1}{2}\right)\right\} + \frac{2}{\pi}\tan^{-1}\left\{\frac{R\sin\left(\frac{\theta_1}{2}\right) + \sqrt{1+\lambda^2}\sin\theta_2}{R\cos\left(\frac{\theta_1}{2}\right) + \sqrt{1+\lambda^2}\cos\theta_2}\right\} + \frac{1}{\pi\eta^2}\left\{2R\sqrt{1+\lambda^2}\sin\left(\frac{\theta_1+2\theta_2}{2}\right) - (1+\lambda^2)\sin 2\theta_2 - R^2\sin\theta_1\right\} \quad (39)$$

where

$$R^4 = (\eta^2+\lambda^2-1)^2 + 4\lambda^2; \quad \lambda = \frac{c}{a}; \quad \eta = \frac{b}{a}$$

$$R\sin\left(\frac{\theta_1}{2}\right) = \left[\frac{1}{2}(R^2+1-\eta^2-\lambda^2)\right]^{1/2}$$

$$R\cos\left(\frac{\theta_1}{2}\right) = \left[\frac{1}{2}(R^2-1+\eta^2+\lambda^2)\right]^{1/2}$$

$$R^2\sin\theta_1 = 2\lambda; \quad \sin\theta_2 = \frac{1}{\sqrt{1+\lambda^2}}; \quad \cos\theta_2 = \frac{\lambda}{\sqrt{1+\lambda^2}}$$

$$\sin 2\theta_2 = \frac{2\lambda}{(1+\lambda^2)} \quad (40)$$

and θ_1 and θ_2 are defined when λ and η are specified.

We note that, as $\lambda \to \infty$, there is no influence of the externally placed load on the displacement of the disc inclusion; as such $(\Delta_0/\Delta^*) \to 0$. Similarly, as $\eta \to 0$, but maintaining $\pi p_0 b^2 \to P_0$, we obtain the displacement of the rigid disc inclusion due to a concentrated force of magnitude P_0, which is located at a distance $z = c$ and directed along the z-axis, i.e.

$$\Delta_0 = \frac{P_0}{16\pi(1-\nu)\mu a}\left[(3-4\nu)\tan^{-1}\left(\frac{1}{\lambda}\right) + \frac{\lambda}{(\lambda^2+1)}\right] \quad (41)$$

As $\lambda \to 0$, we recover from Eq. (41) the result (Eq. (35)) for the displacement of the disc inclusion due to the direct application of the force P_0.

4. Concluding remarks

The effectiveness of Betti's reciprocal theorem is demonstrated by appeal to a disc inclusion problem in the classical theory of elasticity. The method proves to be invaluable when dealing with the evaluation of global responses of inclusions subjected to external loads. The direct formulation of the mixed boundary value problem associated with an externally loaded inclusion yields a complicated set of coupled integral equations of the Fredholm type which can be solved only by recourse to numerical techniques. The auxiliary solution required for the application of Betti's reciprocal theorem usually involves the problem related to the directly loaded inclusion. The solution of this latter problem is much less complicated. In the example discussed here, the use of Betti's reciprocal theorem leads to a closed form solution for the displacement of the inclusion. The procedure can be generalized to determine more complicated situations involving interaction of disc inclusions and other multiple loadings.

Appendix A

Consider the integral

$$I = \int_0^\infty e^{-tx}\frac{\sin x}{x}J_1(\alpha x)\mathrm{d}x \quad (A1)$$

which can be written as the summation of three integrals

$$I_1 = \frac{\alpha}{2}\int_0^\infty e^{-tx}\frac{\sin x}{x}[J_0(\alpha x) - 1]\mathrm{d}x$$

$$I_2 = \frac{\alpha}{2}\int_0^\infty e^{-tx}\frac{\sin x}{x}\mathrm{d}x \quad (A2)$$

$$I_3 = \frac{\alpha}{2}\int_0^\infty e^{-tx}\frac{\sin x}{x}J_2(\alpha x)\mathrm{d}x$$

We use standard results for I_1 and I_2, i.e.

