

## The penny-shaped crack at a bonded plane with localized elastic non-homogeneity

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**Abstract** – This paper examines the axisymmetric problem pertaining to a penny-shaped crack which is located at the bonded plane of two similar elastic halfspace regions which exhibit localized axial variations in the linear elastic shear modulus, which has the form  $G(z) = G_1 + G_2 e^{\pm \zeta z}$ . The equations of elasticity governing this type of non-homogeneity are solved by employing a Hankel transform technique. The resulting mixed boundary value problem associated with the penny-shaped crack is reduced to a Fredholm integral equation of the second kind which is solved in a numerical fashion to generate the crack opening mode stress intensity factor at the tip. © 2000 Éditions scientifiques et médicales Elsevier SAS

**penny-shaped crack / elastic non-homogeneity / bonded interface / integral equations / stress intensity factors**

### 1. Introduction

The classical problem related to the penny-shaped crack embedded in a homogeneous isotropic elastic medium was first investigated by Sneddon (1946) who employed dual integral equation techniques. The problem was subsequently investigated by Sack (1946) who used a formulation based on spheroidal harmonic function techniques. The investigations by Mossakovskii and Rybka (1964), Erdogan (1965), Kassir and Bregman (1972), Lowengrub and Sneddon (1972), Willis (1972) and Erdogan and Arin (1972) extended these studies to include penny-shaped cracks located at the interface of bi-material elastic regions. References to related problems are also given by Kassir and Sih (1975), Atkinson (1979) and Sih and Chen (1981). The present paper examines the problem related to a penny-shaped crack which is located at the bonded plane of two similar elastic solids which exhibit a near interface elastic non-homogeneity. Crack problems related to non-homogeneous elastic media have been examined by a number of investigators. Singh and Dhahliwal (1978) have examined the problem of a Griffith crack in a non-homogeneous elastic medium where the shear modulus varies according to an exponential law. Gerasoulis and Srivastav (1980) have examined the Griffith crack problem for an isotropic elastic medium where the shear modulus varies and is a function of the distance from the plane of the crack. The class of problems involving the internal pressurization of a plane crack located at the boundary of non-homogeneous elastic medium was examined by Delale and Erdogan (1983). Delale and Erdogan (1988a and b) have also extended these studies to include the problem of a crack embedded at the interface between a homogeneous halfplane and a non-homogeneous elastic halfplane. Studies by Erdogan et al. (1991a and b), Erdogan and Ozturk (1992), Ozturk and Erdogan (1993 and 1995) and Clements et al. (1997) also deal with plane and axisymmetric crack problems related to non-homogeneous media. A detailed account of the axisymmetric contact problem related to non-homogeneous media is given by Selvadurai (1996). Selvadurai and Lan (1998) have examined problems related to non-homogeneous media where the non-homogeneity has a harmonic variation. Spencer and Selvadurai (1998) have recently examined a class of crack and dislocation problems related to anti-plane straining of a generalized non-homogeneous elastic solid. Ozturk and Erdogan (1995) cite a number of situations where the elastic non-homogeneity in the vicinity of the crack region is important to engineering applications; these include, chemical reaction zones or diffusion zones

