Boussinesq's Problem for an Elastic Half-Space Reinforced with a Rigid Disk Inclusion

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Abstract: Boussinesq's classical solution for the problem of the normal loading of the surface of a homogeneous elastic half-space is extended to cover the case in which the half-space is reinforced with a fully bonded rigid inclusion of finite radius. The problem is reduced to the solution of two coupled Fredholm integral equations of the second kind. Numerical results presented in the paper illustrate the manner in which the displacement of the rigid disk inclusion is influenced by the depth of the embedment/inclusion radius ratio.

1. INTRODUCTION

The determination of the state of stress in an isotropic, homogeneous elastic half-space that is subjected to a concentrated force normal to an otherwise traction-free surface was first presented by Boussinesq [1], in which the problem is reduced to a boundary value problem in potential theory. Since the problem deals with normal loading of the half-space, the elasticity problem is reduced to that of finding a single harmonic function with the characteristics of a single layer distributed over the plane region and possessing intensity proportion to the applied normal loading. Boussinesq's classical problem was generalized by Michell [2] (see also Love [3]) to include the case of axial loading of a cone at its apex and by Mindlin [4] to include the case in which the concentrated load was located within the half-space region. Other generalizations of this problem to include tangential loading of a half-space by a concentrated force were presented by Cerruti [5]. References to further generalizations of Boussinesq-type problems involving bimaterial elastic regions are given by Mura [6]. The development of many of these solutions is facilitated by the fact that the elastic region under consideration is free of defects such as cracks and inclusions. The problem of the concentrated loading of a half-space region containing low concentrations of cracks and inclusions is of some interest to the study of multiphase composites (Figure 1). The study of such problems can be best attempted through computational schemes that can accommodate the random character of the distribution of such defects in relation to the scale of dimensions of the loaded region. In this paper, we discuss a highly idealized problem of the surface loading of an elastic half-space region containing a bonded rigid disk inclusion. The problem is further simplified by restricting attention to the axisymmetric problem involving a disk-shaped rigid inclusion that is embedded at a finite depth beneath the surface of a half-space that is subjected to a concentrated force (Figure 2).

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The analysis of the axisymmetric problem related to a rigid disk inclusion embedded in an elastic half-space, the surface of which is subjected to a concentrated normal force P, can be reduced to the solution of a pair of coupled Fredholm integral equations of the second kind. These are solved using a numerical technique to evaluate the influence of the depth of embedment on the rigid body displacement of the embedded inclusion. These results also form the auxiliary result, which enables the calculation, via Betti's reciprocal theorem, of the surface displacement of the half-space region due to axial displacement of the bonded rigid disk inclusion.

2. THE EMBEDDED INCLUSION PROBLEM

Following Green and Zerna [7], the solution to the axisymmetric problem in classical elasticity, referred to the cylindrical polar coordinate system, can be expressed in terms of two harmonic functions $\phi(r,z)$ and $\psi(r,z)$, which satisfy

$$\nabla^2\phi = 0 \quad ; \quad \nabla^2\psi = 0, \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is Laplace's operator, referred to the axisymmetric cylindrical polar coordinate system. The displacements and stresses, referred to the cylindrical polar coordinate system, can be expressed in terms of $\phi(r,z)$ and $\psi(r,z)$ as follows:

$$u_r = \frac{\partial\phi}{\partial r} - z\frac{\partial\psi}{\partial r}, \tag{3}$$

$$u_z = (3-4\nu)\psi + \frac{\partial\phi}{\partial z} - z\frac{\partial\psi}{\partial z}, \tag{4}$$

and

$$\sigma_{rr} = 2G\left[\frac{\partial^2\phi}{\partial r^2} + 2\nu\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial r^2}\right], \tag{5}$$

$$\sigma_{\theta\theta} = 2G\left[\frac{1}{r}\frac{\partial\phi}{\partial r} + 2\nu\frac{\partial\psi}{\partial z} - \frac{z}{r}\frac{\partial\psi}{\partial r}\right], \tag{6}$$

$$\sigma_{zz} = 2G\left[2(1-\nu)\frac{\partial\psi}{\partial z} + \frac{\partial^2\phi}{\partial z^2} - z\frac{\partial^2\psi}{\partial z^2}\right], \tag{7}$$

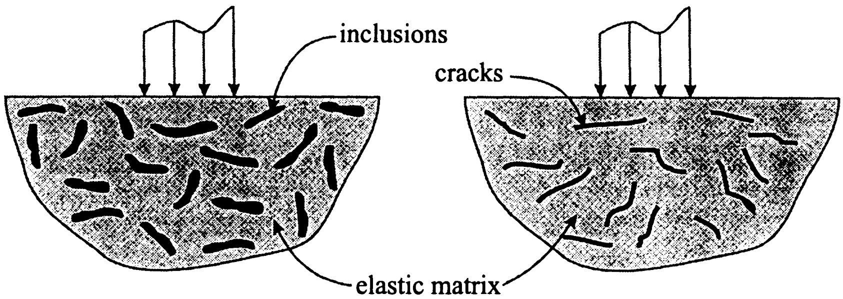

Fig. 1. Half-space region containing low concentrations of inclusions or cracks.