$$\frac{\alpha}{2}\int_0^\infty e^{-sx}[J_0(\alpha x) - 1]\frac{\mathrm{d}x}{x} = \ln\left[\frac{2s}{s+\sqrt{\alpha^2+s^2}}\right] \quad (A3)$$

and

$$\int_0^\infty e^{-tx}\frac{\sin x}{x}\mathrm{d}x = \tan^{-1}\left(\frac{1}{t}\right) \quad (A4)$$

If we put $s = (t - i)$ in Eq. (A3), then the Imaginary part of the integral will give $I_1(2/\alpha)$. With $s = (t - i)$ in Eq. (A3) we get

$$I_4 = \ln(2\sqrt{1+t^2})e^{-i\theta_2}[R^* e^{-i\phi}]^{-1} \quad (A5)$$

where

$$(R^*)^2 = R^2 + t^2 + 1 + 2R\sqrt{1+t^2}\cos\left(\frac{\theta_1}{2} - \theta_2\right)$$

$$R^4 = (\alpha^2 + t^2 - 1)^2 + 4t^2$$

$$\tan\theta_1 = \frac{2t}{\alpha^2 + t^2 - 1} \quad \tan\theta_2 = \frac{1}{t} \tag{A6}$$

$$\tan\phi = \frac{R\sin\dfrac{\theta_1}{2} + \sqrt{1+t^2}\sin\theta_2}{R\cos\dfrac{\theta_1}{2} + \sqrt{1+t^2}\cos\theta_2}$$

Since

$$I_1 = \frac{\alpha}{2}\mathrm{Im}(I_4) = \frac{\alpha}{2}\mathrm{Im}\left(\ln\frac{2\sqrt{1+t^2}}{R^*}\mathrm{e}^{-\mathrm{i}\theta_2 + \mathrm{i}\phi}\right) \tag{A7}$$

we obtain

$$I_1 = \frac{\alpha}{2}(-\theta_2 + \phi) \tag{A8}$$

and

$$I_1 + I_2 = \frac{\alpha}{2}\frac{R\sin\dfrac{\theta_1}{2} + \sqrt{1+t^2}\sin\theta_2}{R\cos\dfrac{\theta_1}{2} + \sqrt{1+t^2}\cos\theta_2} \tag{A9}$$

To evaluate I_3 we use the result

$$\int_0^\infty u^n\,\mathrm{e}^{-su}\,J_m(\alpha u)\mathrm{d}u = \frac{\left(\dfrac{\alpha}{2}\right)^m\Gamma(m+n+1)}{(\alpha^2+s^2)^{(m+n+1)/2}\Gamma(m+1)} \times {}_2F_1\left(\frac{m+n+1}{2}, \frac{m-n}{2}, m+1; \frac{\alpha^2}{\alpha^2+s^2}\right) \tag{A10}$$

where ${}_2F_1(\alpha, \beta, \gamma; \delta)$ is the confluent hypergeometric function (see, e.g. Sneddon [16]).

We note that

$${}_2F_1\left(1, \frac{3}{2,}, 3; \frac{\alpha^2}{\alpha^2+s^2}\right) = \frac{4(\alpha^2+s^2)}{\alpha^4}\left(\sqrt{\alpha^2+s^2} - s\right)^2 \tag{A11}$$

The integral

$$I_5 = \int_0^\infty \frac{\mathrm{e}^{-sx}}{x}J_2(\alpha x)\mathrm{d}x = \frac{(\sqrt{\alpha^2+s^2} - s)^2}{2\alpha^2} \tag{A12}$$

We now let $s = (t - i)$, then the imaginary part of I_5 will give I_3, i.e.

$$I_5 = \frac{1}{2\alpha^2}\left(R^2\,\mathrm{e}^{-\mathrm{i}\theta} + [t^2+1]\mathrm{e}^{-2\mathrm{i}\theta_2} - 2R\sqrt{1+t^2}\,\mathrm{e}^{-\mathrm{i}((\theta_1/2)+\theta_2)}\right) \tag{A13}$$

which gives

$$I_3 = \frac{1}{4\alpha}\left(2R\sqrt{1+t^2}\sin\left(\frac{\theta_1}{2} + \theta_2\right) - (1+t^2)\sin 2\theta_2 - R^2\sin\theta_1\right) \tag{A14}$$