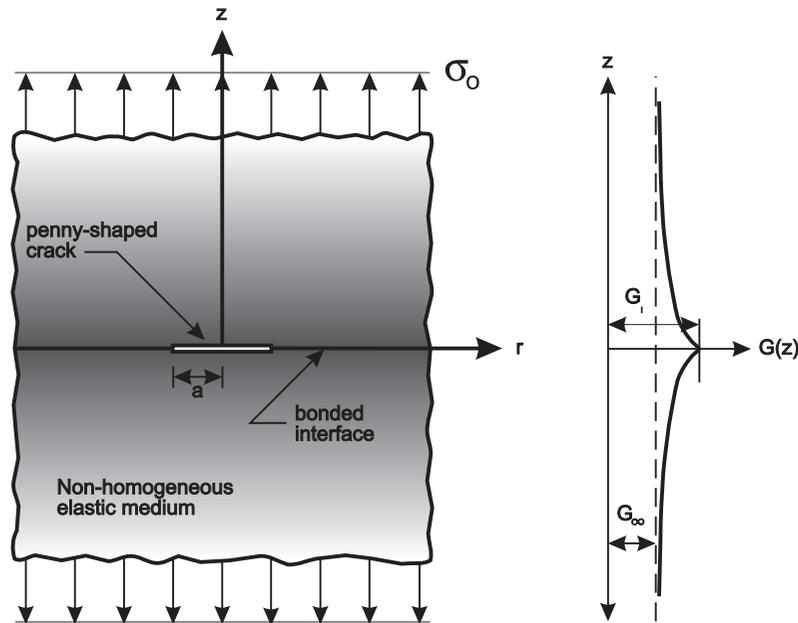


Figure 1. Penny-shaped crack at the bonded plane of two halfspace regions with localized elastic non-homogeneity.

which can be created in an otherwise homogeneous medium as a result of processing techniques such as ion plating, sputtering or plasma spray coating. The interfacial non-homogeneity can also be purposely introduced to minimize concentration of residual and thermal stresses and to increase bonding strength.

In the formulation of a majority of crack problems involving non-homogeneous elastic media, except those involving a non-homogeneous interfacial zone of finite thickness, the variation in the shear modulus is usually specified as an exponential function of the axial coordinate. If, as for most crack problems, the non-homogeneous region is unbounded, the elastic constants can themselves be unbounded as the spatial coordinate(s) approach infinity. Consequently, it is useful to examine the class of crack problems related to non-homogeneous elastic media where the elastic non-homogeneity exhibits a variation but would be bounded in relation to an elastic infinite space. This paper therefore examines a localized inhomogeneity which has an exponential form with a bounded axial variation. The paper focuses on the specific problem related to a penny-shaped crack which is located at the plane boundary between two identical halfspace regions where the shear modulus varies in an exponential manner (*figure 1*). The problem is of particular interest to the examination of the flaws located at the bonded boundary between identical porous elastic media. The changes in the elastic properties, notably the linear elastic shear modulus, can occur as a result of diffusion of the adhesive material into the porous medium, resulting in a localized alteration of the elastic constants. Similar considerations can apply to changes in the elasticity properties in the vicinity of bonded surfaces which can result from chemical changes and moisture-diffusion in the parent material which can be induced by the adhesive (see, e.g., Plueddemann, 1974; Anderson et al., 1977; de Lollis, 1985; Pizzi and Mittal, 1994 and Mittal, 1995).

The paper presents the integral equations governing the penny-shaped crack problem and the numerical results for the stress intensity factor at the tip of the crack. It illustrates the manner in which the stress intensity factor can be influenced by the localized elastic non-homogeneity.

## 2. Governing equations

We consider axisymmetric deformations of an isotropic non-homogeneous elastic material such that the linear elastic shear modulus  $G(r, z)$  and Poisson's ratio  $\nu(r, z)$  have specific variations of the form

$$G(r, z) = G(z); \quad \nu(r, z) = \nu = \text{constant}. \quad (1)$$

For axisymmetric deformations of the linearly elastic material we have

$$u_i = (u_r, 0, u_z) \quad (2)$$

and

$$\sigma_{ij} = 2G(z)\varepsilon_{ij} + \frac{2G(z)}{1-2\nu}\varepsilon_{kk}\delta_{ij}, \quad (3)$$

where  $\sigma_{ij}$  is the stress tensor;  $\varepsilon_{ij}$  is the strain tensor;  $\delta_{ij}$  is Kronecker's delta function and  $\varepsilon_{kk} = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz} = e$ . In the absence of body forces, the equations of equilibrium can be expressed in the forms

$$\nabla^2 u_r + \frac{1}{1-2\nu} \frac{\partial e}{\partial r} - \frac{u_r}{r^2} + \frac{1}{G} \frac{\partial G}{\partial z} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) = 0, \quad (4)$$

$$\nabla^2 u_z + \frac{1}{1-2\nu} \frac{\partial e}{\partial z} + \frac{2}{G} \frac{\partial G}{\partial z} \left( \frac{\partial u_z}{\partial z} + \frac{\nu e}{1-2\nu} \right) = 0, \quad (5)$$

where  $\nabla^2$  is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system, i.e.