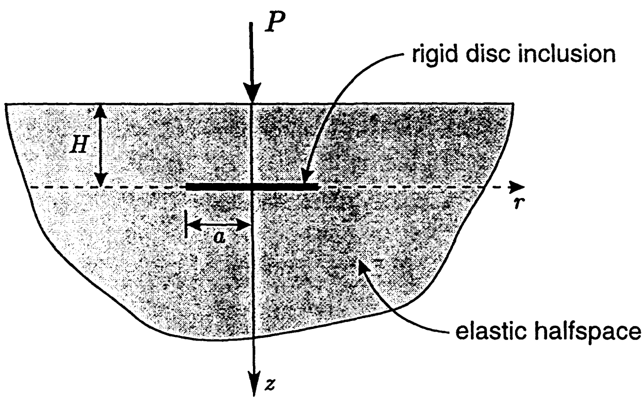

Fig. 2. Boussinesq's problem for a half-space containing a bonded disk inclusion.

$$\sigma_{rz} = 2G\left[(1-2\nu)\frac{\partial\psi}{\partial r} + \frac{\partial^2\phi}{\partial r\partial z} - z\frac{\partial^2\psi}{\partial r\partial z}\right], \tag{8}$$

where G is the linear elastic shear modulus and ν is Poisson's ratio and, owing the axial symmetry, all other displacement and stress components are zero. The boundary value problem associated with the interaction of the concentrated Boussinesq force and the embedded rigid disk inclusion can be posed in several ways. In the ensuing development, however, we commence with the solution to the homogeneous elastic half-space *without* the embedded rigid disk, where the point force P is applied at the origin (0,0). The displacements at plane corresponding to the location of the rigid disk inclusion are given by

$$u_r^0(r,H) = \frac{P}{4\pi G}\left[\frac{rH}{(r^2+H^2)^{\frac{3}{2}}} - \frac{r(1-2\nu)}{[(r^2+H^2)+H\sqrt{r^2+H^2}]}\right] = m(r), \tag{9}$$

$$u_z^0(r,H) = \frac{P}{4\pi G}\left[\frac{2(1-\nu)}{(r^2+H^2)^{\frac{1}{2}}} + \frac{H^2}{(r^2+H^2)^{\frac{3}{2}}}\right] = \ell(r). \tag{10}$$

For the formulation of the boundary value problem, it is convenient to select the origin at the center of the disk inclusion. We shall designate displacements and stresses in the "layer" region above the plane of the disk inclusion for which $r\epsilon\ (0,\infty)$; $z\epsilon\ (0,-H)$ by the superscript $^{(2)}$ and for the region below the plane of the disk inclusion for which $r\epsilon\ (0,\infty)$; $z\epsilon\ (0,\infty)$ by the superscript $^{(1)}$. The "corrective" boundary value problem, which needs to be solved, will be added to the solution for Boussinesq's problem (with an appropriate shift in the coordinate system) for the half-space without an embedded rigid disk inclusion. The boundary value problem corresponding to the corrective solution should satisfy the boundary, continuity, and regularity conditions as follows:

$$\sigma_{zz}^{(2)}(r,z) = 0 \quad ; \quad z=-H \quad ; \quad r>0, \tag{11}$$

$$\sigma_{rz}^{(2)}(r,z) = 0 \quad ; \quad z=-H \quad ; \quad r>0, \tag{12}$$

$$u_r^{(2)}(r,z) = -m(r) \quad ; \quad z=0 \quad ; \quad 0<r\le a, \tag{13}$$

$$u_r^{(1)}(r,z) = -m(r) \quad ; \quad z=0 \quad ; \quad 0<r\le a, \tag{14}$$

$$u_z^{(1)}(r,z) = \Delta-\ell(r) \quad ; \quad z=0 \quad ; \quad 0\le r\le a, \tag{15}$$

$$u_z^{(2)}(r,z) = \Delta-\ell(r) \quad ; \quad z=0 \quad ; \quad 0\le r\le a, \tag{16}$$

$$u_z^{(1)}(r,z) = u_2^{(2)}(r,z) \quad ; \quad z=0 \quad ; \quad a\le r<\infty, \tag{17}$$

$$u_r^{(1)}(r,z) = u_r^{(2)}(r,z) \quad ; \quad z=0 \quad ; \quad a\le r<\infty, \tag{18}$$

$$\sigma_{zz}^{(1)}(r,z) = \sigma_{zz}^{(2)}(r,z) \quad ; \quad z=0 \quad ; \quad a<r<\infty, \tag{19}$$

$$\sigma_{rz}^{(1)}(r,z) = \sigma_{rz}^{(2)}(r,z) \quad ; \quad z=0 \quad ; \quad a<r<\infty, \tag{20}$$

where Δ is an unknown rigid displacement of the embedded disk inclusion. This unknown displacement is determined by the constraint that the net resultant force of tractions acting on the surfaces of the rigid inclusion is zero—that is,

$$\int_0^{2\pi}\int_0^a \{\sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0)\} r dr d\theta = 0. \tag{21}$$

In addition, the stresses and displacements in the half-space region should decay to zero as $r, z \to \infty$.