Hence the integral (A1) can be written as

$$I = \frac{\alpha}{2}\tan^{-1}\frac{R\sin\dfrac{\theta_1}{2} + \sqrt{1+t^2}\sin\theta_2}{R\cos\dfrac{\theta_1}{2} + \sqrt{1+t^2}\cos\theta_2} + \frac{1}{4\alpha}\left(2R\sqrt{1+t^2}\sin\left(\frac{\theta_1}{2} + \theta_2\right) - (1+t^2)\sin 2\theta_2 - R^2\sin\theta_1\right) \tag{A15}$$

where R, θ_1 and θ_2 are defined by Eq. (A6).

The result (A15) can now be used to evaluate the integral (38):

$$\bar{\Delta} = \frac{\Delta_0}{\Delta^*} = \frac{4}{\pi b^2(3-4\nu)} \times \int_0^b r\left(\int_0^\infty \frac{\sin\zeta}{\zeta}[(3-4\nu) + \zeta\lambda]\mathrm{e}^{-\zeta\lambda}J_0\left(\frac{\zeta r}{a}\right)\mathrm{d}\zeta\right)\mathrm{d}r \tag{A16}$$

where $\lambda = c/a$. Changing the order of integration, Eq. (A16) gives

$$\bar{\Delta} = \frac{4\lambda}{\pi\eta^2(3-4\nu)}\int_0^\infty \frac{\sin\zeta}{\varsigma}\mathrm{e}^{-\zeta\lambda}J_1(\zeta\eta)\mathrm{d}\zeta + \frac{4}{\pi\eta}\int_0^\infty \frac{\sin\zeta}{\zeta^2}\mathrm{e}^{-\zeta\lambda}J_1(\zeta\eta)\mathrm{d}\zeta \tag{A17}$$

which, can be expressed in the form

$$\bar{\Delta} = \frac{4\lambda}{\pi\eta^2(3-4\nu)}\left(1 - R\sin\frac{\theta_1}{2}\right) + \frac{2}{\pi}\tan^{-1}\frac{R\sin\dfrac{\theta_1}{2} + \sqrt{1+\lambda^2}\sin\theta_2}{R\cos\dfrac{\theta_1}{2} + \sqrt{1+\lambda^2}\cos\theta_2} + \frac{1}{\pi\eta^2}\left(2R\sqrt{1+\lambda^2}\sin\left(\frac{\theta_1}{2} + \theta_2\right) - (1+\lambda^2)\sin 2\theta_2 - R^2\sin\theta_1\right) \tag{A18}$$

Using the relations (A15), we can rewrite Eq. (A18) in the form

$$\bar{\Delta} = \frac{4\lambda}{\pi\eta^2(3-4\nu)}\left(1 - \sqrt{\tfrac{1}{2}(R^2+1-\eta^2-\lambda^2)}\right) + \frac{2}{\pi}\tan^{-1}\left[\frac{\sqrt{\tfrac{1}{2}(R^2+1-\eta^2-\lambda^2)}+1}{\sqrt{\tfrac{1}{2}(R^2-1+\eta^2+\lambda^2)}+\lambda}\right]\frac{2}{\pi\eta^2} \times\left(\lambda\sqrt{\tfrac{1}{2}(R^2+1-\eta^2-\lambda^2)} + \sqrt{\tfrac{1}{2}(R^2-1+\eta^2-\lambda^2)} - 2\lambda\right) \quad \text{(A19)}$$

or

$$\bar{\Delta} = F_1 + F_2 + F_3 \quad \text{(A20)}$$

In the limit when $\eta \to 0$, we obtain

$$F_1 \to \frac{2\lambda}{\pi(3-4\nu)(1+\lambda^2)};\quad F_2 \to \frac{2}{\pi}\tan^{-1}\left(\frac{1}{\lambda}\right);\quad F_3 \to 0 \quad \text{(A21)}$$

Hence, as $\eta \to 0$, we obtain

$$\bar{\Delta} = \frac{2}{\pi(3-4\nu)}\left[(3-4\nu)\tan^{-1}\left(\frac{1}{\lambda}\right) + \frac{\lambda}{(1+\lambda^2)}\right] \quad \text{(A22)}$$

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