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \quad (6)$$

We now restrict our attention to the specific form of the axial variations in  $G(z)$  such that

$$\begin{aligned} G(z) &= G_1 + G_2 e^{-\zeta z} \quad \text{for } 0 \leq z < \infty, \\ G(z) &= G_1 + G_2 e^{\zeta z} \quad \text{for } -\infty < z \leq 0. \end{aligned} \quad (7)$$

In order to ensure bounded variation in  $G(z)$  as  $|z| \rightarrow \infty$  we require  $\zeta > 0$ . The values of  $G_1$  and  $G_2$  must be prescribed in such a way that the thermodynamic constraints which ensure positive definiteness of the strain energy in the medium is satisfied at every point in the medium; i.e.

$$G(z) > 0; \quad G_1 > 0 \quad \text{for } z \in (-\infty, \infty). \quad (8)$$

It is also possible to interpret  $G_1$  and  $G_2$  in terms of the shear moduli that may be prescribed at  $z = 0$  and  $z \rightarrow \pm\infty$ . If we denote the shear modulus at the interface between the bonded halfspace region by  $G_i$  and the shear modulus at  $z \rightarrow \pm\infty$  by  $G_\infty$ , then

$$G_1 = G_\infty; \quad G_2 = G_i - G_\infty. \quad (9)$$

In the ensuing we shall assume that  $G_i > 0$ ;  $G_\infty > 0$  and that  $(G_i - G_\infty) \neq 0$ . For the examination of the axisymmetric crack problem we consider Hankel transform representations of the displacement components  $u_r$  and  $u_z$  in the following forms (see, e.g., Sneddon, 1951):

$$u_r(r, z) = \mathcal{H}_1[\xi^{-1} A(\xi)U(\xi, z); r], \quad (10)$$

$$u_z(r, z) = \mathcal{H}_0[\xi^{-1} A(\xi)W(\xi, z); r], \quad (11)$$

where  $A(\xi)$  is an arbitrary function. The Hankel operator of order  $\nu$  is given by

$$\mathcal{H}_\nu[f(\xi, z); r] = \int_0^\infty \xi f(\xi, z) J_\nu(\xi r) d\xi, \quad (12)$$

where  $J_\nu$ , is the  $\nu$ th order Bessel function of the first kind. The representations (10) and (11) can be used to reduce the displacement equations of equilibrium to the forms

$$\frac{d^2 U}{dz^2} + q(z) \frac{dU}{dz} - \frac{2(1-\nu)}{1-2\nu} \xi U - \frac{\xi}{1-2\nu} \frac{dW}{dz} - q(z) \xi W = 0, \quad (13)$$

$$\frac{d^2 W}{dz^2} + q(z) \frac{dW}{dz} - \frac{1-2\nu}{2(1-\nu)} \xi^2 W + \frac{\xi}{2(1-\nu)} \frac{dU}{dz} + q(z) \frac{\xi \nu}{1-\nu} U = 0, \quad (14)$$

where

$$q(z) = \frac{1}{G(z)} \frac{dG(z)}{dz}. \quad (15)$$

In order to formulate the mixed boundary value problem associated with the crack problem we require expressions similar to (10) and (11) for the stress components  $\sigma_{zz}$  and  $\sigma_{rz}$ . Considering (3), (10) and (11) we have

$$\sigma_{zz} = \mathcal{H}_0 \left[ \frac{2G(z)(1-\nu)}{1-2\nu} \left\{ \frac{dW}{dz} + \frac{\nu \xi U}{1-\nu} \right\} \xi^{-1} A(\xi); r \right], \quad (16)$$

$$\sigma_{rz} = \mathcal{H}_1 \left[ -G(z) \left\{ -\frac{dU}{dz} + \xi W \right\} \xi^{-1} A(\xi); r \right]. \quad (17)$$

Expressions for the remaining stress components can also be obtained in a similar manner.