By considering a Hankel transform development of the governing equations, we can show that for the half-space region$^{(1)}$, the relevant displacements and stress components take the forms

$$u_r^{(1)}(r,z) = \int_0^\infty [A(\xi) + \xi z B(\xi)] e^{-\xi z} J_1(\xi r) d\xi, \tag{22}$$

$$u_z^{(1)}(r,z) = \int_0^\infty [A(\xi) + (3-4\nu)B(\xi) + \xi z B(\xi)] e^{-\xi z} J_0(\xi r) d\xi, \tag{23}$$

and

$$\sigma_{zz}^{(1)}(r,z) = -2\mu \int_0^\infty \xi [A(\xi) + 2(1-\nu)B(\xi) + \xi z B(\xi)] e^{-\xi z} J_0(\xi r) d\xi, \tag{24}$$

$$\sigma_{rz}^{(1)}(r,0) = -2\mu \int_0^\infty \xi [A(\xi) + (1-2\nu)B(\xi) + \xi z B(\xi)] e^{-\xi z} J_1(\xi r) d\xi, \tag{25}$$

respectively, where $A(\xi)$ and $B(\xi)$ are arbitrary functions. Similarly, for the layer region $^{(2)}$, the relevant displacement and stress components take the forms

$$\begin{aligned} u_r^{(2)}(r,z) &= -\int_0^\infty [\sinh(\xi[z+H])\{C(\xi) - z\xi E(\xi)\} \\ &+ \cosh(\xi[z+H])\{D(\xi) - z\xi F(\xi)\}] \frac{J_1(\xi r) d\xi}{\sinh(\xi H)}, \end{aligned} \tag{26}$$

$$\begin{aligned} u_z^{(2)}(r,z) &= \int_0^\infty [(3-4\nu)\{E(\xi)\sinh(\xi[z+H]) + F(\xi)\cosh(\xi[z+H])\} \\ &+ C(\xi)\cosh(\xi[z+H]) + D(\xi)\sinh(\xi[z+H]) \\ &- \xi z\{E(\xi)\cosh(\xi[z+H]) + F(\xi)\sinh(\xi[z+H])\}] \\ &\times \frac{J_0(\xi r)d\xi}{\sinh(\xi H)}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} \frac{\sigma_{zz}^{(2)}(r,z)}{2G} &= \int_0^\infty [2(1-\nu)\{E(\xi)\cosh(\xi[z+H]) + F(\xi)\sinh(\xi[z+H])\}\xi \\ &+ \xi\{C(\xi)\sinh(\xi[z+H]) + D(\xi)\cosh(\xi[z+H])\} \\ &- \xi^2 z\{E(\xi)\sinh(\xi[z+H]) + F(\xi)\cosh(\xi[z+H])\}] \\ &\times \frac{J_0(\xi r)d\xi}{\sinh(\xi H)}, \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{\sigma_{rz}^{(2)}(r,z)}{2G} &= -\int_0^\infty [(1-2\nu)\{E(\xi)\sinh(\xi[z+H]) + F(\xi)\cosh(\xi[z+H])\} \\ &+ C(\xi)\cosh(\xi[z+H]) + D(\xi)\sinh(\xi[z+H]) \\ &- \xi z\{E(\xi)\cosh(\xi[z+H]) + F(\xi)\sinh(\xi[z+H))\}] \\ &\times \frac{\xi J_1(\xi r)d\xi}{\sinh(\xi H)}, \end{aligned} \tag{29}$$

where $C(\xi), D(\xi)$, and so forth are arbitrary functions. Combining boundary conditions (13) to (18), we note that

$$u_r^{(1)}(r,0) = u_r^{(2)}(r,0) \quad ; \quad 0 < r < \infty, \tag{30}$$

$$u_z^{(1)}(r,0) = u_z^{(2)}(r,0) \quad ; \quad 0 < r < \infty. \tag{31}$$

Using the conditions (29) and (30) and the corresponding integral expressions for $u_r^{(1)}, u_z^{(1)}, u_r^{(2)}$, and $u_z^{(2)}$, given by (21), (22), (25), and (26), we obtain

$$\begin{aligned} A(\xi) &= C(\xi)\,[\xi H \coth(\xi H) + 1 - 2\nu\,] \\ &- D(\xi)\,[\xi H + 2(1-\nu\,)\coth(\xi H)]\,, \end{aligned} \tag{32}$$

$$\begin{aligned} B(\xi) &= \frac{1}{(3-4\nu\,)}\,[C(\xi)\{\coth(\xi H)[2(1-\nu\,) - \xi H] - \xi H - (1-2\nu\,)\} \\ &+ D(\xi)\{\xi H\,[1+\coth(\xi H)] - (1-2\nu\,) + 2(1-\nu\,)\coth(\xi H)\}]\,. \end{aligned} \tag{33}$$