### 3. The penny-shaped crack problem

We examine the problem of a penny-shaped crack which is located at the interface between identical halfspace regions in which the shear modulus varies axially according to the relationships (7). The elastic medium is subjected to a uniform axial stress  $\sigma_0$ . Due to the axial symmetry of the loading and the spatial symmetry of the elastic non-homogeneity about the plane  $z = 0$ , the crack problem can be formulated as a mixed boundary value problem related to the halfspace region, say,  $z \geq 0$ . It can be shown that the mixed boundary value problem governing the penny-shaped crack is equivalent to the following:

$$u_z(r, 0) = 0; \quad a \leq r < \infty, \quad (18)$$

$$\sigma_{zz}(r, 0) = -\sigma_0; \quad 0 < r < a, \quad (19)$$

$$\sigma_{rz}(r, 0) = 0; \quad 0 \leq r < \infty. \quad (20)$$

These boundary conditions in conjunction with the relationships (11), (16) and (17) lead to the following system of dual integral equations:

$$\mathcal{H}_0[W(\xi, 0) \xi^{-1} A(\xi); r] = 0; \quad a \leq r < \infty, \quad (21)$$

$$\mathcal{H}_0 \left[ \frac{2G_i(1-\nu)}{1-2\nu} R(\xi) A(\xi); r \right] = -\sigma_0; \quad 0 \leq r < a, \quad (22)$$

where

$$R(\xi) = \frac{1}{\xi} \left[ \frac{dW}{dz} + \frac{\nu}{1-\nu} \xi U \right]_{z=0}. \tag{23}$$

Introducing a new function  $D(\xi)$  such that

$$D(\xi) = W(\xi, 0)A(\xi) \tag{24}$$

and the finite Fourier transform for  $D(\xi)$  as

$$D(\xi) = -\frac{\sigma_0(1-2\nu)}{G_i\pi(1-\nu)} \int_0^a \psi(t) \sin(\xi t) dt \tag{25}$$

it can be shown that (21) is automatically satisfied and (22) can be reduced to a single Fredholm, integral equation of the second kind:

$$\psi(t) + \frac{2}{\pi} \int_0^a \psi(s) ds \int_0^\infty K(\xi) \sin(\xi s) \sin(\xi t) d\xi = t; \quad 0 < t < a, \tag{26}$$

where the kernel function is defined by

$$K(\xi) = \frac{R(\xi)}{W(\xi, 0)} - 1. \tag{27}$$

The solution of (26) can be used to evaluate the displacement and stress fields within the non-homogeneous elastic medium. A result of primary importance to fracture mechanics of the bonded interface concerns the stress intensity factor at the crack tip which is defined by

$$K_I^a = \lim_{r \rightarrow a^+} [2(r-a)]^{1/2} \sigma_{zz}(r, 0). \tag{28}$$

Using the result for  $\sigma_{zz}$  derived in terms of  $\psi(t)$ , the stress intensity factor (28) can be expressed in the form

$$K_I^a = \frac{2\sigma_0}{\pi} \frac{\psi(a)}{\sqrt{a}}. \tag{29}$$

#### 4. Numerical analysis and results

The form of the kernel function of the Fredholm integral equation (26) is such that it is not amenable to solution in an exact form. This is primarily due to the fact that the functions  $R(\xi)$  and  $W(\xi, 0)$  can only be obtained from an approximate solution of the reduced differential equations (13) and (14). The differential equations (13) and (14) are subject to boundary and regularity conditions

$$\frac{dU}{dz} - \xi W = 0; \quad z = 0, \tag{30}$$

$$U = W = 0; \quad z \rightarrow \infty, \tag{31}$$

$$U = 1; \quad z = 0. \tag{32}$$

The differential equations (13) and (14) subject to the boundary and regularity conditions (30)–(32) can be solved by a trial and error or multiple shooting technique (Fox, 1962). The regularity condition (31) can be

accommodated in the numerical scheme by adopting a special technique. The limit  $z \rightarrow \infty$  can be replaced by a finite limit  $z = L$ . If we consider the variation of  $G(z)$  for  $z \geq 0$  we can write

$$q(z) = \frac{1}{G(z)} \frac{dG(z)}{dz} = \frac{-\eta\zeta e^{-\zeta z}}{1 + \eta e^{-\zeta z}}, \quad (33)$$

where  $\eta = (G_i - G_\infty)/G_\infty$ . Since  $q(z) \rightarrow 0$  as  $z \rightarrow \infty$ ,  $U$  and  $W$  will have solutions of order  $e^{-\xi z}$ . As a result, we can choose  $L$ , such that

$$|e^{-\zeta L}| < \varepsilon; \quad |e^{-\xi L}| < \varepsilon \quad (34)$$

and for each  $\zeta$  and  $\xi$  and a permissible error  $\varepsilon$ ,

$$L = \max \left[ -\frac{\log \varepsilon}{\xi}, -\frac{\log \varepsilon}{\zeta} \right]. \quad (35)$$

It is also convenient, for the purposes of the numerical evaluations, to represent the kernel function  $K(\xi)$  in the following form:

$$K(\xi) = \frac{-2(1-\nu)^2}{1-2\nu} - 1 + \frac{B}{1+\xi^2} + \sum_{j=1}^N C_j e^{-\xi j}, \quad (36)$$

where  $B$  and  $C_j$  are constants which are determined via a least square technique, and the value of  $j$  can be assigned any integer order. Making use of (36) and the integrals

$$\int_0^\infty \sin(x\xi) \sin(xs) dx = \frac{\pi}{2} \delta(\xi - s), \quad (37)$$

$$\int_0^\infty \frac{\sin(x\xi) \sin(xs) dx}{1+x^2} = \frac{\pi}{4} [-e^{-(\xi+s)} + e^{-|\xi-s|}], \quad (38)$$

$$\int_0^\infty e^{-jx} \sin(x\xi) \sin(xs) dx = \frac{1}{2} \left[ \frac{-j}{j^2 + (\xi+s)^2} + \frac{j}{j^2 + (\xi-s)^2} \right], \quad (39)$$

where  $\delta(\xi - s)$  is the Dirac delta function, the integral equation (26) can be transformed into

$$(1 + A^*)\psi(t) + \int_0^a \psi(s)K^*(s, t) ds = t; \quad 0 < t < a, \quad (40)$$

where

$$A^* = - \left\{ 1 + \frac{2(1-\nu)^2}{1-2\nu} \right\} \quad (41)$$

and

$$K^*(s, t) = \frac{B}{2} \{-e^{-(t+s)} + e^{-|t-s|}\} + \frac{1}{\pi} \sum_{j=1}^N C_j \left[ \frac{-j}{j^2 + (t+s)^2} + \frac{j}{j^2 + (t-s)^2} \right]. \quad (42)$$

The solution of the transformed integral equation (40) can be achieved by employing standard techniques outlined by Baker (1977), and Delves and Mohamed (1985), which involves a matrix representation of the form

$$[A_{ij}]\{\psi(t_j)\} = \{\Omega_i\}, \quad (43)$$

where

$$A_{ij} = (1 + A^*)\delta_{ij} + K^*(t_i, t_j)h \tag{44}$$

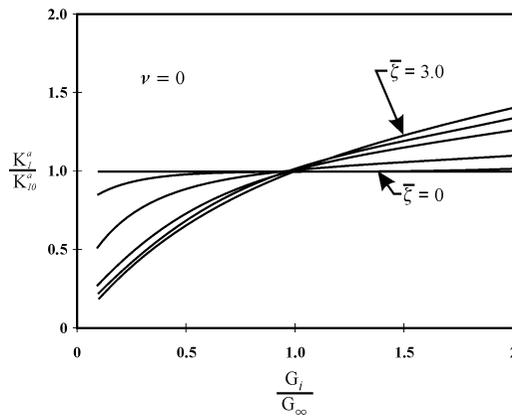
and for the interval  $[0, a]$ , with  $M$  segments,  $h = a/M$ ;  $t_i = (i - \frac{1}{2})h$  with  $i = 1, 2, \dots, M$ .