Using these expressions, we can obtain

$$\begin{aligned} \sigma_{zz}^{(1)}(r,0) - \sigma_{zz}^{(2)}(r,0) &= \frac{4G(1-\nu\,)}{(3-4\nu\,)}\int_0^\infty \xi[1+\coth(\xi H)] \\ &\times \{(2\nu - 1 + \xi H)D(\xi) \\ &+ [2(1-\nu\,) - \xi H]C(\xi)\}J_0(\xi r)d\xi, \end{aligned} \tag{34}$$

$$\begin{aligned} \sigma_{rz}^{(1)}(r,0) - \sigma_{rz}^{(2)}(r,0) &= \frac{4G(1-\nu\,)}{(3-4\nu\,)}\int_0^\infty \xi[1+\coth(\xi H)] \\ &\times \{(\xi H + 1 - 2\nu\,)C(\xi) \\ &- D(\xi)[\xi H + 2(1-\nu\,)]\}J_1(\xi r)d\xi. \end{aligned} \tag{35}$$

The reductions (32) to (35) imply that the remaining boundary conditions can be formulated in terms of two arbitrary functions such as $C(\xi)$ and $D(\xi)$ or a suitable combination of these functions. The remaining boundary conditions now give

$$\begin{aligned} &\int_0^\infty \{C(\xi)[\xi H \coth(\xi H) + (1-2\nu\,)] \\ &- D(\xi)[2(1-\nu\,)\coth(\xi H) + \xi H]\}J_1(\xi r)d\xi \\ &= -m(r) \quad ; \quad 0 < r < a, \end{aligned} \tag{36}$$

$$\int_0^\infty \{C(\xi)[2(1-\nu)\coth(\xi H) - \xi H]$$

$$+ \quad D(\xi)[\xi H \coth(\xi H) - (1-2\nu)]\} J_0(\xi r) d\xi$$

$$= \quad \Delta - \ell(r) \quad ; \quad 0 < r < a, \tag{37}$$

$$\int_0^\infty \xi[1 + \coth(\xi H)]$$

$$\times \quad [C(\xi)(\xi H + 1 - 2\nu) - D(\xi)(\xi H + 2 - 2\nu)] J_1(\xi r) d\xi$$

$$= \quad 0 \quad ; \quad a < r < \infty, \tag{38}$$

$$\int_0^\infty \xi[1 + \coth(\xi H)]$$

$$\times \quad [D(\xi)(\xi H - 1 + 2\nu) + C(\xi)(-\xi H + 2 - 2\nu)] J_0(\xi r) d\xi$$

$$= \quad 0 \quad ; \quad a < r < \infty. \tag{39}$$

Introducing the substitutions

$$\begin{aligned} R_1(\xi) &= C(\xi)\{\xi H + 1 - 2\nu\} - D(\xi)\{\xi H + 2[1-\nu]\}, \\ R_2(\xi) &= C(\xi)\{-\xi H + 2[1-\nu]\} + D(\xi)\{\xi H - 1 + 2\nu\}, \end{aligned} \tag{40}$$

and their finite Fourier transforms in terms of functions $\Phi(t)$ and $\Psi(t)$ such that

$$[1 + \coth(\xi H)] R_1(\xi) = \xi \int_0^a \Phi(t) \cos(\xi t) dt, \tag{41}$$

$$[1 + \coth(\xi H)] R_2(\xi) = \int_0^a \Psi(t) \cos(\xi t) dt, \tag{42}$$

we note that equations (38) and (39) will be identically satisfied and equations (36) and (37) are equivalent to the coupled system of integral equations

$$\Phi(t) + \int_0^a \Phi(u) K_{11}(u,t) du + \int_0^a \Psi(u) K_{12}(u,t) du$$

$$= \frac{4t}{\pi}\int_0^t \frac{m(r)dr}{(t^2-r^2)^{\frac{1}{2}}} \quad ; \quad 0 < t < a, \tag{43}$$

$$\Psi(t) + \int_0^a \Psi(u)K_{22}(u,t)du + \int_0^a \Phi(u)K_{21}(u,t)du$$

$$= \frac{4}{\pi}\left[-\Delta + \frac{d}{dt}\int_0^t \frac{r\ell(r)dr}{(t^2-r^2)^{\frac{1}{2}}}\right] \quad ; \quad 0 < t < a, \tag{44}$$

where $K_{\alpha\beta}(u,t)(\alpha,\beta = 1,2)$ are kernel functions of the integral equations and are given by

$$\begin{aligned} K_{11}(u,t) &= -\frac{4t}{(3-4\nu)\pi}\int_0^\infty \frac{\cos(\xi u)[1-\cos(\xi t)]}{[1+\coth(\xi H)]}\left\{-\frac{(3-4\nu)}{2}[1+\coth(\xi H)]\right. \\ &+ (\xi H - 1 + 2\nu)[\xi H\coth(\xi H) + 1 - 2\nu] \\ &+ \left.(-\xi H + 2 - 2\nu)[2(1-\nu)\coth(\xi H) + \xi H]\right\} d\xi, \end{aligned} \tag{45}$$

$$\begin{aligned} K_{12}(u,t) &= \frac{4t}{(3-4\nu)\pi}\int_0^\infty \frac{\cos(\xi u)[1-\cos(\xi t)]}{\xi[1+\coth(\xi H)]} \\ &\times \{(\xi H + 1 - 2\nu)[\xi H + 2(1-\nu)\coth(\xi H)] \\ &- (\xi H + 2 - 2\nu)[\xi H\coth(\xi H) + 1 - 2\nu]\} d\xi, \end{aligned} \tag{46}$$

$$\begin{aligned} K_{22}(u,t) &= \frac{4}{\pi(3-4\nu)}\int_0^\infty \frac{\xi\cos(\xi u)\cos(\xi t)}{[1+\coth(\xi H)]}\left\{-\frac{(3-4\nu)}{2}[1+\coth(\xi H)]\right. \\ &+ (\xi H + 2 - 2\nu)[-\xi H + 2(1-\nu)\coth(\xi H)] \\ &+ \left.(\xi H + 1 - 2\nu)[\xi H\coth(\xi H) - 1 + 2\nu]\right\} d\xi, \end{aligned} \tag{47}$$

$$\begin{aligned} K_{21}(u,t) &= \frac{-4}{\pi(3-4\nu)}\int_0^\infty \frac{\cos(\xi u)\cos(\xi t)}{[1+\coth(\xi H)]} \\ &\times \{(-\xi H + 2 - 2\nu)[\xi H\coth(\xi H) - 1 + 2\nu] \end{aligned}$$

$$- \quad (\xi H - 1 + 2\nu)[2(1-\nu)\coth(\xi H) - \xi H]\} \, d\xi . \tag{48}$$

Furthermore, using equations (34) and (42), it can be shown that the constraint (21) is equivalent to

$$\int_0^a \Psi(t)dt = 0. \tag{49}$$

This result is used to determine the unknown rigid displacement Δ of the disk inclusion in the axial direction.

3. THE NUMERICAL SOLUTION OF THE GOVERNING INTEGRAL EQUATIONS

The form of the system of coupled integral equations (43) and (44), particularly the forms of the kernel functions $K_{\alpha\beta}(\alpha, \beta = 1, 2)$, is such that it is unlikely that there would be a solution in an exact closed form. As a result, it is necessary to adopt a numerical scheme to generate results of engineering importance. A number of procedures for the solution of coupled systems of integral equations are documented in the literature (Atkinson [8]; Baker [9]; Delves and Mohamed [10]). Considering the expression for $m(r)$ given by (19), we can express the right-hand side of (43) in the form

$$\begin{aligned} F_\Phi(t) &= \frac{4t}{\pi} \int_0^\infty \frac{m(r)dr}{(t^2 - r^2)^{\frac{1}{2}}} \\ &= \frac{P}{G\pi^2} \left[\frac{t^2}{(t^2 + H^2)} - (1 - 2\nu)\ell n \left| \frac{t^2 + (\sqrt{t^2 + H^2} - H)^2}{t^2 - (\sqrt{t^2 + H^2} - H)^2} \right| \right] . \end{aligned} \tag{50}$$

Similarly, using (10), we can express the integral on the right-hand side of (44) in the form

$$F_\Psi(t) = \frac{4}{\pi} \frac{d}{dt} \left[\int_0^t \frac{r\ell(r)dr}{(t^2 - r^2)^{\frac{1}{2}}} \right] = \frac{PH}{G\pi^2(t^2 + H^2)} \left[\frac{(H^2 - t^2)}{(H^2 + t^2)} + 2(1 - \nu) \right] . \tag{51}$$

We can rewrite the coupled system of integral equations (43) and (44) in the form

$$\begin{bmatrix} \Phi(t) \\ \Psi(t) \end{bmatrix} + \int_0^a \begin{bmatrix} K_{11}(u,t) & K_{12}(u,t) \\ K_{21}(u,t) & K_{22}(u,t) \end{bmatrix} \begin{bmatrix} \Phi(u) \\ \Psi(u) \end{bmatrix} du + \begin{bmatrix} 0 \\ \dfrac{4\Delta}{\pi} \end{bmatrix} = \begin{bmatrix} F_\Phi \\ F_\Psi \end{bmatrix} . \tag{52}$$