### 5. Numerical results

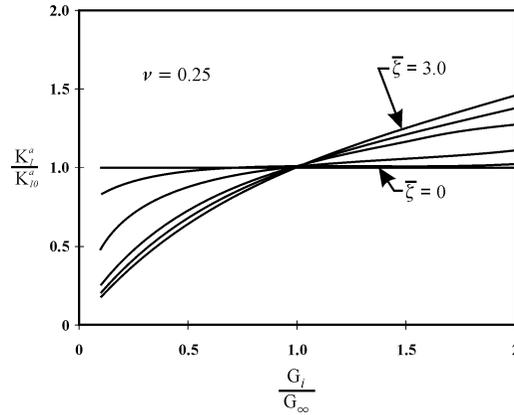
The numerical procedure outlined in the previous section was used to develop results for the stress intensity factor at the crack tip. The choice of the discretization integer  $N$  used in the representation (36) for the kernel function needs to be specified by comparing the accuracy of numerical estimates for increasing values of the integer. It is observed that  $N = 4$  gives results which are sufficiently accurate for the graphical presentation of the results. A further parameter which governs the accuracy of the solution is the value  $L$  which prescribes in the numerical scheme, the numerical limit as  $z \rightarrow \infty$ . It is found that the solution converges when  $L \geq 40a$ . The localized non-homogeneity associated with two bonded regions can be characterized by two non-dimensional parameters. The parameter  $G_i/G_\infty$  which describes either a bonded interface which is stiffer than the parent material ( $G_i > G_\infty$ ) or a bonded interface which has experienced softening ( $G_i < G_\infty$ ). The parameter  $\bar{\zeta}$  ( $= \zeta a$ ) describes the axial variation (either growth or decay) of the shear modulus from the interface. In order to present the numerical results it is also convenient to normalize them in relation to the stress intensity factor for a penny-shaped crack which is located in a homogeneous elastic medium which is given by

$$K_{I0}^a = \frac{2\sigma_0\sqrt{a}}{\pi}. \tag{45}$$

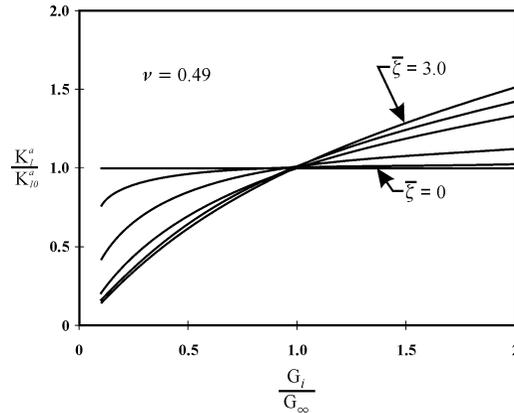
The stress intensity factor for the penny-shaped crack can be obtained from the solution of the integral equation (40) and the result (29). *Figures 2–4* illustrate the variation of the stress intensity factor at the crack tip due to variations in the localized elastic non-homogeneity. In the particular instance when  $\bar{\zeta}$  is large, the stress intensity factor  $K_I^a$  can be obtained by an application of the  $J$ -integral. This limiting case has similarities to the problem of crack path selection in brittle adhesives presented by Fleck et al. (1991). As  $\zeta a \rightarrow \infty$ , the stress field at the vicinity of the crack tip corresponds to a state of plane strain. Considering a plane normal to the crack front, let  $r$  be the distance from the crack tip  $\zeta r \ll 1$ , the stress field scales with the stress intensity



**Figure 2.** Penny-shaped crack at a bounded interface: effects of the localized non-homogeneity on the non-dimensional stress intensity factor  $K_I^a/K_{I0}^a$  for  $\nu = 0$  and for values of  $\bar{\zeta} = (0, 0.1, 1.0, 2.0, \text{ and } 3.0)$ .