The additional equation for Δ is given by (49). In the procedure adopted for the numerical evaluation of (52), the interval $[0, a]$ is divided into N segments, where

$$u_i = \frac{(i-1)a}{N} \quad ; \quad (i = 1, 2, \ldots .N + 1), \tag{53}$$

and

$$t_i = \left(\frac{u_i + u_{i+1}}{2}\right) \quad ; \quad (i = 1, 2, \ldots .N). \tag{54}$$

The coupled system of integral equations and the constraint (49) can be expressed as a matrix equation

$$[\mathbf{A}]\{\mathbf{X}\} = \{\mathbf{B}\} \tag{55}$$

of order $(2N + 1)$. The coefficients of the matrix $[\mathbf{A}]$ are given by

$$\begin{aligned} A_{2n-1,2m-1} &= \delta_{nm} - \frac{4t_n a}{(3-4\nu)\pi N} \int_0^\infty e^{-2\xi H} \Big[(\xi H)^2 - (3-4\nu)\xi H \\ &+ 4(1-\nu)^2 - \frac{(3-4\nu)}{2}\Big] \cos(\xi t_m)[1-\cos(\xi t_n)]d\xi, \end{aligned} \tag{56}$$

$$\begin{aligned} A_{2n,2m-1} &= \frac{4a}{(3-4\nu)\pi N} \int_0^\infty e^{-2\xi H} \\ &\times [(\xi H)^2 - 2(1-\nu)(1-2\nu)] \cos(\xi t_n)\cos(\xi t_m)d\xi, \end{aligned} \tag{57}$$

$$\begin{aligned} A_{2n-1,2m} &= -\frac{4t_n a}{(3-4\nu)\pi N} \int_0^\infty \frac{e^{-2\xi H}}{\xi} \\ &\times [(\xi H)^2 - 2(1-\nu)(1-2\nu)] \cos(\xi t_m)[1-\cos(\xi t_n)]d\xi, \end{aligned} \tag{58}$$

$$\begin{aligned} A_{2n,2m} &= \delta_{nm} + \frac{4a}{(3-4\nu)\pi N} \int_0^a e^{-2\xi H} \Big[(\xi H)^2 + (3-4\nu)\xi H \\ &+ 4(1-\nu)^2 - \frac{(3-4\nu)}{2}\Big] \cos(\xi t_n)\cos(\xi t_m)\xi d\xi, \end{aligned} \tag{59}$$

$$
\begin{aligned}
A_{2N+1,2n} &= A_{2n,2N+1} = \frac{4}{\pi}, \\
A_{2N+1,2n-1} &= A_{2n-1,2N+1} = 0,
\end{aligned}
\tag{60}
$$

with $n, m = 1, 2, 3....N$. The vector on the right-hand side of (55) is given by the elements

$$
B_{2n-1} = F_{\Phi}(t_n) \quad ; \quad B_{2n} = F_{\Psi}(t_n). \tag{61}
$$

The unknowns are

$$
X_{2n-1} = \Phi(t_n) \quad ; \quad X_{2n} = \Psi(t_n) \quad ; \quad X_{2N+1} = \Delta, \tag{62}
$$

with $n = 1, 2, ...N$.

It may be noted that a number of other schemes can be adopted for the numerical solution of a coupled system of integral equations. These can include orthogonal polynomial expansions in which the solution of the system of integral equations is essentially reduced to the determination of the constants characterizing the series expansion for $\Phi(t)$ and $\Psi(t)$. The discretization method adopted here is largely due to the availability of a routine that provides reasonably accurate solutions for the unknown functions. The accuracy of this scheme has been verified by calibration with known results for stress fields with nonoscillatory singular fields. From the point of view of the results of primary interest to this paper, the finite difference scheme yields a very accurate result for load displacement-type results, which can be determined from (49).

4. NUMERICAL RESULTS

Before presentation of the numerical results, it is useful to examine certain limiting cases in which the depth of location of the rigid disk inclusion is varied. As $(H/a) \to \infty$, the interaction between the concentrated force and the inclusion disappears, and the solution to the problem basically reduces to Boussinesq's classical solution. The assessment of this aspect will be considered in the numerical results that will be presented in this section. As the embedment ratio $(H/a) \to 0$, the problem reduces to that of a rigid circular punch in adhesive contact with an isotropic elastic half-space. The exact closed-form solution to the displacement of the bonded circular punch was presented by Ufliand [11], and further expositions of the method of solution are given by Mossakovskii [12] and Gladwell [13]. The result of particular interest to the numerical evaluations considered in the previous section is the displacement of the bonded surface punch, which is given by

$$\Delta_0 = \frac{P(1-2\nu)}{4Ga\,\ell n(3-4\nu)}. \tag{63}$$

It should be noted that the development of the above exact closed-form solution of the bonded circular punch on an elastic half-space is based on the Hilbert problem, which also takes into account the oscillatory form of the stress state (see also Willis [14]; Keer [15]; England [16]) at the boundary of the bonded punch. In the developments presented in this paper, the conditions for bonded displacements are correctly accounted for in the boundary conditions as $(H/a) \to 0$; the stress singularity at the boundary, however, has a regular $1/\sqrt{r}$ form. Selvadurai [17, 18] has examined the degree of error involved in the omission of the oscillatory nature of the state of stress, particularly on the load-displacement relationship for the bonded punch. This is achieved by reformulating the bonded punch, which incorporates the regular stress singularity, which results in the solution of a single Fredholm integral equation of the second kind. It is found that when $\nu = \frac{1}{2}$, the results from the two solution schemes (i.e., the Hilbert problem approach and the Fredholm integral equation formulation) give the same result. When $\nu = 0$, the maximum discrepancy between the two solution schemes is 0.0546%. It is therefore clear that within the scope of a solution scheme that involves numerical computations for the evaluation of the load-displacement relationships, the results from the two solution schemes involve no significant discrepancies for $\nu \in \left(0, \frac{1}{2}\right)$. The numerical solutions were normalized with respect to the exact closed-form solution for the load displacement behavior for the bonded surface punch (63). Figure 3 illustrates the results derived for the displacement of the rigid disk inclusion induced by the concentrated surface load P. It is evident that as H/a increases, the displacements of the rigid disk inclusion decrease, ultimately reducing to zero. For very large values of H/a, an approximate result for the displacement of the embedded rigid disk inclusion can be obtained by considering the solution for $\sigma_{zz}(r,H)$ due to Boussinesq's problem for a homogeneous elastic half-space and the problem for a rigid disk inclusion embedded in an elastic infinite space. The axial stress at a depth H due to a Boussinesq-type surface load is given by

$$\sigma_{zz}(r,H) = \frac{3PH^3}{2\pi(r^2+H^2)^{\frac{5}{2}}}, \tag{64}$$

where H is measured from the plane of application of P. Using the result for the displacement of a rigid disk inclusion embedded in an infinite space (Collins [19]; Kassir and Sih [20]; Selvadurai [21]) and Betti's reciprocal theorem, it can be shown that for a stress distribution of the type (64) acting on the plane of the inclusion, the displacement of the inclusion is given by

$$\Delta = \frac{P(3-4\nu)}{32Ga(1-\nu)}\left[1 - \frac{1}{[1+(a/H)^2]^{\frac{3}{2}}} + \frac{6H^3}{\pi}\int_a^\infty \frac{r\sin^{-1}(a/r)dr}{(r^2+H^2)^{\frac{5}{2}}}\right]. \tag{65}$$

The surface displacement of the inclusion-reinforced half-space can be written in the form

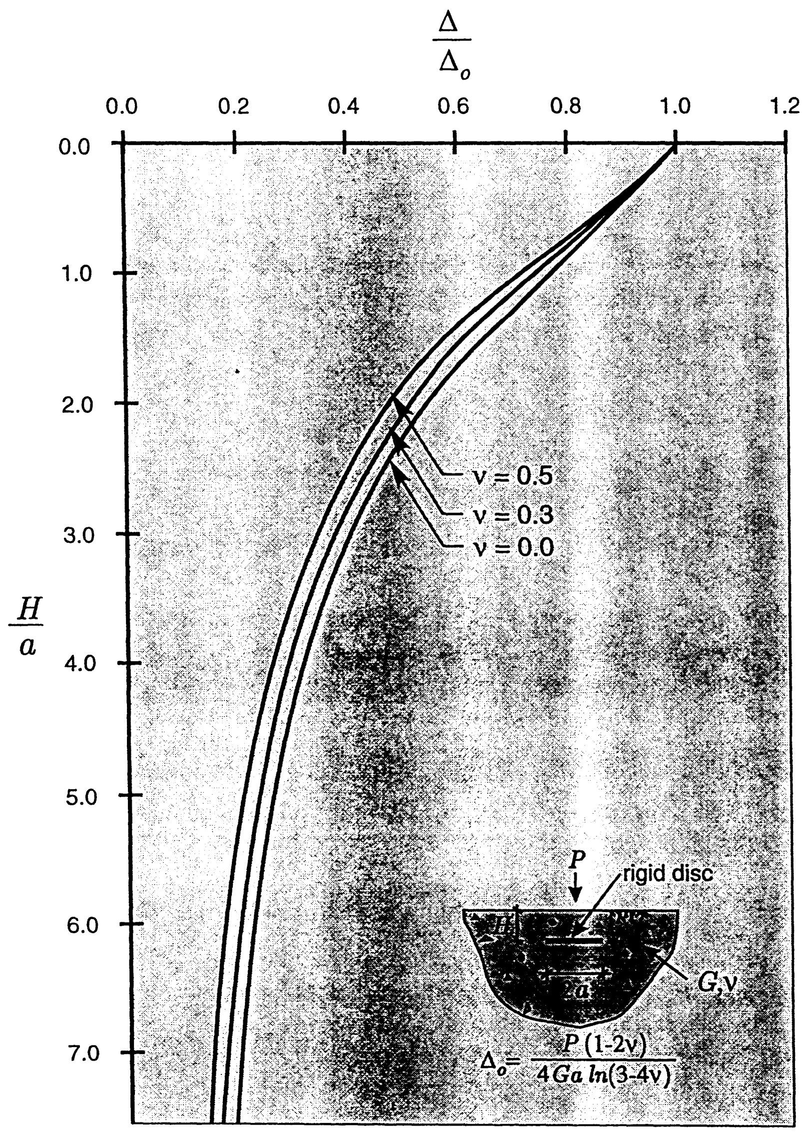

Fig. 3. Axial displacement of the embedded disk inclusion due to the Boussinesq load P.