**Figure 3.** Penny-shaped crack at a bounded interface: effects of the localized non-homogeneity in the shear modulus on the non-dimensional stress intensity factor  $K_I^a/K_{I0}^a$  for  $\nu = 0.25$  and for values of  $\bar{\zeta} = (0, 0.01, 0.1, 1.0, 2.0, \text{ and } 3.0)$ .



**Figure 4.** Penny-shaped crack at a bounded interface: effects of the localized non-homogeneity in the shear modulus on the non-dimensional stress intensity factor  $K_I^a/K_{I0}^a$  for  $\nu = 0.49$  and for values of  $\bar{\zeta} = (0, 0.01, 0.1, 1.0, 2.0, \text{ and } 3.0)$ .

factor  $K_I^a$  which is to be determined. At distances remote from the crack tip ( $1 \ll \zeta r \ll \zeta a$ ) the stress field is the same as that for a penny-shaped crack in a homogeneous material with the stress intensity factor defined by (45). The two stress intensity factors are connected by the conservation  $J$ -integral

$$\frac{(1 - \nu_i)(K_I^a)^2}{G_i} = \frac{(1 - \nu_\infty)(K_{I0}^a)^2}{G_\infty}. \quad (46)$$

When the two Poisson's ratios are identical, (46) gives

$$\frac{K_I^a}{K_{I0}^a} = \left( \frac{G_i}{G_\infty} \right)^{1/2}. \quad (47)$$

This result can be regarded as a limiting result for a bonded interface with a highly localized alteration in the elastic modulus at the bonded plane. The results corresponding to (47) closely resemble the numerical results given in figures 2–4 corresponding to  $\zeta a = 3$ .

## 6. Concluding remarks

The study of the mechanics of bonded surfaces invariably require the evaluation of their resistance to fracture. In previous studies involving the modelling of bonded regions, the junction surface has been modelled either as a distinct zone of differing but constant elastic properties or as a distinct zone with variable elastic properties. The non-homogeneities have largely been restricted to axial variations in the linear elastic shear modulus while the Poisson's ratio is assumed to be constant. In this study an alternative model for the mechanics of a penny-shaped crack at the bonded plane between identical elastic half-spaces is considered. The elastic non-homogeneity allows for the gradual alteration in the elastic characteristics of the halfspace regions, ranging from, softened behaviour ( $G_i < G_\infty$ ) to stiffer behaviour ( $G_i > G_\infty$ ). Both variations are plausible situations which can address alterations in the elasticity parameters which can be induced by long term chemical reaction effects and, most notably, the influence of adhesive migration due to suction and diffusion phenomena in porous elastic materials. In these situations the elastic properties are expected to vary gradually with distance from the bonded plane. The exponential form of the shear modulus has sufficient flexibility to represent near surface variations and to maintain the spatial variations which are bounded within the infinite medium. The results derived from the analysis of the penny-shaped crack problem indicate that in the limit as  $G_i/G_\infty \rightarrow 1$  or when  $\bar{\zeta} \rightarrow 0$ , the result reduces to the classical result for a penny-shaped crack which is located in a homogeneous elastic solid. As the elastic modulus in the vicinity of the bonded plane increases, and provided  $\bar{\zeta} > 0$ , the stress intensity factor is amplified. Similarly, as the modulus at the bonded plane region decreases in relation to the far field elastic modulus, the stress intensity factor is attenuated for all choices of  $\bar{\zeta} > 0$ . In the limiting case when the non-homogeneity is highly localized to the vicinity of the bonded plane, the stress intensity factor can be estimated from the conservation  $J$ -integral approach which indicates that the normalized stress intensity factor varies as  $\sqrt{G_i/G_\infty}$ .

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