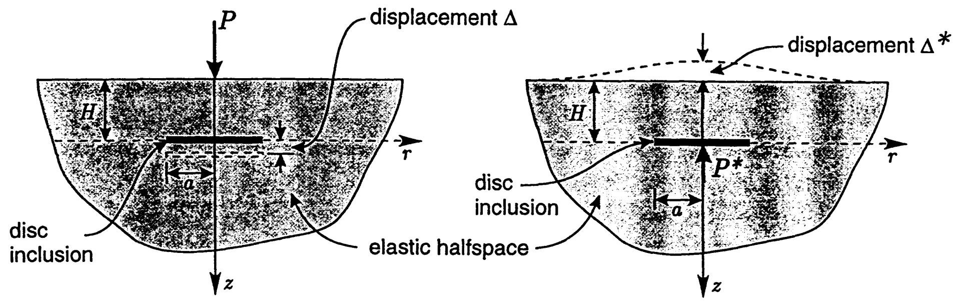

Fig. 4. Application of Betti's reciprocal theorem.

$$
\begin{aligned}
u_z^{(1)}(r,-H) &= \frac{2(1-\nu)}{(3-4\nu)}\left[\int_0^a \Phi(t)dt \int_0^\infty \frac{\xi[\xi H - 1 + 2\nu]J_0(\xi r)\cos(\xi t)}{\sinh(\xi H)[1+\coth(\xi H)]}d\xi \right. \\
&+ \left. \int_0^a \Psi(t)dt \int_0^a \frac{[\xi H + 2(1-\nu)]\cos(\xi t)J_0(\xi r)}{\sinh(\xi H)[1+\coth(\xi H)]}d\xi \right] \\
&+ \frac{P}{4\pi G}\left[\frac{2(1-\nu)}{(r^2+H^2)^{\frac{1}{2}}} + \frac{H^2}{(r^2+H^2)^{\frac{3}{2}}}\right] \quad ; \quad 0 \le r < \infty, \qquad (66)
\end{aligned}
$$

where the last term of (66) corresponds to the surface displacement due to the concentrated force P. Numerical evaluation of the result (66) indicates that when $H/a > 3$, the surface displacement of the half-space region is virtually unaffected by the presence of the rigid disk inclusion.

The solution to the problem of the directly loaded rigid disk inclusion embedded in an elastic half-space was considered by Selvadurai [22]. This problem is of some importance in connection with the study of disk-shaped anchoring devices that are embedded in elastic geomaterials. The magnitude of the maximum displacement of the surface of the half-space due to internally loaded rigid disk inclusion can be obtained by using Betti's reciprocal theorem and the results prescribed in Figure 3. Referring to Figure 4,

$$P\Delta^* = P^*\Delta. \qquad (67)$$

The numerical values for Δ/Δ_0 are given in Figure 3.

5. CONCLUDING REMARKS

Boussinesq's classical problem for the surface loading of a homogeneous elastic half-space has been extended to include the effect of a rigid disk-shaped inclusion that is located beneath the surface load. The numerical evaluation of the governing coupled Fredholm integral equations of the second kind can be used to develop results of engineering interest. In particular, specific numerical results are presented in the rigid displacement of the inclusion due to the Boussinesq force. As the embedded rigid disk inclusion approaches the boundary of the half-space, the problem reduces to that of the axial loading of a half-space by a bonded rigid disk inclusion. In the current study, the stress singularity at the boundary of the inclusion has a regular $\frac{1}{\sqrt{r}}$ form and does not account for the oscillatory form of the stress singularity expected of bonded punch problems examined via a Hilbert problem formulation. It is shown that the contribution of such stress singularities to the overall load-displacement behavior is negligible. The relationship between the Boussinesq force and the displacement of the inclusion also serves as the auxiliary solution that can be used, in conjunction with Betti's reciprocal theorem, to examine the directly loaded rigid disk inclusion embedded in an elastic half-space region. The problem examined in the paper is of course highly idealized in the sense of the axial symmetry of the problem. The result of the surface displacement of the half-space region, however, can be used, again in conjunction with Betti's reciprocal theorem, to determine the estimates for the displacement of the rigid disk inclusion due to other forms of axisymmetric distributed loads.

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